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CORRECTION

Correction to: Towards the exact simulation using hyperbolic Brownian motion

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The authors would like to correct the errors in the publication of the original article. The corrected details are given below for your reading.

Theorem 2 (i) claims that the derivative with respect to x of logarithm of the heat kernel of $\mathbb{H}^2$ is bounded, but its proof presented in the paper has the following gap. In Line-7 of page 837, it is claimed that,

$$e^t(2\pi)p_4(t,r) \leq \frac{3}{2}\frac{1}{1+\cosh(r)}p_2(t,r),$$

which is implied by Gruet's formula:

$$p_n(t,r) = \frac{e^{-(n-1)^2t/8}}{\pi(2\pi)^{n/2}t^{1/2}}\Gamma\left(\frac{n+1}{2}\right)\int_0^\infty \frac{e^{(\pi^2-b^2)/2t}\sinh(b)\sin(\pi b/t)}{(\cosh(b)+\cosh(r))^{(n+1)/2}}db, \qquad (1)$$

where $p_n(t, r(z, z'))$ is the heat kernel of $\mathbb{H}^n$, $r(z, z')$ being the hyperbolic distance between z and z'. However, since the integrand in the left-hand-side of (1) is not always positive, the inequality is not guaranteed.

Actually the claimed inequality is too sharp to prove, we should at this stage admit that Theorem 2 is an error. Instead here we present a corrected version.

The original article can be found online at https://doi.org/10.1007/s13160-017-0265-9.

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Let

$$\theta(t,(x,y),(x',y')) := \mu(x,y)\frac{\frac{\partial q_2}{\partial x}(t,(x,y),(x',y'))}{q_2(t,(x,y),(x',y'))}$$

Theorem 2 (i) *For $z=(x,y), z'=(x',y')\in\mathbb{H}^2$, we have that*

$$|\partial_x q_2(t,z,z')| \le C'(yy')^{-\frac{1}{2}}t^{-\frac{1}{2}}q_2(2t,z,z') \tag{2}$$

for some constant $C'>0$, and for each $n\ge 2$, $t>0$ and $(s_1,\dots,s_{n-1})\in\Delta_{n-1}(t)$

$$\mathbb{E}\left[\prod_{i=1}^{n}\theta\left(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i}\right)\Big|\,Z^0_0=z,Z^0_t=z'\right]q(t,z,z')(y')^{-2}\in L^1(\Delta_{n-1}(t)) \tag{3}$$

where $s_0=0$, $s_n=t$.

(ii) *Set*

$$h_1(t,z,z')=\mu(x,y)\frac{\partial}{\partial x}q_2(t,z,z')(y')^{-2},$$

and

$$h_n(t,z,z') := \int_{\Delta_{n-1}(t)}\mathbb{E}\left[\prod_{i=1}^{n}\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})|Z^0_0=z,Z^0_t=z'\right] \times q_2(t,z,z')/(y')^2 ds_1\dots ds_{n-1}$$

for $n\ge 2$. Then, the series $\sum_{n=1}^{N}h_n(t,z,z')$ is absolutely convergent as $N\to\infty$ uniformly in (t,z,z') on every compact set.

(iii) *The transition density of Z^μ is given by*

$$s(t,z,z') := \frac{q_2(t,z,z')}{(y')^2}+\int_{\mathbb{H}^2}\int_0^t\frac{q_2(t-s,z,z'')}{(y'')^2}\Phi(s,z'',z')dsdz'',$$

where $\Phi(t,z,z')=\sum_{n=1}^{\infty}h_n(t,z,z')$.

To prove Theorem 2, we use the following

Theorem [1]. Let p_n be the heat kernel of the hyperbolic space $\mathbb{H}^n$, and

$$k_{n+1}(t,r) := (2\pi t)^{-\frac{n+1}{2}}e^{-\frac{n^2}{8}t-\frac{r^2}{2t}-\frac{nr}{2}}(1+r+t)^{\frac{n}{2}-1}(1+r), \tag{4}$$

for $n\ge 1$, $t>0$, and $r>0$. For any integer $n>1$, then

$$p_n(t,r)\sim k_n(t,r), \tag{5}$$

uniformly in $r\ge 0$ and $t>0$.

Proof of Theorem 2 By (4) and (5), we have in particular,

$$p_2(t,r) \sim \frac{e^{-\frac{t}{8}-\frac{r^2}{2t}-\frac{r}{2}}}{2\pi t}(1+r+t)^{-\frac{1}{2}}(1+r),$$

and

$$p_4(t,r) \sim \frac{e^{-\frac{9t}{8}-\frac{r^2}{2t}-\frac{3r}{2}}}{(2\pi t)^2}(1+r+t)^{\frac{1}{2}}(1+r).$$

To start with, we calculate the derivative with respect to x of q_2. Using Milson's formula, we have

$$\left|\frac{\partial q_2}{\partial x}(t,(x,y),(x',y'))\right| = \frac{|x-x'|}{yy'}2\pi e^t p_4(t,r).$$

Then,

$$\begin{aligned}
\frac{|x-x'|}{yy'}2\pi e^t p_4(t,r) &\le K\frac{|x-x'|}{yy'}2\pi e^t \frac{e^{-\frac{9t}{8}-\frac{r^2}{2t}-\frac{3r}{2}}}{(2\pi t)^2}(1+r+t)^{\frac{1}{2}}(1+r)\\
&= K\frac{|x-x'|}{yy'}\frac{e^{-\frac{t}{4}-\frac{r^2}{4t}-\frac{r}{2}-\frac{r^2}{4t}+\frac{t}{8}-r}}{2\pi t^2}(1+r+t)^{\frac{1}{2}}(1+r)\\
&\le K'\frac{|x-x'|}{yy'}(1+r+t)^{\frac{1}{2}}(1+r+2t)^{\frac{1}{2}}\frac{e^{-\frac{r^2}{4t}+\frac{t}{8}-r}}{t}p_2(2t,r)\\
&\le K'\frac{|x-x'|}{yy'}(1+r+2t)\frac{e^{-\frac{r^2}{4t}+\frac{t}{8}-r}}{t}p_2(2t,r)\\
&= \frac{K'|x-x'|}{yy'}\frac{e^{-\frac{r^2}{8t}}}{r}(1+r+2t)\frac{re^{-\frac{r^2}{8t}+\frac{t}{8}-r}}{\sqrt{t}}\frac{p_2(2t,r)}{\sqrt{t}},
\end{aligned} \tag{6}$$

where we have used (5) twice. We shall estimate the last term of (6). First, we observe that

$$\begin{aligned}
\frac{|x-x'|}{yy'} &= \sqrt{\frac{2}{yy'}}\sqrt{\frac{|x-x'|^2}{2yy'}}\\
&\le \sqrt{\frac{2}{yy'}}\sqrt{\frac{|x-x'|^2+(y-y')^2}{2yy'}}\\
&= \sqrt{\frac{2}{yy'}}\sqrt{\cosh(r)-1}\\
&\le \sqrt{\frac{2}{yy'}}\sqrt{\cosh^2(r)-1} = \sqrt{\frac{2}{yy'}}\sinh(r).
\end{aligned}$$

Then we see that

$$\frac{|x-x'|}{r\sqrt{yy'}}e^{-\frac{r^2}{8t}}$$

is bounded since $\lim_{r\to 0}\sinh(r)/r = 1$ and $\lim_{r\to\infty}\sinh(r)e^{-\frac{r^2}{8t}}/r = 0$. Noting that $(1+r+2t)e^{\frac{t}{8}-r}$, and $\frac{r}{\sqrt{t}}e^{-\frac{r^2}{8t}}$ with $(t,r)\in[0,T]\times[0,\infty)$, are bounded, we obtain (2).

Subsequently, we see that h_1 is bounded by $Cq_2/\sqrt{yy't}$ with some constant C. Therefore,

$$\begin{aligned}
&|h_n(t,z,z')|\\
&\quad\le \int_{\Delta_{n-1}(t)}\mathbb{E}\left[\prod_{i=1}^{n}\left|\theta(s_i-s_{i-1},Z^0_{s_{i-1}},Z^0_{s_i})\right|\middle|Z^0_0=z,Z^0_t=z'\right]\\
&\quad\times q_2(t,z,z')/(y')^2 ds_1\dots ds_{n-1}\\
&\quad\le \int_{\Delta_{n-1}(t)}\int_{\mathbb{H}}\cdots\int_{\mathbb{H}}|\mu(z)|\left|\frac{\partial_x q_2(s_1,z,z_1)}{(y_1)^2}\right|\times\cdots\times|\mu(z_{n-1})|\left|\frac{\partial_x q_2(t-s_{n-1},z_{n-1},z')}{(y')^2}\right|\\
&\quad\times dz_1\dots dz_{n-1}ds_1\dots ds_{n-1}\\
&\quad\le \int_{\Delta_{n-1}(t)}\int_{\mathbb{H}}\cdots\int_{\mathbb{H}}|\mu(z)\times\cdots\times\mu(z_{n-1})|\\
&\quad\times\frac{q_2(2s_1,z,z_1)/(y_1)^2\times\cdots\times q_2(2(t-s_{n-1}),z_{n-1},z')/(y')^2}{\sqrt{y}y_1\dots y_n\sqrt{y'}\sqrt{s_1\times\cdots\times(t-s_{n-1})}}\\
&\quad\times dz_1\dots dz_{n-1}ds_1\dots ds_{n-1}\\
&\quad\le \int_{\Delta_{n-1}(t)}\int_{\mathbb{H}}\cdots\int_{\mathbb{H}}K_0^nC^n\sqrt{\frac{y}{y'}}\frac{q_2(2s_1,z,z_1)/(y_1)^2\times\cdots\times q_2(2(t-s_{n-1}),z_{n-1},z')/(y')^2}{\sqrt{s_1\times\cdots\times(t-s_{n-1})}}\\
&\quad\times dz_1\dots dz_{n-1}ds_1\dots ds_{n-1}.
\end{aligned}$$

Using the Chapman–Kolmogorov equation, we then obtain that

$$\begin{aligned}
|h_n(t,z,z')| &\le K_0^nC^n\sqrt{\frac{y}{y'}}\frac{q_2(2t,z,z')}{(y')^2}\int_{\Delta_{n-1}(t)}\frac{1}{\sqrt{s_1\times\cdots\times(t-s_{n-1})}}ds_1\dots ds_{n-1}\\
&= K_0^nC^n\sqrt{\frac{y}{y'}}\frac{q_2(2t,z,z')}{(y')^2}t^{\frac{n-1}{2}}\prod_{i=1}^{n}\beta\left(\frac{i}{2},\frac{1}{2}\right).
\end{aligned}$$

Here we have in particular shown (3).

If $n=2m$ with some natural number m, we have

$$\begin{aligned}
\prod_{i=1}^{2m}\beta\left(\frac{i}{2},\frac{1}{2}\right) &= \prod_{i=1}^{2m}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)}\\
&= \prod_{i=2k,k\in\{1\cdots m\}}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)}\prod_{i=2k-1,k\in\{1\cdots m\}}\frac{\Gamma\left(\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \prod_{k=1}^{m} \frac{\Gamma(k)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2k+1}{2}\right)} \prod_{k=1}^{m} \frac{\Gamma\left(\frac{2k-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k)} \\
&= \prod_{k=1}^{m} \left(\frac{\Gamma(k)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2k+1}{2}\right)} \frac{\Gamma\left(\frac{2k-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k)} \right) \\
&= \pi^m \prod_{k=1}^{m} \left(\frac{\Gamma\left(\frac{2k-1}{2}\right)}{\frac{2k-1}{2}\Gamma\left(\frac{2k-1}{2}\right)} \right) = \pi^m \prod_{k=1}^{m} \left(\frac{1}{\frac{2k-1}{2}} \right) = \frac{\pi^{m-1/2}}{\Gamma\left(\frac{2m+1}{2}\right)}.
\end{aligned}
$$

On the other hands, when $n = 2m - 1$ for any $m \in \mathbb{N}$, we have

$$
\begin{aligned}
\prod_{i=1}^{2m-1} \beta\left(\frac{i}{2}, \frac{1}{2}\right) &= \prod_{i=1}^{2m-1} \frac{\Gamma\left(\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)} \\
&= \prod_{i=2k, k\in\{1\cdots m-1\}} \frac{\Gamma\left(\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)} \prod_{i=2k-1, k\in\{1\cdots m\}} \frac{\Gamma\left(\frac{i}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{i+1}{2}\right)} \\
&= \prod_{k=1}^{m-1} \frac{\Gamma(k)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2k+1}{2}\right)} \prod_{k=1}^{m} \frac{\Gamma\left(\frac{2k-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k)} \\
&= \frac{\Gamma\left(\frac{2m-1}{2}\right)}{\Gamma(m)} \prod_{k=1}^{m-1} \left(\frac{\Gamma(k)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{2k+1}{2}\right)} \frac{\Gamma\left(\frac{2k-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k)} \right) \\
&= \frac{\pi^{m-1}\Gamma\left(\frac{2m-1}{2}\right)}{\Gamma(m)} \prod_{k=1}^{m-1} \frac{\Gamma\left(\frac{2k-1}{2}\right)}{\frac{2k-1}{2}\Gamma\left(\frac{2k-1}{2}\right)} \\
&= \frac{\pi^{m-1}\Gamma\left(\frac{2m-1}{2}\right)}{\Gamma(m)} \prod_{k=1}^{m-1} \frac{1}{\frac{2k-1}{2}} = \frac{\pi^{m-3/2}\Gamma\left(\frac{2m-1}{2}\right)}{\Gamma(m)\Gamma\left(\frac{2m-1}{2}\right)} = \frac{\pi^{m-3/2}}{(m-1)!}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\sum_{n=1}^{\infty} |h_n(t, z, z')| &\leq \sum_{n=1}^{\infty} K_0^n C^n \sqrt{\frac{y}{y'}} \frac{q_2(2t, z, z')}{(y')^2} t^{\frac{n-1}{2}} \prod_{i=1}^{n} \beta\left(\frac{i}{2}, \frac{1}{2}\right) \\
&= \sqrt{\frac{y}{y'}} \frac{q_2(2t, z, z')}{(y')^2} \sum_{n=1}^{\infty} \left(K_0^{2n} C^{2n} t^{\frac{2n-1}{2}} \prod_{i=1}^{2n} \beta\left(\frac{i}{2}, \frac{1}{2}\right) \right. \\
&\quad \left. + K_0^{2n-1} C^{2n-1} t^{n-1} \prod_{i=1}^{2n-1} \beta\left(\frac{i}{2}, \frac{1}{2}\right) \right) \\
&= \sqrt{\frac{y}{y'}} \frac{q_2(2t, z, z')}{(y')^2} \sum_{n=1}^{\infty} \left(K_0^{2n} C^{2n} t^{\frac{2n-1}{2}} \frac{\pi^{n-1/2}}{\Gamma\left(\frac{2n+1}{2}\right)} \right. \\
&\quad \left. + K_0^{2n-1} C^{2n-1} t^{n-1} \frac{\pi^{n-3/2}}{(n-1)!} \right),
\end{aligned}
$$

which shows that the series is convergent.

The proof of (iii) in the paper need not be corrected. □

Reference

1. Davies, E.B., Mandouvalos, N.: Heat Kernel bounds on hyperbolic space and Kleinian groups. Proc. Lond. Math. Soc. (3) **52**(1), 182–208 (1988)