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LINCEI CELEBRATIVE ESSAYS

Multifractal approach to fully developed turbulence

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Abstract

The multifractal description of complex phenomena has been introduced in the first half of the 1980s for the characterization of the anomalous scaling of the fully developed turbulence and the structure of the chaotic attractors. From a technical point of view, the idea of the multifractal is basically contained in the large deviations theory; however, the introduction of the multifractal description in 1980s had an important role in statistical physics, chaos, and disordered systems. In particular, to clarify in a neat way that the usual idea, coming from critical phenomena, that just few scaling exponents are relevant, cannot be completely accurate, and an infinite set of exponents is necessary for a complete characterization of the scaling features. We briefly review here the basic aspects and some applications of the multifractal model for turbulence.

Keywords Fully developed turbulence · Self-similarity · Anomalous scaling

Mathematics Subject Classification 76F05 · 82C05

1 Introduction

Let us start by stressing the main difficulties in the building a theory of the fully developed turbulence (FDT) from the first principle; for a general introduction to FDT, we suggest the book by Frisch (1995). As first, we note that the Navier–Stokes equations (NSE)

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{u} + \mathbf{F} \ , \quad \nabla \cdot \mathbf{u} = 0$$

in the limit $\nu \to 0$ and $t \to \infty$, has no relation at all with the Euler equation

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \ , \quad \nabla \cdot \mathbf{u} = 0 \, ,$$

where $\mathbf{u}$ is the velocity field, ρ the fluid density, p the pressure, ν the dynamic viscosity of the fluid, and $\mathbf{F}$ an external forcing driving the system. Actually, the limit $\nu \to 0$, $\mathbf{F} \to 0$, $\mathbf{t} \to \infty$ of the NS equation does not approach to the Euler equations: we are in presence of a singular limit and the statistical features are completely different. This can be understood comparing the statistical features of the Euler equation with those of the FDT.

For the inviscid fluid, once an ultraviolet cutoff is introduced in the Fourier series of the velocity field, it is enough to use the Liouville theorem and the energy conservation and follow the usual approach used for the standard statistical mechanics of Hamiltonian system (Bohr et al. 1998). Consider a fluid in a box L^3 with periodic boundary conditions and a cutoff

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{L^{3/2}} \sum_{|\mathbf{k}|<K_M} \hat{\mathbf{u}}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} \, ;$$

since the incompressibility condition $\hat{\mathbf{u}}(\mathbf{k}, t) \cdot \mathbf{k} = 0$, it is appropriate to use a set of independent variables $\{X_n\}$, evolving according to

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$$\frac{dX_n}{dt} = -\nu k_n^2 X_n + \sum_{j,l} M_{njl} X_j X_l + f_n \quad , \quad n = 1, 2, ..., N \sim K_M^3 .$$

For the Euler equation $\nu = f_n = 0$, it is easy to show that

$$\sum_n \frac{\partial}{\partial X_n} \frac{dX_n}{dt} = 0$$

and

$$\frac{1}{2} \sum_n X_n^2 = E = \text{const} ;$$

therefore, following the same reasoning used for the statistical mechanics of Hamiltonian systems, one obtains

$$P(\{X_n\}) = C\, \delta\Big(\frac{1}{2} \sum_n X_n^2 - E \Big)$$

and equipartition $\langle X_n^2 \rangle = 2E/N$, in the limit of large N, one has a Gaussian distribution

$$P(\{X_n\}) \sim \exp -\frac{\beta}{2} \sum_n X_n^2 .$$

On the other hand, the Euler equation has a rather poor relation with the NSE in the limit $R_e \to \infty$ where R_e is the Reynolds number, U being the r.m.s velocity and L the domain size. The main reason is due to the fact that in the limit $R_e \to \infty$, one has $\epsilon = \frac{\nu}{2} \sum_{i,j} \langle (\partial_j u_i + \partial_i u_j)^2 \rangle = O(1)$ (Frisch 1995; Bohr et al. 1998).

Looking at the experimental data, we see that, instead of the equipartition for $\langle X_n^2 \rangle$, we have a rather different feature; in particular, in a wide range of k_n, we have

$$\langle X_n^2 \rangle \sim k_n^{-\frac{11}{3}} .$$

After the many attempts in the last century, it is now rather clear that an analytical approach to FDT is very difficult (impossible?). The main reason is the nonlinear character of the system: a simple analysis is enough to show that to compute $\langle X_n^2 \rangle$, one has to deal with $\langle X_n X_j X_k \rangle$, and then, for these correlations, one need $\langle X_n X_j X_k X_m \rangle$, and so on; this is the well-known problem of the hierarchy.

If we want to reach some result, it is necessary to close an infinite set of the equations. The situation is similar to the BGGKY hierarchy in kinetic theory for dilute gases, but here a simple approach, e.g., assuming that

$$\langle X_n X_j X_k X_m \rangle = \langle X_n X_j \rangle \langle X_k X_l \rangle + \langle X_n X_k \rangle \langle X_j X_l \rangle + \langle X_n X_l \rangle \langle X_k X_m \rangle,$$

gives inconsistent results.

Therefore, for the closure path, one is forced to use some phenomenological ideas, or some sophisticated theoretical approach: among the most relevant, we can mention the attempts of R. Kraichnan. The reader can find a detailed discussion of the different attempt to the closure problem in the books by Lesieur (1983) and Leslie (1983) .

2 Self-similarity and scaling in turbulence

In the following, we will discuss an alternative approach, which is surely less ambitious but allows for some interesting results.

Perhaps, the scientist who realized in a clear way that it is not necessary to insist too much for a theory from the first principles was Kolmogorov (1941). His point of view had been clearly described by Sinai (2003):

When Kolmogorov was close to 80, I asked him about the history of his discoveries of the scaling laws. He gave me a very astonishing answer by saying that for half a year he studied the results of concrete measurements. ... Kolmogorov was never seriously interested in the problem of existence and uniqueness of solutions of the Navier–Stokes system. He also considered his theory of turbulence as purely phenomenological and never believed that it would eventually have a mathematical framework.

The ideas of Kolmogorov was based on the seminal intuitions of Richardson (1992), who, from just a few empirical data, guessed the self-similar structure of turbulence; here is how he summarised his insight in a verse (inspired by a satirical one by Swift):

Big whirls have little whirls
that feed on their velocity,
and little whirls have lesser whirls
and so on to viscosity
in the molecular sense.

Now, we know that such a kind of behaviours is rather common and it appears in many natural phenomena; for instance, in turbulence, cosmology, and geophysics (Paladin and Vulpiani 1987b; Boffetta et al. 2008; Meakin 1998; Harte 2001).

Let us first summarize the basic idea of the Kolmogorov theory (K41) which can be considered the first modern approach to turbulence. Kolmogorov was able to show an exact result from the Navier–Stokes equation: denoting with $\delta v(\ell)$ the (longitudinal) velocity difference between two points at distance ℓ, in the inertial range $\eta \ll \ell \ll L$, where L and U are the typical length and velocity, respectively, $\eta = L R_e^{-\frac{3}{4}}$ is the Kolmogorov length, $R_e = \frac{UL}{\nu}$ is the Reynolds number, and we have the so-called 4/5 law

$$\langle \delta v(\ell)^3 \rangle = -\frac{4}{5} \epsilon \ell + O(\nu),$$

where ϵ is the mean energy dissipation. Kolmogorov 4/5 law is "exact", provided the limit $\nu \to 0$, $L \to \infty$ can be defined for statistically stationary solutions of the Navier–Stokes equation.

From the above result, some physical arguments as well as the experimental fact that $\epsilon \sim \nu|\nabla \mathbf{u}|^2$ is $O(1)$ in the limit $R_e \to \infty$; it is quite natural to assume

$$\delta v(\ell) \sim \ell^h \ , \ \ h = \frac{1}{3} \, .$$

In mathematical terms, the K41 corresponds to an exact self similarity. A simple example of such a case is the self-similar function of Weierstrass (Edgar 2004; Shen 2018)

$$f(x) = \sum_{n=1}^{\infty} A^{-n} \cos(2\pi B^{n-1} x) \ , \ \ 0 \le x \le 1$$

if B is integer and $A < B$, $f(x)$ is not differentiable in any point. One has $f(x + \Delta x) - f(x) \sim \Delta x^h$ where $h = 2 - D_F = \ln A / \ln B < 1$, being D_F the fractal dimension of the curve; this is a case of self-similarity with a single exponent.

The experimental results, e.g., for the energy spectrum, are in good agreement with the K41 (Frisch 1995). On the other hand is now clear that the K41 cannot be completely correct. Perhaps, the first who realized this was L.D. Landau who noted that the K41 is a "sort of mean field" and therefore cannot be exact: it is necessary to take into account the fluctuations (Frisch 1995; Bohr et al. 1998).

Experimental data about intermittency support Landau's criticism: one exponent is not enough (Anselmet et al. 1984). This is well clear from the anomalous scaling of the structure functions

$$\langle |\delta v(\ell)|^p \rangle \sim \ell^{\zeta_p} \ , \ \ \zeta_p \neq \frac{p}{3} \, .$$

To reply to the Landau criticism in 1962, Kolmogorov proposed a refined version of his previous theory, with a log-normal approach (K62) (Kolmogorov 1962) which gives

$$\zeta_p = \frac{p}{3} + \frac{\mu}{18} p(3 - p),$$

where μ is a measure of the fluctuations. K62 contains two parameters $h = 1/3$ and μ, and it is surely better than K41, but there are still some troubles.

3 The multifractal model

As first, let us note that the NSE are invariant under the scaling transformation

$$\mathbf{x} \to \lambda \mathbf{x} \ , \ \ \mathbf{u} \to \lambda^h \mathbf{u} \ , t \to \lambda^{1-h} t \ , \ \ \nu \to \lambda^{1+h} \nu \, .$$

The exponent h cannot be determined with only symmetry considerations; following the K41, the natural candidate is $h = 1/3$. Notice that under the above scale transformation, the energy dissipation rate transforms as $\epsilon \to \lambda^{3h-1}\epsilon$. We can say that the K41 theory corresponds to a global invariance with $h = 1/3$, which leaves invariant ϵ, and $\zeta_p = p/3$, in disagreement with several experimental investigations (Frisch 1995; Bohr et al. 1998).

Actually, we have a rather interesting phenomenon, which goes under the name of intermittency and is a consequence of the breakdown of self- similarity and implies that the scaling exponents cannot be determined on a simple dimensional basis (Frisch 1995; Bohr et al. 1998). Here, we can use an informal definition of intermittency as relative large (non-Gaussian) fluctuations of $\delta v(l)$ with respect to $\sqrt{\langle \delta v(l)^2 \rangle}$. A more detailed definition is provided below in terms of anomalous scaling of the structure functions $\langle |\delta v(\ell)|^p \rangle$.

The multifractal model of turbulence is an attempt to treat intermittency; the idea has been introduced by Parisi and Frisch (1983), and then developed in Benzi et al. (1984): one assumes that the velocity has a local scale-invariance, i.e., there is continuous spectrum of exponents h, each of which belonging to a given fractal set.

As pioneering works which anticipated some aspects of the multifractal approach to turbulence, we can cite the log-normal theory of Kolmogorov (1962), the contributions of Novikov and Stewart (1964) and Mandelbrot (1974).

In the multifractal model, one assumes that in the inertial range, one has $\delta v(\ell, \mathbf{x}) \sim \ell^h$ if $\mathbf{x} \in S_h$ where S_h is a fractal set with dimension $D(h)$ and $h \in (h_m, h_M)$. Noting that the probability to have a given scaling exponent h at the scale ℓ is

$$P(\ell, h) \sim \ell^{3-D(h)};$$

with simple steepest descent estimation, one has

$$\zeta_p = \inf_h \Big(hp + 3 - D(h) \Big) \, .$$

This means that for each value p, one has a dominant singularity $\tilde{h}$ determined by the equation

$$p = \frac{\mathrm{d}D}{\mathrm{d}h}\Big|_{\tilde{h}} \ \ \to \ \ \zeta_p = \Big(p\tilde{h} + 3 - D(\tilde{h}) \Big) \, .$$

So far, there is no successful idea on how to compute $D(h)$ from the NSE *The computation of $D(h)$ from the NSE is not at present an attainable goal.* A first step is a phenomenological approach using multiplicative processes which generalize the K62 log-normal model, corresponding to a parabolic $D(h)$

$$D(h) = -\frac{9}{2\mu} h^2 + \frac{3}{2}(2 + \mu) h - \frac{4 - 20\mu + \mu^2}{8\mu} \, .$$

4 Multifractals and singular measures

Let us briefly discuss an approach introduced by Halsey et al. (1986) for the description of anomalous scaling, sometimes called $f(\alpha)$ vs α formalism.

Given a singular measure $\mu(\mathbf{x})$, one can introduce a partition with cells of size ℓ

$$P_i(\ell) = \int_{\Lambda_i(\ell)} \mathrm{d}\mu(\mathbf{x}) ,$$

x and then a set of generalized dimensions d_q, called Renyi dimensions defined in terms of scaling properties of moments of $P_i(\ell)$

$$\sum_i P_i(\ell)^q \sim \ell^{(q-1)d_q} .$$

Denoting with $f(\alpha)$ the fractal dimension of the regions such that $P_i(\ell) \sim \ell^\alpha$, one has the Renyi dimensions

$$d_q = \frac{1}{q-1} \inf_\alpha (\alpha q - f(\alpha)).$$

For each value q, one has a dominant singularity $\tilde{\alpha}$ determined by the equation

$$q = \frac{\mathrm{d}f}{\mathrm{d}\alpha}\Big|_{\tilde{\alpha}} \quad \rightarrow \quad d_q = \frac{1}{q-1}\Big(\tilde{\alpha} q - f(\tilde{\alpha})\Big).$$

There is a rather close relation between the two approaches, i.e., $f(\alpha)$ vs α and $D(h)$ vs h, it is enough to remind the K62 theory.

Formulating the K62 approach in term of multifractals one has $\delta v(\ell, \mathbf{x}) \sim (\epsilon_\ell(\mathbf{x})\ell)^{1/3}$ where $\epsilon_\ell(\mathbf{x})$ is the energy dissipation on the cell of size ℓ and center in $\mathbf{x}$. Since the energy dissipation is non-negative, one can introduce a measure

$$\mu(\mathbf{x}) \propto \epsilon(\mathbf{x}) .$$

Simple manipulations show $\zeta_q = \frac{q}{3} + \left(\frac{q}{3} - 1\right)(d_{\frac{q}{3}} - 3)$ in addition

$$h \longleftrightarrow \frac{\alpha - 2}{3} \ , \ D(h) \longleftrightarrow f(\alpha) \ , \ f(\alpha) \le \alpha \longleftrightarrow D(h) \le 3h + 2$$
$$\exists \alpha^* \, : \, \alpha^* = f(\alpha^*) = d_1 \longleftrightarrow \exists h^* \, : \, 3h^* + 3 - D(h^*) = \zeta_3 = 1.$$

5 Toward physics

Coming back to FDT, to go on, one has to find a way to write down the $D(h)$. Let us discuss a multiplicative process, called random β model (Benzi et al. 1984): energy is injected at scale L; at the nth step of the cascade, a mother eddy of size $\ell_n = L2^{-n}$ splits into daughter eddies of size ℓ_{n+1} and the daughter eddies cover a fraction $\beta_j \in (0, 1)$ of the mother volume. Such a model was inspired by the β model (Frisch et al. 1978) which corresponds to assume a unique possible value of β.

Since the energy transfer is constant throughout the cascade, for the velocity difference v_n on the scale ℓ_n, one has $v_n = v_0 \ell_n^{1/3} \prod_{j=1}^n \beta_j^{-1/3}$ where β_j are independent, identically distributed random variables. Simple computations give

$$\zeta_q = \frac{q}{3} - \ln_2 \langle \beta^{1-q/3} \rangle .$$

(Fig. 1).

Phenomenological arguments suggest $\beta_j = 1$ with probability x and $\beta_j = 1/2$ with probability $1 - x$. Notice that $\beta_j = 1/2$ corresponds to a velocity sheet and the phenomenology is borrowed by Saffman (1968). The scaling exponents are

$$\zeta_q = \frac{q}{3} - \ln_2 \left(x + (1-x) 2^{\frac{q}{3} - 1} \right)$$

and

$$D(h) = 3 + (3h-1)\left[1 + \ln_2\left(\frac{1-3h}{1-x}\right)\right] + 3h \ln_2\left(\frac{x}{3h}\right).$$

The two limit cases are $x = 1$ (the K41), and $x = 0$ which is the fractal β -model. Using $x = 7/8$, one has a good fit for the ζ_q of the experimental data at high Reynolds numbers; see Fig. 2. Another very popular phenomenological model providing good fit for the anomalous scaling exponents is the She–Leveque model; see She and Leveque (1994).

Up to now, nobody had been able to obtain $D(h)$ from the first principles, i.e., starting from the NSE, so we can say that the multifractal model is something less than a theory. Even so, it allows for the possibility to do precise previsions in terms of a unique "ingredient" which can be obtained from experimental data. Among the nontrivial previsions from the multifractal model, one has the existence of an intermediate dissipation range (Frisch and Vergassola 1991), the computation of the PdF for the gradient of the velocity (Benzi et al. 1991), as well as the acceleration of particle advected by a turbulent field (Biferale et al. 2004); for a detailed discussion about the statistical properties of the dissipation range, see also Benzi and Biferale (2009).

In the K41, since one has a unique scaling exponent $\delta v(\ell) \sim \ell^{1/3}$, then there is just a unique Kolmogorov length η

$$\frac{\delta v(\eta)\eta}{\nu} \sim 1 \quad \rightarrow \quad \eta \sim \left(\frac{\nu}{U}\right)^{\frac{3}{4}} .$$

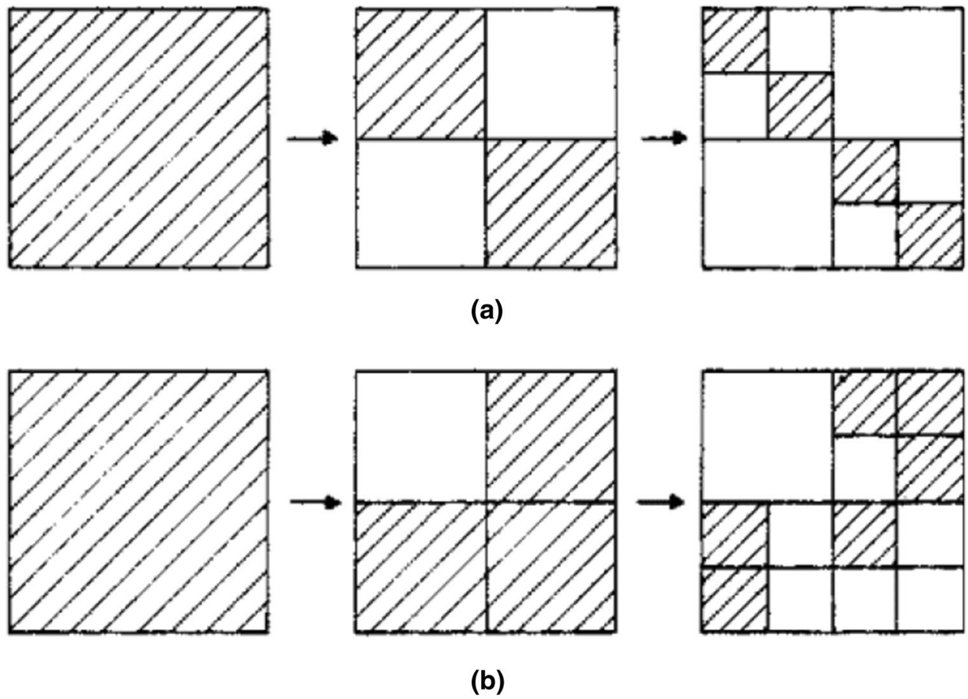

Fig. 1 **a** Schematic view of the β-model; **b** schematic view of the random β-model. The shaded areas are the zones active during the fragmentation process

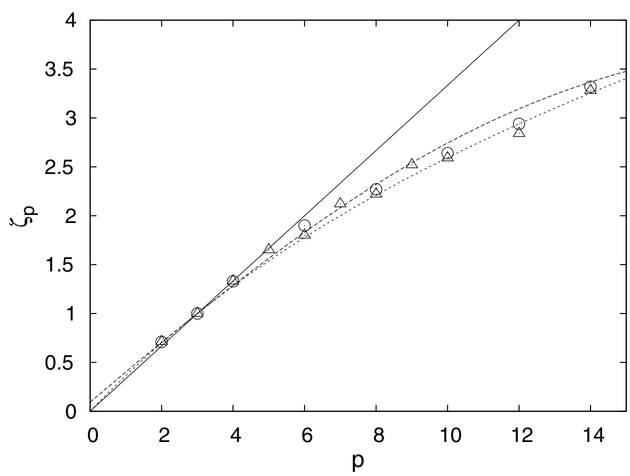

Fig. 2 $\langle|\delta v(\ell)|^p\rangle \sim \ell^{\zeta_p}$ Circles and triangles are the experimental data; the solid line corresponds to K41 scaling $p/3$; the dashed line is the random β-model prediction; the dotted line is another moltiplicative model due to She and Leveque (1994)

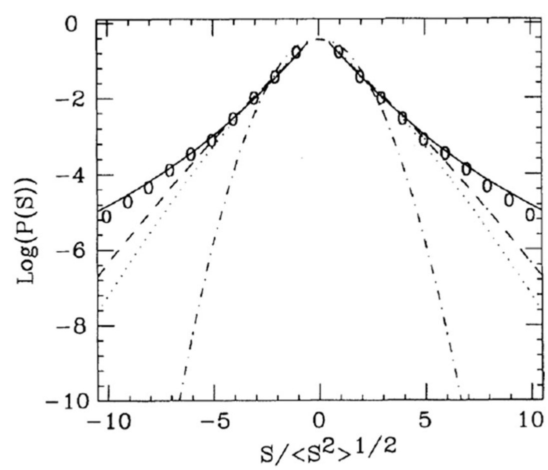

Fig. 3 PdF of the velocity gradient. Points represent experimental data, solid line is the multifractal prediction with the random β-model, dotted and dashed lines represent the K41 and β-model results, respectively, and the dash-dotted line is the Gaussian distribution

In the multifractal model, one has different Kolmogorov lengths $\eta(h)$ (Paladin and Vulpiani 1987) which are selected imposing the effective Reynolds number to be order one

$$R_e(\eta(h)) = \frac{\eta(h)\delta v(\eta(h))}{\nu} \sim 1 \;\rightarrow\; \eta(h) \sim L R_e^{-\frac{1}{1+h}} \sim \left(\frac{\nu}{U}\right)^{\frac{1}{1+h}}.$$

Assuming, in agreement with the experiments, that at large scale, one has a Gaussian statistics, from the knowledge of the $D(h)$, one can derive the PdF of the velocity gradient s (Benzi et al. 1991)

$$p(s) \sim \int \mathrm{d}h \left(\frac{\nu}{|s|}\right)^{2-\frac{h+D(h)}{2}} \exp\left(-\frac{\nu^{1-h}|s|^{1+h}}{2U^2}\right);$$

see Fig. 3.

In a similar way, it is possible to find the PdF for the acceleration (Biferale et al. 2004), Fig. 4 shows the comparison with

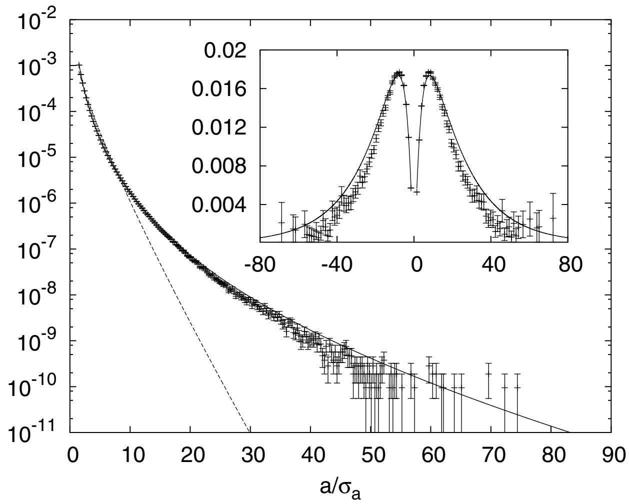

Fig. 4 Pdf of the rescaled acceleration. Points are the DNS data, the solid line is the multifractal prediction, and the dashed line is the K41 prediction. The inset shows $\tilde{a}^4 P(\tilde{a})$ vs $\tilde{a} = a/\sigma_a$

the direct numerical simulation (DNS), and note the very large values of a/σ_a.

6 Intermediate dissipative range

Frisch and Vergassola (1991) showed that the presence of a range of Kolmogorov lengths $\eta(h)$, as discussed in the previous section, has rather interesting consequences. Let us indicate η_m and η_M the minimal and the maximal Kolmogorov lengths, respectively, if, $\ell \gg \eta_M$ for the structure functions of order p, one has the usual scaling, determined by $\tilde{h}(p)$; on the contrary, in the intermediate dissipative range $\eta_m \ll \ell \ll \eta_M$, one has a less simple behaviour which involves not only $\tilde{h}(p)$ but the whole $D(h)$. Remarkably, such a result has rather interesting consequences which can be tested in experiments and numerical simulations.

Translating the above result for the energy spectrum, one has that the standard scaling (valid in the K41) gives

$$E(k) = F(k/k_D)\ , \ \ k_D = \frac{1}{\eta} \sim R_e^{3/4},$$

where $k_D \sim \eta^{-1}$, with $F(z) \sim z^{-5/3}$ for $z \ll 1$, $F(z) \sim e^{-cz}$ for $z \gg 1$.

In presence of multifractality, one has a generalized scaling

$$\frac{\ln E(k)}{\ln R_e} = G\Big(\frac{\ln k}{\ln R_e}\Big);$$

the shape of $G(z)$ depends on the $D(h)$ for $z < z_*$; one has the usual scaling shape, depending only by the exponent ζ_2, while for larger value of z, the shape of $D(h)$ plays a role.

The above generalized scaling had been observed in the experimental data (Gagne et al. 1993). Similar results hold for the statistical Lagrangian features (Arnéodo 2008), and let us mention a very accurate test of the intermediate dissipative range, for the (rescaled) structure function

$$\frac{\langle (v_i(t+\tau) - v_i(t))^4 \rangle}{\langle (v_i(t+\tau) - v_i(t))^2 \rangle} \sim \tau^{\zeta(4,\tau)},$$

where $v_i(t)$ is the Lagrangian velocity of a particles advected by the turbulent field. One has a correspondence with the behaviour observed for the Eulerian properties with ℓ replaced by τ. For large value of τ, the exponent $\zeta(4,\tau)$ is constant, while for small values, it depends on the shape of $D(h)$).

7 Conclusions

Starting from the seminal ideas of Richardson and Kolmogorov, we discussed the statistical features of the fully developed turbulence in the framework of the multifractal model. The still open problem is how to determine $D(h)$ from first principles. On the other hand, it is possible use multiplicative models motivated by phenomenological arguments. We have the nontrivial result that once $D(h)$ is obtained with a fit of the experimental data from the scaling exponents ζ_p, then one can obtain accurate predictions in the multifractal framework, e.g., the PDF of the velocity gradient, the existence of an intermediate dissipative range, the scaling of Lagrangian structure functions, are well verified.

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