Letters in Mathematical Physics (2021) 111:90
https://doi.org/10.1007/s11005-021-01432-y

3-dimensional F-manifolds

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Received: 20 December 2020 / Revised: 16 June 2021 / Accepted: 22 June 2021 /
Published online: 5 July 2021

Abstract

F-manifolds are complex manifolds with a multiplication with unit on the holomorphic tangent bundle with a certain integrability condition. Here, the local classification of 3-dimensional F-manifolds with or without Euler fields is pursued.

Keywords F-manifold · Multiplication on the tangent bundle · Analytic spectrum · Lagrange variety

Mathematics Subject Classification 34M35 · 53D12 · 57R15 · 32B10

Contents

In remembrance of Boris Dubrovin.

This work was funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)–242588615.

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1 Introduction

Boris Dubrovin defined and studied *Frobenius manifolds* [10,11]. A Frobenius manifold is a complex manifold M with a holomorphic flat metric g and a holomorphic commutative and associative multiplication $\circ$ with unit e on the holomorphic tangent bundle TM such that $g(X \circ Y, Z) = g(X, Y \circ Z)$ and such that locally a holomorphic function Φ (a *potential*) with $g(X \circ Y, Z) = XYZ(\Phi)$ for flat vector fields X, Y, Z exists. Often one has additionally an Euler field E, a holomorphic vector field with $\mathrm{Lie}_E(\circ) = 1 \cdot \circ$ and $\mathrm{Lie}_E(g) = D \cdot g$ for some $D \in \mathbb{C}$.

This seemingly purely differential geometric object has many different facets and lies at the crossroads of very different mathematical areas, integrable systems, meromorphic connections, singularity theory, quantum cohomology and thus mirror symmetry. Boris Dubrovin explored many of these crossroads.

Manin and the second author defined the notion of an *F-manifold* [16]. It is a complex manifold M with a holomorphic commutative and associative multiplication $\circ$ with a unit e on the holomorphic tangent bundle which satisfies the integrability condition

$$\mathrm{Lie}_{X \circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ) \text{ for } X, Y \in \mathcal{O}(TM). \tag{1.1}$$

Here, an Euler field is a holomorphic vector field E with $\mathrm{Lie}_E(\circ) = 1 \cdot \circ$.

Frobenius manifolds are F-manifolds, and this is the original motivation for the definition of F-manifolds. However, there are also F-manifolds which cannot be enriched to Frobenius manifolds. The paper [9] starts with F-manifolds and studies how and when they can be enriched to Frobenius manifolds. Crucial is the existence of a certain bundle with a meromorphic connection (called (TE)-structure in [9]) over an F-manifold.

Slightly weaker, but almost as strong as a Frobenius manifold is the notion of a *flat F-manifold*, which was defined by Manin [23]. It is an F-manifold with flat connection D on TM with $D(C^M) = 0$ and $D(e) = 0$, where C^M is the Higgs field from the multiplication, so $C^M_X = X\circ : TM \to TM$ for $X \in \mathcal{O}(TM)$. Then, an Euler field E is an Euler field of the F-manifold such that $D_\bullet E : TM \to TM$ (with $D_\bullet E : X \mapsto D_X E$) is a flat endomorphism.

Recently, flat F-manifolds with Euler fields were subject to work by Arsie and Lorenzoni [2–4,22], Kato, Mano and Sekiguchi [18], Kawakami and Mano [19], Konishi, Minabe and Shiraishi [20,21]. They established such structures on orbit spaces of complex reflection groups. And especially they observed a beautiful correspondence between *regular* flat 3-dimensional F-manifolds and solutions of the Painlevé equations of types VI, V and IV [4,18,19].

A *regular* F-manifold is an F-manifold with Euler field such that the endomorphism $E\circ$ on TM has everywhere for each eigenvalue only one Jordan block. This notion was defined and studied by David and the second author [7]. The classification of germs of regular F-manifolds is given in Theorem 1.3 in [7]: Each such germ is a product of irreducible such germs, and in each dimension, there is (up to isomorphism) only one irreducible germ of a regular F-manifold. Furthermore, a small representative of it is everywhere irreducible. However, the classification of *generically* regular F-manifolds is an open and interesting problem, to which this paper contributes in the case of dimension 3.

The second author studied F-manifolds in [15, ch. 1–5]. There he classified all germs of 2-dimensional F-manifolds with or without Euler fields. This classification is easy, see below. However, already the classification of the germs of 3-dimensional F-manifolds is rich. It was not pursued systematically in [15] or anywhere else.

This paper aims at a systematic classification of germs of 3-dimensional F-manifolds. It succeeds in the majority of the cases, but not in all cases. The classification is up to isomorphism, i.e., up to isomorphisms of germs of complex manifolds, which respect the multiplication.

In order to distinguish different cases, the 3-dimensional algebras over $\mathbb{C}$ have to be listed.

Remarks 1.1 Here, the commutative and associative algebras with unit over $\mathbb{C}$ of dimensions 1, 2 and 3 are listed. In dimension 1, the only algebra is $\mathbb{C}$. In dimension 2, there exist up to isomorphism two algebras

$$P^{(1)} := \mathbb{C}[x]/(x^2),$$
$$P^{(2)} := \mathbb{C} \oplus \mathbb{C}.$$

In dimension 3, there exist up to isomorphism four algebras,

$$Q^{(1)} := \mathbb{C}[x_1, x_2]/(x_1^2, x_1x_2, x_2^2),$$
$$Q^{(2)} := \mathbb{C}[x]/(x^3),$$
$$Q^{(3)} := \mathbb{C} \oplus \mathbb{C}[x]/(x^2) = \mathbb{C} \oplus P^{(1)},$$
$$Q^{(4)} := \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}.$$

A sum $\bigoplus_{j=1}^n \mathbb{C}$ of 1-dimensional algebras is called *semisimple*, so $\mathbb{C}$, $P^{(2)}$ and $Q^{(4)}$ are semisimple. The algebras $\mathbb{C}$, $P^{(1)}$, $Q^{(1)}$ and $Q^{(2)}$ are irreducible. The decomposition of each algebra into irreducible algebras is unique. The algebras $\mathbb{C}$, $P^{(1)}$, $P^{(2)}$, $Q^{(2)}$, $Q^{(3)}$ and $Q^{(4)}$ are Gorenstein rings; $Q^{(1)}$ is not a Gorenstein ring. (The notations $Q^{(1)}$ and $Q^{(2)}$ are opposite to those in [15, 5.5].)

Now let $(M, \circ, e)$ be a connected 3-dimensional complex manifold with a commutative and associative multiplication on TM with unit field e, but not necessarily with (1.1). Choose local coordinates $t = (t_1, t_2, t_3)$, and denote by $y = (y_1, y_2, y_3)$ the fiber coordinates on T^*M such that y_j corresponds to the coordinate vector field $\partial_j := \partial/\partial t_j$. Then, $\alpha := \sum_{j=1}^3 y_j \mathrm{d}t_j$ is the canonical 1-form on T^*M. The multiplication is given by $\partial_i \circ \partial_j = \sum_{k=1}^3 a_{ij}^k \partial_k$ with coefficients $a_{ij}^k \in \mathcal{O}_M$. We suppose

$e = \partial_1$. The multiplication gives rise to the sheaf of ideals $\mathcal{I}_M \subset \mathcal{O}(T^*M)$ with

$$\mathcal{I}_M := \left(y_1 - 1, y_i y_j - \sum_{k=1}^{3} a_{ij}^k y_k \mid i, j \in \{1, 2, 3\} \right) \subset \mathcal{O}(T^*M), \tag{1.2}$$

and the complex space $L_M \subset T^*M$ which is as a set the zero set of $\mathcal{I}_M$ and which has the complex structure $\mathcal{O}_{L_M} = (\mathcal{O}_{T^*M}/\mathcal{I}_M)|_{L_M}$. The projection $\pi_L : L_M \to M$ is flat and finite of degree 3. For each $t \in M$, the points in $\pi_L^{-1}(t) \subset T_t^*M$ are the simultaneous eigenvalues of all endomorphisms $X|_t\circ : T_tM \to T_tM$ for $X \in T_tM$. They correspond to the irreducible subalgebras of T_tM.

The numbering $Q^{(1)}$, ..., $Q^{(4)}$ above was chosen so that for each $j \in \{1, 2, 3, 4\}$, the subset $\bigcup_{i\leq j}\{t \in M \mid T_tM \cong Q^{(i)}\}$ is empty or an analytic subvariety of M or equal to M (Lemma 4.3 gives more precise statements). The algebra $Q^{(j)}$ with $T_tM \cong Q^{(j)}$ for generic $t \in M$ is called the *generic type* of M (M is connected). M is called *generically semisimple* if the generic type is $Q^{(4)}$.

L_M is called *analytic spectrum* of $(M, \circ, e)$. It encodes the multiplication and is crucial for its understanding. The integrability condition (1.1) of an F-manifold is equivalent to $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$, where $\{.,.\}$ is the Poisson bracket on $\mathcal{O}(T^*M)$ [17] (cited in Theorem 2.12). In the generically semisimple case, this is equivalent to $L_M^{reg} \subset T^*M$ being Lagrange. This connects the generically semisimple F-manifolds with the Lagrange fibrations and Lagrange maps of Arnold [1, ch. 18]. Givental's paper [13] on Lagrange maps contains implicitly many results and examples of generically semisimple F-manifolds.

In the case of an F-manifold, the integrability condition (1.1) implies that at a point $t \in M$ such that T_tM decomposes into several irreducible algebras, also the germ of the F-manifold decomposes uniquely into a product of germs of F-manifolds, one for each summand of T_tM [15, Theorem 2.11] (cited in Theorem 2.5), and an Euler field decomposes accordingly. Therefore in the classification of germs of F-manifolds, we can restrict to the classification of the irreducible germs, which are the germs $(M, 0)$ such that T_0M is irreducible.

A rough distinction of classes is given by the isomorphism class T_0M and the generic type. It turns out, that in a germ of dimension ≤ 3, only one or two types arise, the type of T_0M and the generic type. They may coincide. If they do not coincide, the type of T_0M arises in codimension 1 or 2, most often in codimension 1.

The following table shows which examples, lemmas and theorems in this paper concern which class of irreducible germs $(M, 0)$ of F-manifolds of dimensions 1 or 2 or 3. It also indicates the parameters, functional or holomorphic or discrete, in the families of F-manifolds. It does not take into account the possible Euler fields. Though the theorems do.

The results for dimension 1 and 2 are cited from [15], and they are easy. The classification in dimension 3 is surprisingly rich. The cases with $T_0M \cong Q^{(2)}$ are easier than those with $T_0M \cong Q^{(1)}$. In the two cases with $T_0M \cong Q^{(1)}$ and generic type $Q^{(3)}$ or $Q^{(4)}$, we have no complete classification, but just some examples. The reason is that then the integrability condition $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$ is much more difficult to control than in the cases with $T_0M \cong Q^{(2)}$, where we have Lemma 4.6.

Table 1 Table of results

T_0M	Generic type	
$\mathbb{C}$	$\mathbb{C}$	Lemma 2.6: 1 F-manifold A_1
$P^{(1)}$,	$P^{(1)}$	Theorems 3.1, 3.2: 1 F-manifold $\mathcal{N}_2$
$P^{(1)}$	$P^{(2)}$	Theorem 3.1: 1 series $I_2(m)$, $m \in \mathbb{Z}_{\geq 3}$
$Q^{(1)}$	$Q^{(1)}$	Theorem 5.1: 1 functional parameter
$Q^{(1)}$	$Q^{(2)}$	Theorem 5.3 (b)+(c): 1 or 2 functional parameters
$Q^{(2)}$	$Q^{(2)}$	Theorem 5.3 (a): 1 F-manifold
$Q^{(1)}$	$Q^{(3)}$	no complete classification, Lemma 5.7: one family of examples
$Q^{(2)}$	$Q^{(3)}$	Theorem 5.5: 1 series with parameter $p \in \mathbb{Z}_{\geq 2}$
$Q^{(1)}$	$Q^{(4)}$	no complete classification, Theorem 6.3: a structural result, Lemmas 6.4, 6.5 and Examples 6.7: examples
$Q^{(2)}$	$Q^{(4)}$	Examples 6.2: the ADE F-manifolds, Theorem 6.3: a structural result, Theorem 7.1: all other germs, namely 3 families with 1 discrete parameter $p \in \mathbb{Z}_{\geq 2}$, 2 families with 2 discrete parameters $p, q \in \mathbb{Z}_{\geq 2}$ with $q \geq p$, in all 5 families $p-1$ holomorphic parameters

In Theorem 5.3 (c), the type of T_0M arises in codimension 1 or 2. In all other cases in Sects. 5 to 7, the type of T_0M arises in codimension 1 or coincides with the generic type.

Most not generically semisimple F-manifolds appear here for the first time. And also most of the generically semisimple F-manifolds, namely most of those in Theorem 7.1 with $T_0M \cong Q^{(2)}$, are new. Their classification is linked to the classification of germs of plane curves of multiplicity 3 (see the Remarks 7.3).

The germs with $T_0M \cong Q^{(1)}$ and generic type $Q^{(4)}$ are related to certain germs of Lagrange surfaces with embedding dimension 4 which are Cohen–Macaulay, but not Gorenstein (Theorem 6.3 (d)). We do not have a classification of them.

Possibly the most interesting germs $(M, 0)$ of 3-dimensional F-manifolds are the generically semisimple germs with Euler field. Those with $T_0M \cong Q^{(2)}$ are given in Corollary 7.2.

Section 2 collects general facts on F-manifolds from [15]. Section 3 recalls the classification of the 2-dimensional germs of F-manifolds. Section 4 provides basic formulas for 3-dimensional F-manifolds, which are used in the proofs of the classification results in Sects. 5, 6, and 7. Section 5 classifies the not generically semisimple germs (except those with $T_0M \cong Q^{(1)}$ and generic type $Q^{(3)}$). It proceeds by explicit coordinate changes. Section 6 gives examples of generically semisimple F-manifolds and states the structural result Theorem 6.3 for the generically semisimple F-manifolds. Section 7 classifies the generically semisimple germs with $T_0M \cong Q^{(2)}$. Sections 6 and 7 work a lot with the analytic spectrum.

2 General facts on *F*-manifolds

F-manifolds were first defined in [16]. Their basic properties were developed in [15]. This section reviews the main basic properties from [15] and an additional fact from [17].

Definition 2.1 [16] (a) An F*-manifold* $(M, \circ, e)$ (without Euler field) is a holomorphic manifold M with a holomorphic commutative and associative multiplication $\circ$ on the holomorphic tangent bundle TM and with a global holomorphic vector field $e \in \mathcal{T}_M := \mathcal{O}(TM)$ with $e\circ = \mathrm{id}$ (e is called a *unit field*), which satisfies the integrability condition (1.1).

(b) Given an F-manifold $(M, \circ, e)$, an *Euler field* on it is a global vector field $E \in \mathcal{T}_M$ with $\mathrm{Lie}_E(\circ) = \circ$.

Remark 2.2 The integrability condition (1.1) looks surprising at first sight. Though it is natural from several points of view. Here are four of them.

(i) Theorem 2.12 rewrites condition (1.1) as a natural condition on the ideal giving the analytic spectrum in T^*M.
(ii) Theorem 2.5 gives a decomposition result for germs of F-manifolds. Condition (1.1) is crucial in its proof in [15].
(iii) The potentiality condition in a Frobenius manifold with holomorphic metric g is equivalent to (1.1) plus the closedness of the 1-form (called coidentity) $g(e, .)$ [15, Theorem 2.15].
(iv) If the Higgs field of a (TE)-structure over a manifold M is primitive, it induces on M the structure of an F-manifold with Euler field, see, e.g., [9].

Remark 2.3 [15, Proposition 2.10] If one has l F-manifolds $(M_k, \circ_k, e_k)$, $k \in \{1, ..., l\}$, their product $M = \prod_{k=1}^l M_k$ inherits a natural structure of an F-manifold $(M, \bigoplus_{k=1}^l \circ_k, \sum_{k=1}^l (\text{lift of } e_k \text{ to } M))$. And if there are Euler fields E_k, then the sum $E = \sum_{k=1}^l (\text{lift of } E_k \text{ to } M)$ is an Euler field on the product M.

Remark 2.4 A finite dimensional commutative and associative $\mathbb{C}$-algebra A with unit $e \in A$ decomposes uniquely into a direct sum $A = \bigoplus_{k=1}^l A_k$ of local and irreducible algebras A_k with units e_k with $e = \sum_{k=1}^l e_k$ and $A_{k_1} \circ A_{k_2} = 0$ for $k_1 \neq k_2$ (of course, the choice, which summand gets which label in $\{1, ..., l\}$, is arbitrary). This is elementary (linear) algebra. The decomposition is obtained as the simultaneous decomposition into generalized eigenspaces of all endomorphisms $a\circ : A \to A$ for $a \in A$ (see, e.g., Lemma 2.1 in [15]). The algebra A is called *semisimple* if $l = \dim A$ (so then $A_k = \mathbb{C} \cdot e_k$ for all k).

Thanks to the condition (1.1), this pointwise decomposition extends in the case of an F-manifold to a local decomposition, see Theorem 2.5. This is the first important step in the local classification of F-manifolds.

Theorem 2.5 [15, Theorem 2.11] *Let* $((M,t^0),\circ,e)$ *be the germ at* t^0 *of an* F*-manifold.*

(a) The decomposition of the algebra $(T_{t^0}M,\circ|_{t^0},e|_{t^0})$ *with unit into local algebras extends into a canonical decomposition* $(M,t^0)=\prod_{k=1}^{l}(M_k,t^{0,k})$ *as a product of germs of* F*-manifolds.*
(b) If E *is an Euler field of* M*, then* E *decomposes as* $E=\sum_{k=1}^{l}E_k$ *with* E_k *(the canonical lift of) an Euler field on* M_k*.*

Lemma 2.6 [15, Example 2.12 (i)] *In dimension 1, (up to isomorphism) there is only one germ of an* F*-manifold, the germ* $(M,0)=(\mathbb{C},0)$ *with* $e=\partial/\partial_{u_1}$*, where* u_1 *is the coordinate on* $\mathbb{C}$*. Any Euler field on it has the shape* $E=(u_1+c_1)e$ *for some* $c_1\in\mathbb{C}$*.*

Definition 2.7 (a) Fix $n\in\mathbb{N}=\{1,2,..\}$ and define the set of its partitions,

$$\mathcal{P}_n:=\left\{\beta=(\beta_1,...,\beta_{l(\beta)})\,|\,\beta_i\in\mathbb{N},\ \beta_i\geq\beta_{i+1},\sum_{i=1}^{l(\beta)}\beta_i=n\right\}.$$

For $\beta,\gamma\in\mathcal{P}_n$ define

$$\beta\geq\gamma:\iff\ \exists\,\sigma:\{1,...,l(\gamma)\}\to\{1,...,l(\beta)\}\text{ s.t. }\beta_j=\sum_{i\in\sigma^{-1}(j)}\gamma_i,$$
$$\beta>\gamma:\iff\ \beta\geq\gamma\text{ and }\beta\neq\gamma.$$

(b) Let $(M,\circ,e)$ be an F-manifold of dimension n. Consider the map

$$P:M\to\mathcal{P}_n,\ P(t):=\{\text{the partition of } n \text{ by the dimensions of the irreducible subalgebras of } T_tM\}$$

(c) An F-manifold is called *generically semisimple* if $P(t)=(1,...,1)$ ($\Leftrightarrow$ $l(P(t))=n$) for generic t (In [15] such an F-manifold is called *massive*). An F-manifold is called *semisimple* if it is semisimple at all points.

Lemma 2.8 *Let* $(M,\circ,e)$ *be an* F*-manifold of dimension* n*.*

(a) [15, Proposition 2.5] *For any* $\beta\in\mathcal{P}_n$*, the set* $\{t\in M\,|\,P(t)\geq\beta\}$ *is an analytic subset of* M *or empty.*
(b) [15, Proposition 2.6] *Suppose that* M *is connected. Then, there is a unique partition* $\beta_0\in\mathcal{P}_n$ *such that the set* $\{t\in M\,|\,P(t)=\beta_0\}$ *is open. Its complement is called* caustic *and is denoted by* $\mathcal{K}:=\{t\in M\,|\,P(t)\neq\beta_0\}$*. The caustic is an analytic hypersurface or empty. If* $t\in\mathcal{K}$*, then* $P(t)>\beta_0$*.*
(c) By Theorem 2.5 and Lemma 2.6, a semisimple germ of an F*-manifold is isomorphic to* $(\mathbb{C}^n,0)$ *with coordinates* $u=(u_1,...,u_n)$ *and partial units* $e_k=\partial_{u_k}$*, which determine the multiplication by* $e_k\circ e_k=e_k$ *and* $e_{k_1}\circ e_{k_2}=0$ *for* $k_1\neq k_2$*. The global unit field is* $e=\sum_{k=1}^{n}e_k$*. The semisimple germ of dimension* n *is*

said to be of type A_1^n. *The coordinates* u_k *or their shifts* $u_k + c_k$ *for any constants* $c_1, .., c_n \in \mathbb{C}$ *are* Dubrovin's canonical coordinates. *Any Euler field on this* F*-manifold has the shape* $E = \sum_{k=1}^n (u_k + c_k)e_k$ *for some* $c_1, ..., c_n \in \mathbb{C}$. *If an Euler field* E *is fixed, the eigenvalues* $u_k + c_k$ *of* $E\circ$ *can be used as canonical coordinates. This fixes their ambiguity.*

A generically semisimple F-manifold M has canonical coordinates locally on $M - \mathcal{K}$, so there it can be described easily. A description near K is more difficult and more interesting.

Three notions from the theory of isolated hypersurface singularities generalize to F-manifolds, the *μ-constant stratum*, the *modality*, and *simpleness*.

Definition 2.9 Let $(M, \circ, e)$ be an F-manifold.

(a) For $p \in M$, the *μ-constant stratum* of p is the subvariety $S_\mu(p) := \{t \in M \mid P(t) \geq P(p)\}$. The *modality* $\mathrm{mod}_\mu(M, p)$ is

$$\mathrm{mod}_\mu(M, p) := \dim(S_\mu(p), p) - l(P(p)). \tag{2.1}$$

(b) The F-manifold is *simple* if $\mathrm{mod}_\mu(M, p) = 0$ for any $p \in P$. A simple F-manifold is generically semisimple because for any F-manifold $\mathrm{mod}_\mu(M, p) = n - l(\beta_0)$ for $p \in M - \mathcal{K}$.

The definition of the modality is motivated by the following. If $(M, p) = \prod_{j=1}^{l(P(p))} (M_j, p^{(j)})$ as a germ of an F-manifold with idempotent vector fields $e_1, ..., e_{l(P(p))}$, then $\mathrm{Lie}_{e_j}(\circ) = 0 \cdot \circ$, so the germs (M, q) for q in one integral manifold of $e_1, ..., e_{l(P(p))}$ are isomorphic as germs of F-manifolds.

In the case of a generically semisimple F-manifold with Euler field, the Euler field gives rise to a complementary result.

Theorem 2.10 [15, Corollary 4.16] *Let* $(M, \circ, e, E)$ *be a generically semisimple* F*-manifold with Euler field. For any* $p \in M$, *the set*

$$\{t \in M \mid ((M, t), \circ, e, E) \cong ((M, p), \circ, e, E)\}$$

is discrete and closed in M.

All the information of an F-manifold is carried also by its analytic spectrum, which will be introduced now.

Definition 2.11 Let $(M, \circ, e)$ be a complex manifold of dimension n with a holomorphic commutative and associative multiplication $\circ$ on the holomorphic tangent bundle and with a unit field e. (For the moment, the condition (1.1) is not imposed.)

(a) We need some standard data on T^*M: Let $\pi : T^*M \to M$ denote the projection. Let $t = (t_1, ..., t_n)$ be local coordinates on M, and define $\partial_k := \partial/\partial t_k$. Let $y = (y_1, ..., y_n)$ be the fiber coordinates on T^*M which correspond to $(\partial_1, ..., \partial_n)$.

Then, the canonical 1-form α takes the shape $\sum_{i=1}^{n} y_i \mathrm{d}t_i$, and $\omega = \mathrm{d}\alpha$ is the standard symplectic form. The Hamilton vector field of $f \in \mathcal{O}_{T^*M}$ is

$$H_f = \sum_{k=1}^{n} \Big(\frac{\partial f}{\partial t_k} \cdot \frac{\partial}{\partial y_k} - \frac{\partial f}{\partial y_k} \cdot \frac{\partial}{\partial t_k} \Big). \tag{2.2}$$

The Poisson bracket $\{.,.\}$ on $\mathcal{O}_{T^*M}$ is defined by

$$\{f, g\} := H_f(g) = \omega(H_f, H_g) = -H_g(f). \tag{2.3}$$

(b) Define an ideal sheaf $\mathcal{I}_M \subset \mathcal{O}_{T^*M}$ as follows. We choose coordinates t_k and y_k as in part (a) and such that $e_1 = \partial_1$. Write

$$\partial_i \circ \partial_j = \sum_{k=1}^{n} a_{ij}^k \partial_k \text{ with } a_{ij}^k \in \mathcal{O}_M. \tag{2.4}$$

Then

$$\mathcal{I}_M := \left(y_1 - 1, y_i y_j - \sum_{k=1}^{n} a_{ij}^k y_k \right) \subset \mathcal{O}_{T^*M}. \tag{2.5}$$

The *analytic spectrum* (or *spectral cover*) $L_M := \mathrm{Specan}_{\mathcal{O}_M}(TM, \circ) \subset T^*M$ of $(M, \circ, e)$ is as a set the set at which the functions in $\mathcal{I}_M$ vanish. It is a complex subspace of T^*M with complex structure given by $\mathcal{O}_{L_M} = (\mathcal{O}_{T^*M}/\mathcal{I}_M)|_{L_M}$.

The analytic spectrum L_M was studied in [15, 2.2 and 3.2]. Though, the following result was missed there.

Theorem 2.12 [17, 2.5 Theorem] *A manifold $(M, \circ, e)$ with holomorphic commutative and associative multiplication $\circ$ on the holomorphic tangent bundle and unit field e is an F-manifold if and only if $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$.*

Remarks 2.13 (i) The points in L_M above a point $t \in M$ are the 1-forms, which are the simultaneous eigenvalues for all multiplication endomorphisms in $T_t M$. They are in 1-1 correspondence with the irreducible subalgebras of $T_t M$.

(ii) Let $(M, \circ, e)$ be a complex manifold of dimension n with commutative and associative multiplication on the holomorphic tangent bundle. The projection $\pi|_{L_M} : L_M \to M$ is finite and flat of degree n. The map

$$\mathbf{a} : \mathcal{T}_M \to \pi_* \mathcal{O}(L_M), \quad X \mapsto \alpha(X)|_{L_M}, \tag{2.6}$$

is an isomorphism of $\mathcal{O}_M$-algebras. In this way, the multiplication on $\pi_*\mathcal{O}(L_M)$ determines the multiplication on the tangent bundle. The value $\alpha(X)(y, t) \in \mathbb{C}$ at a point $(y, t) \in L_M$ is the eigenvalue of $X\circ$ on the irreducible subalgebra of $T_t M$ which corresponds to (y, t).

(iii) In the case of a manifold with a multiplication and unit field, such that the multiplication is generically semisimple, the restriction $L_M|_{M-\mathcal{K}}$ of L_M to $M-\mathcal{K}$ is obviously smooth with dim M sheets above $M-\mathcal{K}$. Theorem 3.2 in [15] says that then L_M is reduced everywhere, so also above $L_M \cap \pi|_{L_M}^{-1}(\mathcal{K})$.

(iv) In this situation, $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$ says that L_M is at smooth points a Lagrange submanifold of T^*M.

(v) However, in the case of a manifold with multiplication and unit field, such that the multiplication is nowhere semisimple, the analytic spectrum L_M is nowhere reduced. Then, $\mathcal{I}_M$ is quite different from the reduced ideal $\sqrt{\mathcal{I}_M}$. Especially, the conditions

$$\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M \quad \text{and} \quad \{\sqrt{\mathcal{I}_M}, \sqrt{\mathcal{I}_M}\} \subset \sqrt{\mathcal{I}_M} \tag{2.7}$$

do not imply one another. The second condition in (2.7) is equivalent to the condition that L_M^{red} (the reduced space underlying L_M) is at smooth points a Lagrange submanifold of T^*M. The examples 2.5.2 and 2.5.3 in [17] and the examples below in Theorem 5.1 and Remark 5.2 (ii) with $b_2 \neq 0$ are examples of F-manifolds (so $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$ holds) with $\{\sqrt{\mathcal{I}_M}, \sqrt{\mathcal{I}_M}\} \not\subset \sqrt{\mathcal{I}_M}$. The example (with $n = 4$) in [9, 2.13 (v)] is an example of a manifold $(M, \circ, e)$ with $\{\mathcal{I}_M, \mathcal{I}_M\} \not\subset \mathcal{I}_M$ and $\{\sqrt{\mathcal{I}_M}, \sqrt{\mathcal{I}_M}\} \subset \sqrt{\mathcal{I}_M}$.

We are mainly interested in the case of generically semisimple F-manifolds. There the following result of Givental is relevant. The embedding dimension of a complex space germ $(X, 0)$ is the minimal number $k \in \mathbb{N} \cup \{0\}$ such that an embedding $(X, 0) \hookrightarrow (\mathbb{C}^k, 0)$ exists.

Theorem 2.14 [13, ch. 1.1] *An n-dimensional germ $(L, 0)$ of a Lagrange variety with embedding dimension* $\mathrm{embdim}(L, 0) = n + k$ *with $k < n$ is a product of a k-dimensional Lagrange germ $(L', 0)$ with* $\mathrm{embdim}(L', 0) = 2k$ *and a smooth $(n-k)$-dimensional Lagrange germ $(L'', 0)$; here, the decomposition of $(L, 0)$ corresponds to a decomposition*

$$((S, 0), \omega) \cong ((S', 0), \omega') \times ((S'', 0), \omega'')$$

of the symplectic space germ $(S, 0)$ which contains $(L, 0)$.

Existence of an Euler field for a given F-manifold is a problem with many facets. Some F-manifolds have many Euler fields, others few, others none. The cases of all 2- and many 3-dimensional F-manifolds will be discussed in Sects. 3 to 7. We are mainly interested in the generically semisimple F-manifolds. There the following holds.

Theorem 2.15 *(a)* [15, Theorem 3.3] *Let $(M, \circ, e)$ be a generically semisimple F-manifold. A vector field E is an Euler field if and only if*

$$d(\mathbf{a}(E))|_{L_M^{reg}} = \alpha|_{L_M^{reg}}. \tag{2.8}$$

(b) [15, Lemma 3.4] *Let M be a sufficiently small representative of an irreducible germ (M, t^0) of a generically semisimple F-manifold. For any $c \in \mathbb{C}$, there is a unique function $F : (L, (y^0, t^0)) \to (\mathbb{C}, c)$ which is holomorphic on L_M^{reg} and continuous on L (with value c at (y^0, t^0)) and which satisfies $dF|_{L_M^{reg}} = \alpha|_{L_M^{reg}}$.*

(c) The parts (a) and (b) imply that in the situation of (b), for any $c \in \mathbb{C}$, there is a unique Euler field E_c on $M - \mathcal{K}$ such that for $t \to t^0$ all eigenvalues of $E_c\circ$ tend to c. We have $E_c = E_0 + c \cdot e$. The characteristic polynomial of $E_c\circ$ extends holomorphically to t^0 and has there the value $(x - c)^n$. The Euler field E_c extends holomorphically to M if and only if the function F in part (b) is holomorphic on L_M.

Theorem 2.16 (d) will rephrase the question whether the function F in part (b) is holomorphic on L_M. A special case will be singled out in Theorem 2.16 (e). Now we consider the germ $(S, 0)$ of an N-dimensional manifold and the germ $(L, 0) \subset (S, 0)$ of an n-dimensional reduced subvariety. $H^\bullet_{Giv}(S, L, 0)$ denotes the cohomology of the de Rham complex

$$(\Omega^\bullet_{S,0}/\{\omega \in \Omega^\bullet_{S,0} \mid \omega|_{L^{reg}} = 0\}, \tag{2.9}$$

which was considered first by Givental [13, ch. 1.1].

Theorem 2.16 *(a)* [13, ch. 1.1] *If $(L, 0)$ is quasihomogeneous then $H^\bullet_{Giv}(S, L, 0) = 0$.*

(b) [26] *If $(S, 0) = (\mathbb{C}^2, 0)$ and $(L, 0) = (f^{-1}(0), 0)$ for a holomorphic function germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0, then* $\dim H^1_{Giv}(S, L, 0) = \mu - \tau$, *where*

$$\mu := \dim \mathcal{O}_{\mathbb{C}^2,0}/\Big(\frac{\partial f}{\partial x_i}\Big) \quad \text{and} \quad \tau := \dim \mathcal{O}_{\mathbb{C}^2,0}/\Big(f, \frac{\partial f}{\partial x_i}\Big).$$

(c) [13, ch. 1.2] *In the situation of (b), $\mu > \tau \iff (L, 0)$ is not quasihomogeneous. And if $(L, 0)$ is not quasihomogeneous, then $\eta \in \Omega^1_{\mathbb{C}^2,0}$ satisfies* $[\eta] \in H^1_{Giv}(\mathbb{C}^2, L, 0) - \{0\}$ *if* $\mathrm{d}\eta = u(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$ *with* $u(0) \neq 0$ *(i.e.,* $\mathrm{d}\eta$ *is a volume form).*

(d) [13, ch 1.1] *In part (b) in Theorem 2.15, F is holomorphic on L_M if and only if* $[\alpha] = 0 \in H^1_{Giv}(T^*M, L_M, (y^0, t^0))$.

(e) In part (b) in Theorem 2.15, suppose that $\mathrm{embdim}(L_M, (y^0, t^0)) \leq n + 1$. *Then, $(L_M, (y^0, t^0)) \cong (\mathbb{C}^{n-1}, 0) \times (C, 0)$ where $(C, 0)$ is the germ of a plane curve. And then F is holomorphic on L if and only if $(C, 0)$ is quasihomogeneous.*

(f) A germ (M, t^0) of a simple F-manifold has a (holomorphic) Euler field.

Proof of the parts (e) and (f): (e) The first statement follows from Theorem 2.14, and the decomposition is compatible with a decomposition of the symplectic germ $(T^*M, (y^0, t^0))$. The second statement follows from the first statement and from the parts (a), (c) and (d).

(f) We can restrict to an irreducible germ (M, t^0) of a simple F-manifold. The caustic $\mathcal{K}$ is a hypersurface. At a generic point $p \in \mathcal{K}$,

$$0 = \dim(S_\mu(p), p) - l(P(p)) = n - 1 - l(P(p)), \quad \text{so } l(P(p)) = n - 1,$$

so $P(p) = (2, 1, ..., 1)$, and (M, t^0) is a product of $n-2$ 1-dimensional and 1 2-dimensional F-manifolds. They have Euler fields, so F is holomorphic on $M - \mathcal{K}^{sing}$. However, codim $\mathcal{K}^{sing} \geq 2$, so F is holomorphic on M, and the Euler field E_0 from Theorem 2.15 extends to M. □

A generalization of the generically semisimple F-manifolds are the generically regular F-manifolds.

Definition 2.17 [7, Definition 1.2] Let $(M, \circ, e, E)$ be an F-manifold with Euler field.

(a) The Euler field is regular at a point $t \in M$ if $E \circ |_t : T_t M \to T_t M$ is a regular endomorphism, i.e., it has for each eigenvalue only one Jordan block.
(b) The F-manifold with Euler field $(M, \circ, e, E)$ is called a [generically] regular F-manifold if the Euler field is regular at all [respectively, at generic] points.

Theorem 1.3 in [7] provides a generalization of the canonical coordinates of a semisimple F-manifold with Euler field to the case of a regular F-manifold.

3 2-dimensional *F*-manifolds

The 2-dimensional germs of F-manifolds were classified in [15].

Theorem 3.1 [15, Theorem 4.7] *In dimension 2, (up to isomorphism) the germs of F-manifolds fall into three types:*

(a) The semisimple germ (of type A_1^2). See Lemma 2.8 (c) for it and for the Euler fields on it.

(b) Irreducible germs, which (i.e., some holomorphic representatives of them) are at generic points semisimple. They form a series $I_2(m)$, $m \in \mathbb{Z}_{\geq 3}$. The germ of type $I_2(m)$ can be given as follows.

$$(M, 0) = (\mathbb{C}^2, 0) \text{ with coordinates } t = (t_1, t_2) \text{ and } \partial_k := \frac{\partial}{\partial t_k},$$
$$e = \partial_1, \quad \partial_2 \circ \partial_2 = t_2^{m-2} e. \tag{3.1}$$

Any Euler field takes the shape

$$E = (t_1 + c_1)\partial_1 + \frac{2}{m} t_2 \partial_2 \quad \text{for some } c_1 \in \mathbb{C}. \tag{3.2}$$

(c) An irreducible germ, such that the multiplication is everywhere irreducible. It is called $\mathcal{N}_2$, and it can be given as follows.

$$(M, 0) = (\mathbb{C}^2, 0) \text{ with coordinates } t = (t_1, t_2) \text{ and } \partial_k := \frac{\partial}{\partial t_k},$$

$$e = \partial_1, \quad \partial_2 \circ \partial_2 = 0. \tag{3.3}$$

Any Euler field takes the shape

$$\begin{aligned} E = (t_1 + c_1)\partial_1 + g(t_2)\partial_2 \quad &\text{for some } c_1 \in \mathbb{C} \\ &\text{and some function } g(t_2) \in \mathbb{C}\{t_2\}. \end{aligned} \tag{3.4}$$

However, in the case of $\mathcal{N}_2$, one has still freedom in the choice of the coordinate t_2, and one can use this to put an Euler field into a normal form. This was not studied in [15], but in [8].

Theorem 3.2 [8, Theorem 48] *(a) The automorphism group of the germ $\mathcal{N}_2$ of an F-manifold is*

$$\begin{aligned} \mathrm{Aut}(\mathcal{N}_2) &= \mathrm{Aut}((M,0), \circ, e, E) \\ &= \{(t_1, t_2) \mapsto (t_1, f(t_2)) \mid f(t_2) \in \mathbb{C}\{t_2\} \text{ with } f(0) = 0, f'(0) \neq 0\}. \end{aligned} \tag{3.5}$$

(b) Let $\widetilde{E}$ be an Euler field on $\mathcal{N}_2$. Its orbit under the automorphism group $\mathrm{Aut}(\mathcal{N}_2)$ *contains precisely one of the Euler fields in the following list,*

$$E = (t_1 + c)\partial_1 + \partial_2, \tag{3.6}$$
$$E = (t_1 + c)\partial_1, \tag{3.7}$$
$$E = (t_1 + c)\partial_1 + c_0 t_2 \partial_2, \tag{3.8}$$
$$E = (t_1 + c)\partial_1 + t_2^r(1 + c_1 t_2^{r-1})\partial_2, , \tag{3.9}$$

where $c, c_1 \in \mathbb{C}$, $c_0 \in \mathbb{C}^*$ and $r \in \mathbb{Z}_{\geq 2}$.

$\mathcal{N}_2$ with the Euler field E in (3.6) is regular (Definition 2.17). $\mathcal{N}_2$ with the Euler field E in (3.8) or (3.9) is generically regular. $\mathcal{N}_2$ with the Euler field in (3.7) is not even generically regular.

4 Basic formulas for 3-dimensional *F*-manifolds

Notations 4.1 In Sects. 4 and 5, we consider a 3-dimensional complex manifold M with a holomorphic commutative multiplication on the holomorphic tangent bundle and with a unit field e (so $e\circ = \mathrm{id}$) with $\mathrm{Lie}_e(\circ) = 0$. This last condition $\mathrm{Lie}_e(\circ) = 0$ is weaker than (and implied by) the integrability condition (1.1) of an F-manifold. We do not suppose (1.1) at the beginning, though we suppose $\mathrm{Lie}_e(\circ) = 0$ from the beginning.

We work locally near a point $p \in M$ and suppose to have coordinates $t = (t_1, t_2, t_3)$ with $\partial_1 = e$ and $(M, p) \cong (\mathbb{C}^3, 0)$ and coordinate vector fields $\partial_j = \partial/\partial t_j$. Let $y = (y_1, y_2, y_3)$ be the fiber coordinates on T^*M which correspond to $\partial_1, \partial_2, \partial_3$. Then, the canonical 1-form α takes the shape $\alpha = \sum_{i=1}^3 y_i \mathrm{d}t_i$.

We write

$$\partial_2 \circ \partial_2 = \widetilde{a}_1 \partial_1 + \widetilde{a}_2 \partial_2 + a_3 \partial_3, \tag{4.1}$$
$$\partial_2 \circ \partial_3 = \widetilde{b}_1 \partial_1 + b_2 \partial_2 + b_3 \partial_3, \tag{4.2}$$
$$\partial_3 \circ \partial_3 = \widetilde{c}_1 \partial_1 + c_2 \partial_2 + \widetilde{c}_3 \partial_3, \tag{4.3}$$

with $\widetilde{a}_1, \widetilde{a}_2, a_3, \widetilde{b}_1, b_2, b_3, \widetilde{c}_1, c_2, \widetilde{c}_3 \in \mathcal{O}_M$. Many formulas take a simpler shape if we rewrite the formulas above as follows,

$$(\partial_2 - b_3\partial_1) \circ (\partial_2 - b_3\partial_1) = a_1\partial_1 + a_2(\partial_2 - b_3\partial_1) + a_3(\partial_3 - b_2\partial_1), \tag{4.4}$$
$$(\partial_2 - b_3\partial_1) \circ (\partial_3 - b_2\partial_1) = b_1\partial_1, \tag{4.5}$$
$$(\partial_3 - b_2\partial_1) \circ (\partial_3 - b_2\partial_1) = c_1\partial_1 + c_2(\partial_2 - b_3\partial_1) + c_3(\partial_3 - b_2\partial_1), \tag{4.6}$$

with $a_j, b_j, c_j \in \mathcal{O}_M$. The condition $\mathrm{Lie}_e(\circ) = 0$ is equivalent to $a_j, b_j, c_j \in \mathbb{C}\{t_2, t_3\}$, so we suppose this from now on (and it also implies $\widetilde{a}_1, \widetilde{a}_2, \widetilde{b}_1, \widetilde{c}_1, \widetilde{c}_3 \in \mathbb{C}\{t_2, t_3\}$). We denote

$$\partial_i a_j := a_{ji},\ \partial_i b_j := b_{ji},\ \partial_i c_j := c_{ji},\ \text{and analogously for } \widetilde{a}_j, \widetilde{b}_j, \widetilde{c}_j. \tag{4.7}$$

If $s = (s_1, s_2, s_3)$ is another system of coordinates on (M, p) with $t = t(s)$ and $s = s(t)$ and $s(p) = 0$, write $\widetilde{\partial}_j := \partial/\partial s_j$ for the coordinate vector fields of this system of coordinates, and write $z = (z_1, z_2, z_3)$ for the fiber coordinates which correspond to $\widetilde{\partial}_1, \widetilde{\partial}_2, \widetilde{\partial}_3$. Then

$$\mathrm{d}t_i = \sum_{j=1}^{3} \widetilde{\partial}_j t_i \cdot \mathrm{d}s_j, \quad z_j = \sum_{i=1}^{3} \widetilde{\partial}_j t_i \cdot y_i. \tag{4.8}$$

We suppose $\widetilde{\partial}_1 = e = \partial_1$. This is equivalent to $t_i(s) \in (\delta_{i1} \cdot s_1 + \mathbb{C}\{s_2, s_3\})$ and also to $s_j(t) \in (\delta_{j1} \cdot t_1 + \mathbb{C}\{t_2, t_3\})$. Often it is useful to make first a special coordinate change of the type $t_2 = s_2, t_3 = s_3, t_1 = s_1 + \tau$ with $\tau \in \mathbb{C}\{t_2, t_3\} = \mathbb{C}\{s_2, s_3\}$. Then

$$z_1 = y_1, \quad z_2 = \partial_2\tau \cdot y_1 + y_2, \quad z_3 = \partial_3\tau \cdot y_1 + y_3. \tag{4.9}$$

Lemma 4.2 *In the situation of the Notations 4.1, the multiplication is associative if and only if*

$$a_1 = -a_3c_3, \quad b_1 = a_3c_2, \quad c_1 = -a_2c_2. \tag{4.10}$$

Proof Straightforward calculations with (4.4)–(4.6) of both sides of the equations

$$\begin{aligned} &\big((\partial_2 - b_3\partial_1) \circ (\partial_2 - b_3\partial_1)\big) \circ (\partial_3 - b_2\partial_1) \\ =\ &\big((\partial_2 - b_3\partial_1) \circ (\partial_3 - b_2\partial_1)\big) \circ (\partial_2 - b_3\partial_1), \\ &\big((\partial_2 - b_3\partial_1) \circ (\partial_3 - b_2\partial_1)\big) \circ (\partial_3 - b_2\partial_1) \end{aligned}$$

$$= \big((\partial_3 - b_2\partial_1) \circ (\partial_3 - b_2\partial_1)\big) \circ (\partial_2 - b_3\partial_1).$$

□

The next lemma starts with M as in the Notations 4.1, but with associative multiplication, and tells to which of the four algebras $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$ or $Q^{(4)}$ in the Remarks 1.1 the algebra T_tM for $t \in M$ is isomorphic.

Lemma 4.3 *Let $(M, \circ, e)$ be as in the Notations 4.1 with coordinates $t = (t_1, t_2, t_3)$, and suppose that the multiplication $\circ$ is associative. Define $R_1, R_2, R_3 \in \mathbb{C}\{t_2, t_3\}$ by*

$$R_1 := a_3c_3 - \frac{1}{3}a_2^2,\ \ R_2 := a_2c_2 - \frac{1}{3}c_3^2,\ \ R_3 := a_3c_2 - \frac{1}{9}a_2c_3. \tag{4.11}$$

For a point $t \in M$, the following statements hold.

$$T_tM \cong Q^{(1)} \iff (a_2, a_3, c_2, c_3)(t) = 0. \tag{4.12}$$
$$T_tM \cong Q^{(2)} \iff (R_1, R_2, R_3)(t) = 0,\ (a_3, c_2)(t) \neq 0. \tag{4.13}$$
$$T_tM \cong Q^{(3)} \iff (9R_3^2 - 4R_1R_2)(t) = 0, (R_1, R_2, R_3)(t) \neq 0. \tag{4.14}$$
$$T_tM \cong Q^{(4)} \iff (9R_3^2 - 4R_1R_2)(t) \neq 0. \tag{4.15}$$
$$a_3(t) \neq 0 \ \text{ and } \ (R_1, R_3)(t) = 0 \Rightarrow R_2(t) = 0. \tag{4.16}$$
$$c_2(t) \neq 0 \ \text{ and } \ (R_2, R_3)(t) = 0 \Rightarrow R_1(t) = 0. \tag{4.17}$$

Proof Define

$$\psi_1 := \partial_2 - b_3\partial_1 - \frac{1}{3}a_2\partial_1 \ \text{ and } \ \psi_2 := \partial_3 - b_2\partial_1 - \frac{1}{3}c_3\partial_1. \tag{4.18}$$

One calculates

$$\psi_1^{\circ 2} = \frac{1}{3}a_2\psi_1 + a_3\psi_2 - \frac{2}{3}R_1\partial_1, \tag{4.19}$$
$$\psi_1 \circ \psi_2 = -\frac{1}{3}c_3\psi_1 - \frac{1}{3}a_2\psi_2 + R_3\partial_1, \tag{4.20}$$
$$\psi_2^{\circ 2} = c_2\psi_1 + \frac{1}{3}c_3\psi_2 - \frac{2}{3}R_2\partial_1, \tag{4.21}$$
$$0 = \psi_1^{\circ 3} + R_1\psi_1 + (\frac{2}{9}a_2R_1 - a_3R_3)\partial_1, \tag{4.22}$$
$$0 = \psi_2^{\circ 3} + R_2\psi_2 + (\frac{2}{9}c_3R_2 - c_2R_3)\partial_1. \tag{4.23}$$

The lack of a quadratic term $\psi_1^{\circ 2}$ in (4.22) shows that the sum of the three eigenvalues of $\psi_1 \circ |_t : T_tM \to T_tM$ is zero for any $t \in M$, and similarly for $\psi_2 \circ |_t$. If T_tM is irreducible, then $\psi_1 \circ |_t$ and $\psi_2 \circ |_t$ have only one eigenvalue, which is then zero. Therefore, they are nilpotent, so $\psi_1^{\circ 3}|_t = 0$ and $\psi_2^{\circ 3}|_t = 0$, so $(R_1, R_2, R_3)(t) = 0$.

Vice versa, if $(R_1, R_2, R_3)(t) = 0$, then $\psi_1^{\circ 3}|_t = 0$ and $\psi_2^{\circ 3}|_t = 0$, so $\psi_1 \circ |_t$ and $\psi_2 \circ |_t$ are nilpotent, and $T_t M$ is an irreducible algebra. We proved

$$T_t M \cong Q^{(1)} \text{ or } T_t M \cong Q^{(2)} \iff (R_1, R_2, R_3)(t) = 0.$$

Suppose that $T_t M$ is an irreducible algebra. If $a_3(t) \neq 0$ then

$$(\partial_3 - b_2\partial_1)(t) = a_3(t)^{-1}\Big((\partial_2 - b_3\partial_1)^{\circ 2} - a_2(\partial_2 - b_3\partial_1) - a_1\partial_1\Big)(t),$$
$$\text{so} \quad T_t M = \bigoplus_{j=0}^{2} \mathbb{C} \cdot \partial_2^{\circ j}(t), \quad \text{and thus} \quad T_t M \cong Q^{(2)},$$

and in the same way $c_2(t) \neq 0$ implies $T_t M \cong Q^{(2)}$. The other way round, if $(a_3, c_2)(t) = 0$, then $(R_1, R_2)(t) = 0$ implies also $(a_2, c_3)(t) = 0$, and then $T_t M \cong Q^{(1)}$. This finishes the proof of (4.12) and (4.13).

Next we want to show (4.14). (4.22) and (4.23) generalize as follows for arbitrary $\psi := \lambda_1\psi_1 + \lambda_2\psi_2$ with $\lambda_1, \lambda_2 \in \mathbb{C}\{t_2, t_3\}$:

$$\begin{aligned} 0 &= \psi^{\circ 3} + [R_1\lambda_1^2 - 3R_3\lambda_1\lambda_2 + R_2\lambda_2^2] \cdot \psi \\ &\quad + \Big[(\frac{2}{9}a_2R_1 - a_3R_3)\lambda_1^3 - (\frac{2}{3}c_3R_1 - a_2R_3)\lambda_1^2\lambda_2 \\ &\quad -(\frac{2}{3}a_2R_2 - c_3R_3)\lambda_1\lambda_2^2 + (\frac{2}{9}c_3R_2 - c_2R_3)\lambda_2^3\Big] \cdot \partial_1. \end{aligned} \tag{4.24}$$

A lengthy calculation shows that the discriminant of (4.24) is

$$\begin{aligned} &4(\text{coefficient of } \psi)^3 + 27(\text{coefficient of } \partial_1)^2 \\ &= (9R_3^2 - 4R_1R_2) \cdot 3(a_3\lambda_1^3 - a_2\lambda_1^2\lambda_2 + c_3\lambda_1\lambda_2^2 - c_2\lambda_2^3)^2. \end{aligned} \tag{4.25}$$

First suppose $T_t M \cong Q^{(3)}$. Then, $\psi|_t$ has at most two different eigenvalues, and the discriminant in (4.25) must vanish at t. Because λ_1 and $\lambda_2 \in \mathbb{C}\{t_2, t_3\}$ are arbitrary and $(a_2, a_3, c_2, c_3) \neq 0$, this shows $(9R_3^2 - 4R_1R_2)(t) = 0$. Vice versa, if $(9R_3^2 - 4R_1R_2)(t) = 0$, then the discriminant in (4.25) vanishes at t for any ψ. Therefore, $\psi \circ |_t$ has at most two eigenvalues for any ψ. This shows $T_t M \not\cong Q^{(4)}$. Then, the condition $(R_1, R_2, R_3) \neq 0$ yields $T_t M \cong Q^{(3)}$. This proves (4.14).

(4.15) is a consequence of (4.12)–(4.14). The implications (4.16) and (4.17) are trivial. □

Lemma 4.4 *Let $(M, \circ, e)$ be as in the Notations 4.1 with coordinates $t = (t_1, t_2, t_3)$ and fiber coordinates $y = (y_1, y_2, y_3)$ of T^*M, and suppose that the multiplication $\circ$ is associative. With the result from Lemma 4.2, the ideal $\mathcal{I}_M \subset \mathcal{O}(T^*M)$ which defines the analytic spectrum is*

$$\begin{aligned} \mathcal{I}_M &= \big(y_1 - 1, Y_{22}, Y_{23}, Y_{33}\big), \quad \text{where} \\ Y_{22} &:= (y_2 - b_3)(y_2 - b_3) + a_3c_3 - a_2(y_2 - b_3) - a_3(y_3 - b_2), \end{aligned}$$

$$
\begin{aligned}
Y_{23} &:= (y_2 - b_3)(y_3 - b_2) - a_3c_2, \\
Y_{33} &:= (y_3 - b_2)(y_3 - b_2) + a_2c_2 - c_2(y_2 - b_3) - c_3(y_3 - b_2),
\end{aligned} \tag{4.26}
$$

with $a_j, b_j, c_j \in \mathcal{O}_M$. Recall the notation in formula (4.7). Define

$$
\begin{aligned}
A_2 &:= a_2(-b_{22} + b_{33} + a_{23}) + a_3(-2c_{22} - c_{33}) - a_{32}c_2 - a_{33}c_3, \\
A_2^{dual} &:= c_3(-b_{33} + b_{22} + c_{32}) + c_2(-2a_{33} - a_{22}) - c_{23}a_3 - c_{22}a_2, \\
A_3 &:= -3b_{22} + 3b_{33} + a_{23} - c_{32}.
\end{aligned} \tag{4.27}
$$

Then

$$
\begin{aligned}
\{y_1 - 1, Y_{ij}\} &= \partial_1 Y_{ij} \quad (= 0 \text{ because of } a_k, b_k, c_k \in \mathbb{C}\{t_2, t_3\}), \\
\{Y_{22}, Y_{23}\} &= Y_{22}[-2b_{22} + 2b_{33} + a_{23}] + \quad Y_{23}[a_{22} + a_{33}] + Y_{33}[a_{33}] \\
&\quad + (y_2 - b_3)A_2 + (y_3 - b_2)a_3A_3 + [-a_3A_2^{dual} - a_3c_3A_3], \\
\{Y_{33}, Y_{23}\} &= Y_{33}[-2b_{33} + 2b_{22} + c_{32}] + Y_{23}[c_{33} + c_{22}] + Y_{22}[c_{22}] \\
&\quad + (y_3 - b_2)A_2^{dual} - (y_2 - b_3)c_2A_3 + [-c_2A_2 + c_2a_2A_3], \\
\{Y_{22}, Y_{33}\} &= Y_{22}[-2c_{22}] + \quad Y_{23}[2(-2b_{22} + 2b_{33} + a_{23} - c_{32})] \\
&\quad + Y_{33}[2a_{33}] + (y_2 - b_3)[-A_2^{dual} - c_3A_3] \\
&\quad + (y_3 - b_2)[A_2 - a_2A_3] \\
&\quad + [-c_3A_2 + a_2A_2^{dual} + (a_2c_3 + a_3c_2)A_3].
\end{aligned} \tag{4.28}
$$

Therefore, $(M, \circ, e)$ is an F-manifold if and only if

$$
(a_2, a_3, c_2, c_3) = 0, \tag{4.29}
$$

$$
\text{or} \qquad (A_2, A_2^{dual}, A_3) = 0. \tag{4.30}
$$

The intersection of these two cases is the case (4.29) with additionally $b_{22} - b_{33} = 0$.

Proof The calculation of the Poisson brackets in (4.28) is straightforward and leads to the claimed formulas in (4.28). By Theorem 2.12, $(M, \circ, e)$ is an F-manifold if and only if $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$, so if and only if

$$
A_2 = A_2^{dual} = a_3A_3 = c_2A_3 = c_3A_3 = a_2A_3 = 0. \tag{4.31}
$$

This leads to the two cases (4.29) and (4.30). □

Remark 4.5 In the case (4.30), the condition $A_3 = 0$ can be used to make a specific special coordinate change as in (4.9), namely we choose the new coordinates $s = (s_1, s_2, s_3)$ such that

$$
\begin{aligned}
&t_2 = s_2, \quad t_3 = s_3, \quad t_1 = s_1 + \tau, \quad \tau \in \mathbb{C}\{t_2, t_3\} \text{ with} \\
&\qquad \partial_2\tau = -b_3 - \frac{1}{3}a_2, \;\; \partial_3\tau = -b_2 - \frac{1}{3}c_3.
\end{aligned} \tag{4.32}
$$

With the notation $\widetilde{\partial}_j := \partial/\partial s_j$ and with ψ_1, ψ_2 as in (4.18), we obtain

$$
\begin{aligned}
\widetilde{\partial}_1 = \partial_1 = e,\ \widetilde{\partial}_2 = \psi_1,\ \widetilde{\partial}_3 = \psi_2, \\
\partial_2 - b_3\partial_1 = \widetilde{\partial}_2 + \frac{1}{3}a_2\partial_1, \quad \partial_3 - b_2\partial_1 = \widetilde{\partial}_3 + \frac{1}{3}c_3\partial_1.
\end{aligned}
\tag{4.33}
$$

Formula (4.24) in the proof of Lemma 4.4 tells that for any $\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ with $\lambda_1, \lambda_2 \in \mathbb{C}\{t_2, t_3\}$, the sum of the eigenvalues of $\psi\circ$ is zero at any $t \in M$.

If we call now the new coordinates again $t = (t_1, t_2, t_3)$, the new and old coefficients a_2, a_3, c_2, c_3 coincide, and the new coefficients $b_2^{(new)}, b_3^{(new)}, A_2^{(new)}, (A_2^{dual})^{(new)}$, $A_3^{(new)}$ become

$$
b_3^{(new)} = -\frac{1}{3}a_2,\ b_2^{(new)} = -\frac{1}{3}c_3,\ A_3^{(new)} = 0, \tag{4.34}
$$

$$
A_2^{(new)} = -\partial_3 R_1 + \frac{1}{3}a_2c_{32} - 2a_3c_{22} - a_{32}c_2, \tag{4.35}
$$

$$
(A_2^{dual})^{(new)} = -\partial_2 R_2 + \frac{1}{3}a_{23}c_3 - 2a_{33}c_2 - a_3c_{23}. \tag{4.36}
$$

In the case (4.30), we will often, but not always, assume that the coordinates $t = (t_1, t_2, t_3)$ have been chosen as in this remark.

If a germ $(M, 0)$ of a 3-dimensional F-manifold satisfies $T_0M \cong Q^{(2)}$ or $Q^{(3)}$ or $Q^{(4)}$, then life is easier than in the case $T_0M \cong Q^{(1)}$. The next lemma makes this explicit in one way.

Lemma 4.6 *Let $((M, 0), \circ, e)$ be as in the Notations 4.1 with coordinates $t = (t_1, t_2, t_3)$ with $t(0) = 0$ and fiber coordinates (y_1, y_2, y_3) of T^*M, and suppose that the multiplication $\circ$ is associative. Suppose $T_0M \not\cong Q^{(1)}$. The coordinates t can and will be chosen such that*

$$
\mathbb{C}\cdot\partial_1|_0 \oplus \mathbb{C}\cdot\partial_2|_0 \oplus \mathbb{C}\cdot\partial_2^{\circ 2}|_0 = T_0M. \tag{4.37}
$$

Then,

$$
\partial_2^{\circ 3} = g_2\cdot\partial_2^{\circ 2} + g_1\cdot\partial_2 + g_0\cdot\partial_1, \tag{4.38}
$$

$$
\partial_3 = h_2\cdot\partial_2^{\circ 2} + h_1\cdot\partial_2 + h_0\cdot\partial_1, \tag{4.39}
$$

for suitable coefficients $g_2, g_1, g_0, h_2, h_1, h_0 \in \mathbb{C}\{t_2, t_3\}$. We denote similarly to (4.7)

$$
\partial_i g_j =: g_{ji}, \quad \partial_i h_j =: h_{ji}.
$$

The ideal $\mathcal{I}_M \subset \mathcal{O}_{T^*M}$ which defines the analytic spectrum is

$$
\begin{aligned}
\mathcal{I}_M &= (y_1 - 1, Z_2, Z_3), \quad \textit{where} \\
Z_2 &:= y_2^3 - g_2y_2^2 - g_1y_2 - g_0,
\end{aligned}
$$

$$Z_3 := y_3 - h_2 y_2^2 - h_1 y_2 - h_0. \tag{4.40}$$

Then,

$$\begin{aligned}
&\{y_1 - 1, Z_j\} = \partial_1 Z_j \quad (= 0 \text{ because of } g_i, h_i \in \mathbb{C}\{t_2, t_3\}),\\
&\{Z_3, Z_2\} = Z_2[2g_{22}h_2 + (3y_2 + g_2)h_{22} + 3h_{12}]\\
&+ y_2^2[\partial_2((g_2^2 + 2g_1)h_2 + g_2 h_1 + 3h_0) - g_{23}] \qquad (4.41)\\
&+ y_2[(2g_{22}g_1 + 2g_{02})h_2 + (g_2 g_1 + 3g_0)h_{22} + g_{12}h_1\\
&\quad +2g_1 h_{12} - 2g_2 h_{02} - g_{13}] \qquad (4.42)\\
&+ [2g_{22}g_0 h_2 + g_2 g_0 h_{22} + g_{02}h_1 + 3g_0 h_{12} - g_1 h_{02} - g_{03}]. \qquad (4.43)
\end{aligned}$$

Therefore $((M, 0), \circ, e)$ *is a germ of an* F*-manifold if and only if the terms in square brackets in* (4.41)–(4.43) *vanish.*

Proof In each of the algebras $Q^{(j)}$ for $j \in \{2, 3, 4\}$, a generic element a satisfies $Q^{(j)} = \mathbb{C} \cdot 1 \oplus \mathbb{C} \cdot a \oplus \mathbb{C} \cdot a^{\circ 2}$. One can choose the coordinates t on $(M, 0)$ such that $\partial_1 = e$ and $\partial_2|_0$ is such a generic element. This implies (4.37)–(4.40). The calculation of $\{Z_3, Z_2\}$ is straightforward. □

Corollary 4.7 *Let* $g_2^{(0)}, g_1^{(0)}, g_0^{(0)} \in \mathbb{C}\{t_2\}$ *and* $h_2, h_1, h_0 \in \mathbb{C}\{t_2, t_3\}$ *be arbitrary. There exist unique* $g_2, g_1, g_0 \in \mathbb{C}\{t_2, t_3\}$ *such that* $g_j|_{t_3=0} = g_j^{(0)}$ *and such that the 3-dimensional germ* $(M, 0)$ *of a manifold with multiplication* $\circ$ *on* TM *defined by* $\partial_1 = e$, (4.38) *and* (4.39) *is a germ of an* F*-manifold.*

Proof The Cauchy–Kowalevski theorem in the following form [12, (1.31), (1.40), (1.41)] will be applied (there the setting is real analytic, but the proofs and statements hold also in the complex analytic setting): Given $N \in \mathbb{N}$ and matrices $A_i, B \in M_{N\times N}(\mathbb{C}\{s_1, ..., s_m, y, x_1, ..., x_N\})$, there exists a unique vector

$$\Phi \in M_{N\times 1}(\mathbb{C}\{s_1, ..., s_m, y\})$$

with

$$\begin{aligned}
\frac{\partial \Phi}{\partial y} &= \sum_{i=1}^{m} A_i(s, y, \Phi)\frac{\partial \Phi}{\partial s_i} + B(s, y, \Phi),\\
\Phi(s, 0) &= 0.
\end{aligned} \tag{4.44}$$

In our situation $y = t_3$, $(s_1, ..., s_m) = (t_2)$, $\Phi = (g_2 - g_2^{(0)}, g_1 - g_1^{(0)}, g_0 - g_0^{(0)})^t$, and A_1, A_2, A_3 and B come from the terms in (4.41)–(4.43) without g_{23}, g_{13}, g_{03}, more precisely, (4.44) is here

$$\partial_3 \begin{pmatrix} g_2 \\ g_1 \\ g_0 \end{pmatrix} = \begin{pmatrix} \partial_2((g_2^2 + 2g_1)h_2 + g_2 h_1 + 3h_0) \\ (2g_{22}g_1 + 2g_{02})h_2 + (g_2 g_1 + 3g_0)h_{22} + g_{12}h_1 \\ +2g_1 h_{12} - 2g_2 h_{02} \\ 2g_{22}g_0 h_2 + g_2 g_0 h_{22} + g_{02}h_1 + 3g_0 h_{12} - g_1 h_{02} \end{pmatrix}. \tag{4.45}$$

The Cauchy–Kowalevski theorem tells that there exist unique $g_2, g_1, g_0 \in \mathbb{C}\{t_2, t_3\}$ such that $g_j|_{t_3=0} = g_j^{(0)}$ and such that the terms in (4.41)–(4.43) vanish. The multiplication on TM which is defined by (4.38) and (4.39), is automatically associative. The condition $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$ is equivalent to the vanishing of the terms in (4.41)–(4.43). By Theorem 2.12, $((M, 0), \circ, e)$ is an F-manifold if and only if $\{\mathcal{I}_M, \mathcal{I}_M\} \subset \mathcal{I}_M$. □

Remarks 4.8 (i) The corollary makes it easy to construct a 3-dimensional germ $(M, 0)$ of an F-manifold. Arbitrary initial data $g_2^{(0)}, g_1^{(0)}, g_0^{(0)} \in \mathbb{C}\{t_2\}$ and $h_2, h_1, h_0 \in \mathbb{C}\{t_2, t_3\}$ give a unique germ of an F-manifold. However, this approach does not tell easily which properties such constructed F-manifolds have, which families exist and what are their parameters.

(ii) The condition $A_3 = 0$ in Lemma 4.4 corresponds to the first line of equation (4.45). We can make a coordinate change as in Remark 4.5. In the case of an F-manifold in Lemma 4.6, we can choose new coordinates $s = (s_1, s_2, s_3)$ such that

$$
\begin{aligned}
t_2 = s_2, t_3 = s_3, t_1 = s_1 + \tau \quad &\text{with} \quad \tau \in \mathbb{C}\{t_2, t_3\} \text{ with} \\
\partial_2\tau = -\frac{1}{3}g_2, \ \partial_3\tau = -\frac{1}{3}&((g_2^2 + 2g_1)h_2 + g_2h_1 + 3h_0), \\
\widetilde{\partial}_2 = \partial_2 + \partial_2\tau \cdot \partial_1, \ &\widetilde{\partial}_3 = \partial_3 + \partial_3\tau \cdot \partial_1.
\end{aligned}
\tag{4.46}
$$

If we now call the new coordinates again $t = (t_1, t_2, t_3)$, the new coefficients $g_j^{(new)}$ and $h_j^{(new)}$ satisfy $g_2^{(new)} = 0$ and $2g_1^{(new)}h_2^{(new)} + 3h_0^{(new)} = 0$. This says that the sum of the eigenvalues of $\partial_2\circ$ is zero, and that the sum of the eigenvalues of $\partial_3\circ$ is 0. The last statement follows from the facts that the sum of the eigenvalues of $h_0\partial_1\circ$ is $3h_0$ and that the sum of the eigenvalues of $h_2\partial_2^{\circ 2}\circ$ is $2h_2g_1$, because $\lambda_1^2+\lambda_2^2+\lambda_3^2 = -2(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)$ for any $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ with $\sum_{i=1}^3 \lambda_i = 0$.

5 3-dimensional not generically semisimple *F*-manifolds

We want to classify all 3-dimensional germs of F-manifolds. The reducible ones are products of 1- and 2-dimensional germs of F-manifolds by Theorem 2.5. Those are classified in Lemma 2.6 and Theorem 3.1. The 3-dimensional reducible germs of F-manifolds are A_1^3, $A_1I_2(m)$ for $m \geq 3$ and $A_1\mathcal{N}_2$, and the Euler fields are as described in Theorem 2.5, Lemma 2.6 and Theorem 3.2. It remains to classify the irreducible germs $(M, 0)$ of F-manifolds, i.e., those where T_0M is irreducible. We start with those with $T_tM \cong Q^{(1)}$ for any $t \in M$.

Theorem 5.1 *Any 3-dimensional germ $(M, 0)$ of an F-manifold with $T_tM \cong Q^{(1)}$ for all $t \in M$ can be given as follows:*

$$
\begin{aligned}
(M, 0) = (\mathbb{C}^3, 0) &\text{ with coordinates } t = (t_1, t_2, t_3), \\
e = \partial_1, \ \partial_2^{\circ 2} &= \partial_2 \circ (\partial_3 - b_2\partial_1) = (\partial_3 - b_2\partial_1)^{\circ 2} = 0, \\
&\text{where } b_2 \text{ is arbitrary in } t_2\mathbb{C}\{t_2, t_3\}.
\end{aligned}
\tag{5.1}
$$

The ideal $(\partial_2 b_2) \subset \mathbb{C}\{t_2, t_3\}$ is up to coordinate changes an invariant of the germ of an F-manifold. A vector field $E = \varepsilon_1\partial_1 + \varepsilon_2\partial_2 + \varepsilon_2\partial_3$ with $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{C}\{t_1, t_2, t_3\}$ is an Euler field if and only if $\varepsilon_1 \in t_1 + \mathbb{C}\{t_2, t_3\}$, $\varepsilon_2, \varepsilon_3 \in \mathbb{C}\{t_2, t_3\}$ and

$$\partial_2(\varepsilon_1) = -b_2\partial_2(\varepsilon_2), \quad \partial_3(\varepsilon_1) = -\varepsilon_2\partial_2(b_2) - \partial_3(\varepsilon_3 b_2) + b_2. \tag{5.2}$$

Proof Let $((M, 0), \circ, e)$ be a germ of an F-manifold with $T_t M \cong Q^{(1)}$ for any $t \in M$. We choose coordinates $t = (t_1, t_2, t_3)$ with $t(0) = 0$ and use the Notations 4.1 and Lemma 4.2. By Lemma 4.3, $(a_3, a_2, c_3, c_2) = 0$. We are in the case (4.29) in Lemma 4.4. The F-manifold condition gives no constraint on $b_2, b_3 \in \mathbb{C}\{t_2, t_3\}$.

We make a specific special coordinate change as in (4.9), namely we choose the new coordinates $s = (s_1, s_2, s_3)$ such that

$$\begin{aligned} t_2 = s_2, t_3 = s_3, t_1 = s_1 + \tau \quad \text{with} \quad \tau \in \mathbb{C}\{t_2, t_3\} \text{ with} \\ \partial_2\tau = -b_3,\ \partial_3\tau + b_2 \in t_2\mathbb{C}\{t_2, t_3\}. \end{aligned} \tag{5.3}$$

Then τ exists and is unique. With the notation $\widetilde{\partial}_j := \partial/\partial s_j$ we obtain

$$\begin{aligned} \widetilde{\partial}_1 = \partial_1 = e,\ \widetilde{\partial}_2 = \partial_2 - b_3\partial_1,\ \widetilde{\partial}_3 = \partial_3 + \partial_3\tau \cdot \partial_1, \\ \partial_3 - b_2\partial_1 = \widetilde{\partial}_3 - (\partial_3\tau + b_2\partial_1). \end{aligned} \tag{5.4}$$

If we call now the new coordinates again $t = (t_1, t_2, t_3)$, the new coefficients $b_2^{(new)}$ and $b_3^{(new)}$ are

$$b_3^{(new)} = 0 \quad \text{and} \quad b_2^{(new)} = \partial_3\tau + b_2 \in t_2\mathbb{C}\{t_2, t_3\}.$$

Now we want to show that the ideal $(\partial_2 b_2) \subset \mathbb{C}\{t_2, t_3\}$ is up to coordinate changes an invariant of the germ of an F-manifold. We consider a coordinate change as in (4.8) with $z_1 = y_1$, which implies $\widetilde{\partial}_1 t_1 = 1$ and $\widetilde{\partial}_1 t_2 = \widetilde{\partial}_1 t_3 = 0$. Then

$$\begin{aligned} z_2 = \widetilde{\partial}_2 t_1 \cdot y_1 + \widetilde{\partial}_2 t_2 \cdot y_2 + \widetilde{\partial}_2 t_3 \cdot y_3, \\ z_3 = \widetilde{\partial}_3 t_1 \cdot y_1 + \widetilde{\partial}_3 t_2 \cdot y_2 + \widetilde{\partial}_3 t_3 \cdot y_3. \end{aligned}$$

$z_2 - (\widetilde{\partial}_2 t_1 + \widetilde{\partial}_2 t_3 \cdot b_2)y_1$ and $z_3 - (\widetilde{\partial}_3 t_1 + \widetilde{\partial}_3 t_3 \cdot b_2)y_1$ are nilpotent in $\mathcal{O}_{T^*M}/\mathcal{I}_M|_{L_M}$. We need z_2 to be nilpotent, so the coordinate change satisfies

$$\widetilde{\partial}_2 t_1 = -\widetilde{\partial}_2 t_3 \cdot b_2.$$

And then

$$\widetilde{b}_2 := \widetilde{\partial}_3 t_1 + \widetilde{\partial}_3 t_3 \cdot b_2.$$

takes the role of b_2 for the new coordinates. A short calculation shows

$$\widetilde{\partial}_2\widetilde{b}_2 = (\partial_2 b_2)(t(s)) \cdot (\widetilde{\partial}_3 t_3 \cdot \widetilde{\partial}_2 t_2 - \widetilde{\partial}_2 t_3 \cdot \widetilde{\partial}_3 t_2).$$

The second factor is a unit. Therefore, the ideal $(\partial_2 b_2)$ is up to coordinate changes an invariant of the germ $(M, 0)$ of the F-manifold.

The constraint (5.2) for an Euler field $E = \varepsilon_1\partial_1 + \varepsilon_2\partial_2 + \varepsilon_3\partial_3$ follows straightforwardly from the explicit version

$$0 = [E, \partial_i \circ \partial_j] - [E, \partial_i] \circ \partial_j - [E, \partial_j] \circ \partial_i - \partial_i \circ \partial_j \tag{5.5}$$

for $i, j \in \{1, 2, 3\}$ of the condition $\mathrm{Lie}_E(\circ) = 1\cdot\circ$ (where we assume the multiplication to be as in (5.1)). □

Remarks 5.2 (i) The ideal $(\partial_2 b_2) \subset \mathbb{C}\{t_2, t_3\}$ up to coordinate changes is a rich invariant. It shows that there is a functional parameter in the family of 3-dimensional germs of F-manifolds with $T_t M \cong Q^{(1)}$ for all $t \in M$.

(ii) Though all these germs except the one with $b_2 = 0$ have the unpleasant property $\{\sqrt{\mathcal{I}_M}, \sqrt{\mathcal{I}_M}\} \not\subset \sqrt{\mathcal{I}_M}$: Here $\sqrt{\mathcal{I}_M} = (y_1 - 1, y_2, y_3 - b_2\}$ and

$$\{y_2, y_3 - b_2\} = -\partial_2(b_2),$$

and this is in $\sqrt{\mathcal{I}_M}$ only if $b_2 = 0$.

(iii) Therefore, the germ of an F-manifold with $b_2 = 0$ is the most important one of those in Theorem 5.1. In the case $b_2 = 0$, the compatibility condition (5.2) for the coefficients of the Euler field says $\varepsilon_1 \in t_1 + \mathbb{C}$. So, then $\varepsilon_1 \in t_1 + \mathbb{C}$, and $\varepsilon_2, \varepsilon_3 \in \mathbb{C}\{t_2, t_3\}$ are arbitrary.

(iv) Theorem 3.2 improved the classification of the Euler fields for $\mathcal{N}_2$ in Theorem 3.1 (c) by exploiting coordinate changes which do not change the multiplication. We expect that a similar reduction of Euler fields to normal forms is possible for the case $b_2 = 0$ in Theorem 5.1. However, we do not pursue it here.

Next we classify the irreducible germs $(M, 0)$ of F-manifolds with $T_t M \cong Q^{(2)}$ for generic (or all) $t \in M$. It is also surprisingly rich. There is also a functional parameter.

Theorem 5.3 *The following three constructions give (up to isomorphism) all germs $(M, 0)$ of 3-dimensional F-manifolds with $T_t M \cong Q^{(2)}$ for generic $t \in M$. The three constructions do not overlap.* $\mathbf{m} \subset \mathbb{C}\{t_2, t_3\}$ *denotes the maximal ideal.*

(a) Up to isomorphism, there is only one germ of a 3-dimensional F-manifold with $T_t M \cong Q^{(2)}$ for all $t \in M$. In suitable coordinates $t = (t_1, t_2, t_3)$, it looks as follows.

$$(M, 0) = (\mathbb{C}^3, 0),\ e = \partial_1,\ \partial_2^{\circ 2} = \partial_3,\ \partial_2 \circ \partial_3 = \partial_3^{\circ 2} = 0. \tag{5.6}$$

An Euler field is a vector field of the shape

$$\begin{aligned} E = (t_1 + c_1)\partial_1 + \varepsilon_2\partial_2 + (\varepsilon_{3,0} + t_3(2\partial_2\varepsilon_2 - 1))\partial_3, \\ \text{with}\quad c_1 \in \mathbb{C},\ \varepsilon_2, \varepsilon_{3,0} \in \mathbb{C}\{t_2\}. \end{aligned} \tag{5.7}$$

(b) *Consider an arbitrary $f \in \mathbf{m}-\{0\}$. Then, $(M,0)=(\mathbb{C}^3,0)$ with $e=\partial_1$ and with the multiplication given by*

$$\partial_2^{\circ 2}=f\cdot\partial_3,\ \partial_2\circ\partial_3=\partial_3^{\circ 2}=0 \tag{5.8}$$

is an F-manifold with $T_tM\cong Q^{(2)}$ for generic $t\in M$. Here $\mathbb{C}\times f^{-1}(0)=\{t\in M\,|\,T_tM\cong Q^{(1)}\}$. The ideal $(f)\subset\{\mathbf{m}\}$ is up to coordinate changes an invariant of the germ of an F-manifold. An Euler field is a vector field of the shape

$$E=(t_1+c_1)\partial_1+\varepsilon_2\partial_2+\varepsilon_3\partial_3, \tag{5.9}$$
$$\text{with}\quad c_1\in\mathbb{C},\ \varepsilon_2\in\mathbb{C}\{t_2\},\ \varepsilon_3\in\mathbb{C}\{t_2,t_3\},\quad\text{and}$$
$$0=(\varepsilon_2\partial_2+\varepsilon_3\partial_3)(f)+f(2\partial_2(\varepsilon_2)-\partial_3(\varepsilon_3)-1). \tag{5.10}$$

(c) *Consider arbitrary $f_1,f_2\in\mathbf{m}$ with $\gcd(f_1,f_2)=1$ and an arbitrary $h\in\mathbb{C}\{t_2,t_3\}-\{0\}$. Define for $(M,0)=(\mathbb{C}^3,0)$ the vector field $\sigma:=hf_2\partial_2+hf_1\partial_3$. Then, $(M,0)$ with $e=\partial_1$ and with the multiplication given by*

$$\partial_2^{\circ 2}=f_1^2\sigma,\ \partial_2\circ\partial_3=-f_1f_2\sigma,\ \partial_3^{\circ 2}=f_2^2\sigma, \tag{5.11}$$

is an F-manifold with $T_tM\cong Q^{(2)}$ for generic $t\in M$. Here $\{t\in M\,|\,T_tM\cong Q^{(1)}\}$ is equal to $\mathbb{C}\times h^{-1}(0)$ if $h\in\mathbf{m}$, and equal to $\mathbb{C}\times\{0\}$ if $h(0)\neq 0$. Here, $\partial_2\circ\sigma=\partial_3\circ\sigma=0$. The ideals $(f_1,f_2)\subset\mathbf{m}$ and $(h)\subset\mathbb{C}\{t_2,t_3\}$ are up to coordinate changes invariants of the germ of an F-manifold. An Euler field is a vector field of the shape

$$E=(t_1+c_1)\partial_1+\varepsilon_2\partial_2+\varepsilon_3\partial_3, \tag{5.12}$$
$$\text{with}\quad c_1\in\mathbb{C},\ \varepsilon_2,\varepsilon_3\in\mathbb{C}\{t_2,t_3\},\quad\text{and}$$
$$\begin{aligned}0=\;&3h(\varepsilon_2\partial_2+\varepsilon_3\partial_3)(f_1)+f_1(\varepsilon_2\partial_2+\varepsilon_3\partial_3)(h)\\&+2f_1\partial_2(\varepsilon_2)-3f_2\partial_2(\varepsilon_3)-f_1\partial_3(\varepsilon_3)-f_1,\end{aligned} \tag{5.13}$$
$$\begin{aligned}0=\;&3h(\varepsilon_2\partial_2+\varepsilon_3\partial_3)(f_2)+f_2(\varepsilon_2\partial_2+\varepsilon_3\partial_3)(h)\\&+2f_2\partial_3(\varepsilon_3)-3f_1\partial_3(\varepsilon_2)-f_2\partial_2(\varepsilon_2)-f_2.\end{aligned} \tag{5.14}$$

Proof Let $((M,0),\circ,e)$ be a germ of an F-manifold with $T_tM\cong Q^{(2)}$ for generic $t\in M$. We choose coordinates $t=(t_1,t_2,t_3)$ with $t(0)=0$. and use the Notations 4.1 and Lemma 4.2. By Lemma 4.3, $(R_1,R_2,R_3)=0$, but $(a_3,a_2,c_3,c_2)\neq 0$. We are in the case (4.30) in Lemma 4.4.

The coordinates can and will be chosen as in Remark 4.5. Therefore for $\partial_2\circ$ as well as for $\partial_3\circ$, the sum of the eigenvalues is 0. As each algebra T_tM is irreducible, in both cases there is only one eigenvalue. Therefore, it is 0, and $\partial_2\circ$ and $\partial_3\circ$ are nilpotent.

For generic t, $T_tM\cong Q^{(2)}$, and at least one of $\partial_2|_t$ and $\partial_3|_t$ is not in the (1-dimensional) socle of the algebra T_tM. Suppose $\partial_2|_t$ is not in the socle. Then, $\partial_2|_t\circ\partial_2|_t\neq 0$, but it is in the socle. Therefore, the section $\partial_2\circ\partial_2$ is $\neq 0$, and for any $t\in M$ its value is in the socle of T_tM (remark that 0 is in the socle). Write $\partial_2\circ\partial_2=\widetilde{f}_2\partial_2+\widetilde{f}_1\partial_3$ with $\widetilde{f}_1,\widetilde{f}_2\in\mathbb{C}\{t_2,t_3\}$.

Recall that $\mathbb{C}\{t_2, t_3\}$ is a factorial ring (e.g., [14, Theorem 1.16]). Divide out joint factors of $\widetilde{f_1}$ and $\widetilde{f_2}$ and obtain a section $\rho := f_2\partial_2 + f_1\partial_3$ with $\gcd(f_1, f_2) = 1$. Then for each $t \in M - \mathbb{C} \times \{0\}$, $\gcd(f_1, f_2) = 1$ implies that $\rho|_t \neq 0$, and $\rho|_t$ is in the socle of T_tM. Therefore, any section $\widetilde{\rho}$ with $\widetilde{\rho}|_t$ in the socle for $t \in M - \mathbb{C} \times \{0\}$ has the shape $\widetilde{\rho} = g \cdot \rho$ with $g \in \mathcal{O}_{M-\mathbb{C}\times\{0\}} = \mathcal{O}_M$. Especially $\partial_2^{\circ 2}, \partial_2 \circ \partial_3, \partial_3^{\circ 2} \in \mathcal{O}_M \cdot \rho$.

Now we consider two cases. In the 1st case, $(f_1, f_2)(0) \neq (0, 0)$, and then we suppose first $f_1(0) \neq 0$, and then by multiplying ρ with a unit, we can arrange $f_1 = 1$. In the 2nd case $(f_1, f_2)(0) = (0, 0)$.

1st case, $f_1 = 1$: We make a coordinate change $t = t(s)$ with $t_1 = s_1$, $t_3 = s_3$ and $t_2 = t_2(s_2, s_3)$ such that $\widetilde{\partial}_3 t_2(s) = f_2(t(s))$. Then

$$\begin{aligned}\widetilde{\partial}_2 &= \widetilde{\partial}_2 t_2 \cdot \partial_2 + \widetilde{\partial}_2 t_3 \cdot \partial_3 = \widetilde{\partial}_2 t_2 \cdot \partial_2,\\ \widetilde{\partial}_3 &= \widetilde{\partial}_3 t_2 \cdot \partial_2 + \widetilde{\partial}_3 t_3 \cdot \partial_3 = f_2(t(s)) \cdot \partial_2 + \partial_3 = \rho(t(s)).\end{aligned}$$

We call the new coordinates again t. Then, $\partial_3 = \rho$. This shows (5.8) for a function $f \in \mathbb{C}\{t_2, t_3\}$.

In the case $f(0) \neq 0$, a coordinate change $t = t(s)$ with $t_1 = s_1$, $t_2 = s_2$ and $t_3 = t_3(s_2, s_3)$ such that $\widetilde{\partial}_3 t_3 = f(t(s))$ exists and gives

$$\begin{aligned}\widetilde{\partial}_2 = \partial_2 + \widetilde{\partial}_2 t_3 \cdot \partial_3, \quad \widetilde{\partial}_3 = \widetilde{\partial}_3 t_3 \cdot \partial_3 \quad \text{with} \quad \widetilde{\partial}_3 t_3 \in \mathbb{C}\{s_2, s_3\}^*,\\ \widetilde{\partial}_2^{\circ 2} = \partial_2^{\circ 2} = f(t(s)) \cdot \partial_3 = f(t(s))(\widetilde{\partial}_3 t_3)^{-1} \cdot \widetilde{\partial}_3 = \widetilde{\partial}_3,\end{aligned}$$

so we obtain (5.6).

In order to show that the ideal (f) up to coordinate changes is an invariant of the germ $(M, 0)$ of an F-manifold, we have to consider all coordinate changes which respect the shape of (5.8). These are coordinate changes such that $\widetilde{\partial}_3$ is a multiple by a unit of ∂_3, and $\widetilde{\partial}_2\circ$ is still nilpotent. Thus, $t_1 = s_1$ and $t_2 = t_2(s_2, s_3)$ such that $\widetilde{\partial}_3 t_2 = 0$, so $t_2 = t_2(s_2)$. Then, $\widetilde{\partial}_2 t_2$ and $\widetilde{\partial}_3 t_3$ are units in $\mathbb{C}\{t_2, t_3\}$, and

$$\begin{aligned}\widetilde{\partial}_2 = \widetilde{\partial}_2 t_2 \cdot \partial_2 + \widetilde{\partial}_2 t_3 \cdot \partial_3, \quad \widetilde{\partial}_3 = \widetilde{\partial}_3 t_3 \cdot \partial_3,\\ \widetilde{\partial}_2^{\circ 2} = (\widetilde{\partial}_2 t_2)^2 (\widetilde{\partial}_3 t_3)^{-1} \cdot f \cdot \widetilde{\partial}_3, \quad \text{so} \quad \widetilde{f} = (\widetilde{\partial}_2 t_2)^2 (\widetilde{\partial}_3 t_3)^{-1} \cdot f(t(s)).\end{aligned}$$

f and $\widetilde{f}$ generate the same ideal up to a coordinate change.

Now let $(M, 0)$ be the germ of a manifold with the multiplication in (5.8) for some $f \in \mathbb{C}\{t_2, t_3\} - \{0\}$ on TM. We have to show that it is an F-manifold. With the notations in (4.4)–(4.6), we have $a_3 = f$ and $a_2 = b_2 = b_3 = c_2 = c_3 = a_1 = b_1 = c_1$ and therefore $A_2 = A_2^{dual} = A_3 = 0$ in (4.27). Lemma 4.3 applies and shows that M is an F-manifold. Also Lemma 4.4 applies, the vanishing of R_1, R_2, R_3 and the nonvanishing of $a_3 = f$ show $T_tM \cong Q^{(2)}$ for generic $t \in M$.

For the shape of the Euler field $E = \varepsilon_1\partial_1 + \varepsilon_2\partial_2 + \varepsilon_3\partial_3$, one has to study the explicit version (5.5) of the condition $\mathrm{Lie}_E(\circ) = 1 \cdot \circ$. The case $(i, j) = (1, 1)$ gives $[e, E] = e$ and $\varepsilon_j \in \delta_{j1} + \mathbb{C}\{t_2, t_3\}$. The cases $(i, j) \in \{(2, 1), (3, 1)\}$ give nothing. The case $(i, j) = (3, 3)$ gives $\partial_3\varepsilon_1 = 0$. The case $(i, j) = (2, 3)$ gives this again and additionally $\partial_2\varepsilon_1 + f\partial_3\varepsilon_2 = 0$. The case $(i, j) = (2, 2)$ gives $2\partial_2\varepsilon_1 - f\partial_3\varepsilon_2 = 0$ and

(5.10). We obtain (5.9) and (5.10). The case $f = 1$ specializes this to (5.7). The parts (a) and (b) are proved.

2nd case, $f_1, f_2 \in \mathbf{m}$: We have

$$\partial_2^{\circ 2} = g_1\rho,\ \partial_2 \circ \partial_3 = g_2\rho,\ \partial_3^{\circ 2} = g_3\rho \quad \text{for some} \quad g_1, g_2, g_3 \in \mathbb{C}\{t_2, t_3\}.$$

One calculates

$$\begin{aligned} 0 &= \partial_2 \circ \rho = (f_2 g_1 + f_1 g_2)\rho, \quad \text{so} \quad 0 = f_2 g_1 + f_1 g_2, \\ 0 &= \partial_3 \circ \rho = (f_2 g_2 + f_1 g_3)\rho, \quad \text{so} \quad 0 = f_2 g_2 + f_1 g_3. \end{aligned}$$

As $\mathbb{C}\{t_2, t_3\}$ is a factorial ring and $\gcd(f_1, f_2) = 1$, this implies

$$(g_1, g_2, g_3) = (hf_1^2, -hf_1 f_2, hf_2^2) \quad \text{for some} \quad h \in \mathbb{C}\{t_2, t_3\}.$$

We define $\sigma := h \cdot \rho$ and obtain the multiplication in (5.11).

In order to show that the ideals (f_1, f_2) and (h) up to coordinate changes are invariants of the germ M of an F-manifold, we have to consider all coordinate changes which respect the shape of (5.11). The arguments above show that it is sufficient that $\widetilde{\partial}_2\circ$ and $\widetilde{\partial}_3\circ$ are nilpotent. Therefore, we consider a coordinate change which satisfies $t_1 = s_1$ and $(t_2, t_3) = (t_2(s_2, s_3), t_3(s_2, s_3))$. Then,

$$\begin{aligned} (\text{unit})(f_2\partial_2 + f_1\partial_3) &= (\text{unit}) \cdot \rho(t(s)) = \widetilde{\rho} = \widetilde{f_2}\widetilde{\partial}_2 + \widetilde{f_1}\widetilde{\partial}_3 \\ &= (\widetilde{f_2}\widetilde{\partial}_2 t_2 + \widetilde{f_1}\widetilde{\partial}_3 t_2)\partial_2 + (\widetilde{f_2}\widetilde{\partial}_2 t_3 + \widetilde{f_1}\widetilde{\partial}_3 t_3)\partial_3. \end{aligned}$$

The equality $(f_2(t(s)), f_1(t(s))) = (\widetilde{f_2}, \widetilde{f_1})$ of ideals follows. Consider the set $\{g \in \mathbb{C}\{t_2, t_3\} \,|\, \exists$ a vector field X with $[e, X] = 0$ and $X\circ$ nilpotent and $X^{\circ 2} = g\rho\}$. The function h is a greatest common divisor of all functions g in this set. This property is coordinate independent. Therefore, the ideal (h) up to coordinate changes is an invariant of the germ $(M, 0)$ of an F-manifold.

Now let $(M, 0)$ be the germ of a manifold with the multiplication in (5.11) for some $f_1, f_2 \in \mathbf{m}$ with $\gcd(f_1, f_2) = 1$ and for some $h \in \mathbb{C}\{t_2, t_3\}$. We have to show that it is an F-manifold. One calculates immediately $\partial_2 \circ \sigma = 0$ and $\partial_3 \circ \sigma = 0$, so the section σ is everywhere in the socle. Because of this and (5.11), $\partial_2\circ$ and $\partial_3\circ$ are nilpotent. Calculation and comparison with (4.19)–(4.21) and Remark 4.5 give

$$\begin{aligned} \partial_2^{\circ 2} &= hf_1^2 f_2\partial_2 + hf_1^3\partial_3 = \frac{1}{3}a_2\partial_2 + a_3\partial_3, \\ \partial_2 \circ \partial_3 &= -hf_1 f_2^2\partial_2 - hf_1^2 f_2\partial_3 = -\frac{1}{3}c_3\partial_2 - \frac{1}{3}a_2\partial_3, \\ \partial_2^{\circ 2} &= hf_2^3\partial_2 + hf_1 f_2^2\partial_3 = c_2\partial_2 + \frac{1}{3}c_3\partial_3, \end{aligned}$$

so

$$\begin{array}{c|c|c|c|c|c} a_2 & a_3 & b_2 = -\frac{1}{3}c_3 & b_3 = -\frac{1}{3}a_2 & c_2 & c_3 \\ \hline 3hf_1^2 f_2 & hf_1^3 & -hf_1 f_2^2 & -hf_1^2 f_2 & hf_2^3 & 3hf_1 f_2^2 \end{array} \tag{5.15}$$

Easy calculations show $A_2 = A_2^{dual} = A_3 = 0$ for A_2, A_2^{dual} and A_3 as in Lemma 4.3 (or, better, in Remark 4.5). Lemma 4.3 applies and shows that M is an F-manifold. Also Lemma 4.4 applies, the vanishing of R_1, R_2, R_3 and the nonvanishing of a_2, a_3, c_2, c_3 show $T_tM \cong Q^{(2)}$ for generic $t \in M$.

For the shape of the Euler field $E = \varepsilon_1\partial_1 + \varepsilon_2\partial_2 + \varepsilon_3\partial_3$, one has to study the explicit version (5.5) of the condition $\mathrm{Lie}_E(\circ) = 1 \cdot \circ$. The case $(i, j) = (1, 1)$ gives $[e, E] = e$ and $\varepsilon_j \in_{j1} +\mathbb{C}\{t_2, t_3\}$. The cases $(i, j) \in \{(2, 1), (3, 1)\}$ give nothing. The cases $(i, j) \in \{(2, 2), (2, 3), (3, 3)\}$ give with some tedious calculations $\partial_2\varepsilon_1 = 0$, $\partial_3\varepsilon_1 = 0$, (5.13) and (5.14). Part (c) proved. □

Remarks 5.4 (i) The ideal (f) in part (b) and the ideals (f_1, f_2) and (h) in part (c) up to coordinate changes are rich invariants of the germ of an F-manifold. They show that there is a functional parameter in the family of 3-dimensional germs of F-manifolds with $T_tM \cong Q^{(2)}$ for generic $t \in M$ if $T_0M \cong Q^{(1)}$. This is surprising, as part (a) says that the F-manifold is near points $t \in M$ with $T_tM \cong Q^{(2)}$ unique up to isomorphism.

(ii) If in part (c) h is chosen in $\mathbf{m}$, then h has a clear meaning, namely $\mathbb{C} \times h^{-1}(0) = \{t \in M \mid T_tM \cong Q^{(1)}\}$. The meaning of the ideal (f_1, f_2) is more subtle. It tells how the rank 1 bundle of socles of the algebras T_tM on $M - \{t \in M \mid T_tM \cong Q^{(1)}\}$ approaches 0.

(iii) The case $h(0) \neq 0$ in Theorem 5.3 (c) is the only case in Sects. 5 to 7 of an irreducible germ $(M, 0)$ of an F-manifold where the type of T_0M arises in codimension 2. In all other cases, it arises in codimension 1 or is equal to the generic type.

(iv) In all three parts of Theorem 5.3, $\mathcal{I}_M \supset (y_2^3, y_2^2y_3, y_2y_3^2, y_3^3)$ and $\sqrt{\mathcal{I}_M} = (y_1 - 1, y_2, y_3)$, so here $\{\sqrt{\mathcal{I}_M}, \sqrt{\mathcal{I}_M}\} \subset \sqrt{\mathcal{I}_M}$.

(v) The F-manifold in part (a) with an Euler field $E = (t_1 + c_1)\partial_1 + \varepsilon_2\partial_2 + \varepsilon_3\partial_3$ is a regular F-manifold if and only if $\varepsilon_2(0) \neq 0$. In fact, up to the choice of $c_1 \in \mathbb{C}$, there is only one regular germ of a 3-dimensional and everywhere irreducible F-manifold [7, Theorem 1.3]. A germ $(M, 0)$ of an F-manifold with $T_0M \cong Q^{(1)}$ has no regular Euler field, because the socle of $Q^{(1)}$ has dimension 2.

Next we classify the irreducible germs $(M, 0)$ of F-manifolds with $T_0M \cong Q^{(2)}$ and $T_tM \cong Q^{(3)}$ for generic $t \in M$.

Theorem 5.5 *The irreducible germs $(M, 0)$ of F-manifolds with $T_0M \cong Q^{(2)}$ and $T_tM \cong Q^{(3)}$ for generic $t \in M$ form a family with the only parameter $p \in \mathbb{Z}_{\geq 2}$. For fixed $p \in \mathbb{Z}_{\geq 2}$, the germ of the F-manifold can be given as follows:*

$$\begin{aligned} (M, 0) &= (\mathbb{C}^3, 0) \text{ with coordinates } t = (t_1, t_2, t_3),\, e = \partial_1, \\ \partial_2^{\circ 2} &= \varphi^2 \cdot \partial_3,\ \partial_2 \circ \partial_3 = t_2^{p-2}\varphi \cdot \partial_3,\ \partial_3^{\circ 2} = t_2^{2p-2} \cdot \partial_3, \\ &\text{with } \varphi := p + (2p - 2)t_2^{p-2}t_3. \end{aligned} \tag{5.16}$$

The caustic is $\mathcal{K} = \{t \in M \mid t_2 = 0\} = \{t \in M \mid T_tM \cong Q^{(2)}\}$. A vector field $E = \varepsilon_1\partial_1 + \varepsilon_2\partial_2 + \varepsilon_3\partial_3$ is an Euler field if and only if ε_1, ε_2 and ε_3 have the following *shape, here* $c_1 \in \mathbb{C}$ *and* $\varepsilon_{3,0} \in \mathbb{C}\{t_2\}$ *are arbitrary,*

$$\begin{aligned}
\varepsilon_1 &= t_1 + c_1, \\
\varepsilon_2 &= t_2 p^{-1}(1 - t_2^{p-2}\varepsilon_{3,0}), \\
\varepsilon_3 &= \varepsilon_{3,0} + t_3 p^{-1}(2 - p + (2p-2)t_2^{p-2}\varepsilon_{3,0}).
\end{aligned} \tag{5.17}$$

The following Remarks 5.6 make the geometry of the F-manifolds in Theorem 5.5 more transparent. The proof of Theorem 5.5 will be given after these remarks and will contain the proof of these remarks.

Remarks 5.6 Let $((M, 0), \circ, e)$ be one of the germs of F-manifolds in Theorem 5.5. On $M - \mathcal{K}$, the bundle TM of algebras decomposes into the direct sum $TM|_{M-\mathcal{K}} = T_{A_1} \oplus T_{\mathcal{N}_2}$ of bundles of algebras isomorphic to $\mathbb{C}$, respectively, to $\mathbb{C}[x]/(x^2)$. Write $\sigma|_{A_1}$, respectively, $\sigma|_{\mathcal{N}_2}$ for the summands of a section σ of $TM|_{M-\mathcal{K}}$ in T_{A_1}, respectively, in $T_{\mathcal{N}_2}$. Then,

$$(\partial_2|_{\mathcal{N}_2})^{\circ 2} = 0, \ \ \partial_3|_{\mathcal{N}_2} = 0, \tag{5.18}$$

$$\partial_2|_{A_1} = \partial_2 f \cdot e|_{A_1}, \ \ \partial_3|_{A_1} = \partial_3 f \cdot e|_{A_1}, \tag{5.19}$$

$$\text{with } f = t_2^p + t_2^{2p-2} t_3,$$

$$\text{so } \partial_2 f = t_2^{p-1}\varphi, \ \ \partial_3 f = t_2^{2p-2}. \tag{5.20}$$

Also

$$\partial_2^{\circ 3} = \partial_2 f \cdot \partial_2^{\circ 2}, \tag{5.21}$$

$$\partial_3 = h_2 \cdot \partial_2^{\circ 2} \quad \text{with} \quad h_2 = \varphi^{-2}. \tag{5.22}$$

The Euler field has freedom in $\varepsilon_{3,0} \in \mathbb{C}\{t_2\}$. This is not obvious, but it is also not surprising, as at $t \in M - \mathcal{K}$ the germ of the F-manifold is $A_1\mathcal{N}_2$, and the Euler fields of $\mathcal{N}_2$ have a similar freedom, see (3.4). Though in the case of $\mathcal{N}_2$, one can normalize the Euler fields by changing the variable t_2, see Theorem 3.2. This is not possible here. The functional freedom in $\varepsilon_{3,0} \in \mathbb{C}\{t_2\}$ cannot be get rid of.

Proof of Theorem 5.5 and the Remarks 5.6: Let $(M, 0)$ be a germ of a 3-dimensional F-manifold with $T_0M \cong Q^{(2)}$ and $T_tM \cong Q^{(3)}$ for generic $t \in M$. Then the caustic $\mathcal{K}$ is a hypersurface and $\mathcal{K} = \{t \in M \mid T_tM \cong Q^{(2)}\}$ and $M - \mathcal{K} = \{t \in M \mid T_tM \cong Q^{(3)}\}$. The bundle $TM|_{M-\mathcal{K}}$ decomposes into $T_{A_1} \oplus T_{\mathcal{N}_2}$ as described in the Remarks 5.6, and we write $\sigma = \sigma|_{A_1} + \sigma|_{\mathcal{N}_2}$ for the summands of a section σ of $TM|_{M-\mathcal{K}}$.

Because of $T_0M \cong Q^{(2)}$, Lemma 4.6 applies. We choose the coordinates t as in this lemma, so with (4.37)–(4.43) for suitable coefficients $g_2, g_1, g_0, h_2, h_1, h_0 \in \mathbb{C}\{t_2, t_3\}$. Let $\mathbf{m} \subset T_0M$ be the maximal ideal in T_0M. Refining (4.37), we can choose the coordinates t even such that

$$\begin{aligned}
\mathbb{C} \cdot \partial_2|_0 \oplus \mathbb{C} \cdot \partial_2^{\circ 2}|_0 &= \mathbf{m} \subset T_0M, \\
\mathbb{C} \cdot \partial_3|_0 = \mathbb{C} \cdot \partial_2^{\circ 2} &= \mathbf{m}^3 \subset T_0M
\end{aligned} \tag{5.23}$$

holds. Then, $h_2(0) \neq 0$ (and also $g_2(0) = g_1(0) = g_0(0) = h_1(0) = h_0(0) = 0$). In the following, five coordinate changes will be made in order to reach the normal form in Theorem 5.5.

The eigenvalue of $\partial_2|_{\mathcal{N}_2^\circ}$ is a holomorphic function on $M - \mathcal{K}$ which extends continuously and thus holomorphically to M. It can be written as $t_1 + \lambda$ with $\lambda \in \mathbb{C}\{t_2, t_3\}$. We make a special coordinate change as in (4.9), namely we choose the new coordinates $s = (s_1, s_2, s_3)$ such that

$$t_2 = s_2, t_3 = s_3, t_1 = s_1 + \tau \quad \text{with} \quad \tau \in \mathbb{C}\{t_2, t_3\} = \mathbb{C}\{s_2, s_3\}$$
$$\text{such that } \partial_2\tau = -\lambda \in \mathbb{C}\{t_2, t_3\}.$$

We obtain

$$\widetilde{\partial}_1 = \partial_1,\ \widetilde{\partial}_2 = \partial_2 + \partial_2\tau \cdot \partial_1 = \partial_2 - \lambda \cdot \partial_1,\ \widetilde{\partial}_3 = \partial_2 + \partial_3\tau \cdot \partial_1.$$

We call the new coordinates again t and denote also the new coefficients again as g_j, h_j. Now $\partial_2|_{\mathcal{N}_2^\circ}$ is nilpotent. This implies $g_1 = g_0 = 0$ and $\partial_2|_{A_1} = g_2 \cdot e|_{A_1}$.

The term in square brackets in line (4.42) in Lemma 4.6 vanishes because $(M, 0)$ is an F-manifold. Because of $g_1 = g_0 = 0$, it boils down to $g_2 h_{02} = 0$. Though $g_2 \neq 0$ because $T_t M \cong Q^{(3)}$ for generic t. Therefore, $\partial_2 h_0 = 0$. We make again a special coordinate change as in (4.9), now with $\tau \in \mathbb{C}\{t_3\}$ such that $\partial_3\tau = -h_0$. Then,

$$\widetilde{\partial}_1 = \partial_1, \widetilde{\partial}_2 = \partial_2, \widetilde{\partial}_3 = \partial_3 + \partial_3\tau \cdot \partial_1 = \partial_3 - h_0\partial_1.$$

We call the new coordinates again t and denote also the new coefficients again as g_j, h_j. Now $g_1 = g_0 = h_0 = 0$.

We make a coordinate change $t = t(s)$ with $t_1 = s_1$, $t_3 = s_3$ and $t_2 = t_2(s)$ with $\widetilde{\partial}_3 t_2(s) = -h_1(t(s))$. Then,

$$\widetilde{\partial}_1 = \partial_1, \widetilde{\partial}_2 = \widetilde{\partial}_2 t_2 \cdot \partial_2, \widetilde{\partial}_3 = \widetilde{\partial}_3 t_2 \cdot \partial_2 + \widetilde{\partial}_3 t_3 \cdot \partial_3 = \partial_3 - h_1\partial_2.$$

Here, $\widetilde{\partial}_2 t_2$ is a unit. We call the new coordinates again t and denote also the new coefficients as g_j, h_j. Now $g_1 = g_0 = h_0 = h_1 = 0$ and $\partial_3 = h_2\partial_2^{\circ 2}$. And now $\partial_3|_{\mathcal{N}_2} = 0$ because $\partial_2|_{\mathcal{N}_2^\circ}$ is nilpotent.

The term in square brackets in line (4.41) in Lemma 4.6 vanishes. Here, it is $\partial_2(h_2 g_2^2) - \partial_3 g_2$. Therefore, a function $f \in \mathbb{C}\{t_2, t_3\}$ with

$$\partial_2 f = g_2 \quad \text{and} \quad \partial_3 f = h_2 g_2^2 = h_2(\partial_2 f)^2 \tag{5.24}$$

exists. Here, $f|_{t_3=0}$ has a zero of an order $p \in \mathbb{Z}_{\geq 2}$ because $g_2(0) = 0$.

A coordinate change $t = t(s)$ with $t_1 = s_1$, $t_3 = s_3$ and $t_2 = t_2(s_2) \in \mathbb{C}\{t_2\}$ exists such that (after calling the new coordinates again t)

$$f|_{t_3=0} = t_2^p, \quad \text{so} \quad f = t_2^p + \sum_{k\geq 1} f_k t_3^k \text{ with } f_k \in \mathbb{C}\{t_2\}.$$

The equation $\partial_3 f = h_2(\partial_2 f)^2$ and $h_2(0) \neq 0$ and $p \geq 2$ imply inductively $f_k \in t_2^{2p-2}\mathbb{C}\{t_2\}$ for all $k \geq 1$ and especially $f_1 \in t_2^{2p-2} \cdot \mathbb{C}\{t_2\}^*$. Therefore

$$f = t_2^p\Big(1 + t_2^{p-2}t_3 \cdot (\text{a unit in } \mathbb{C}\{t_2\}\Big).$$

We can and will change the coordinate t_3 such that $f = t_2^p(1 + t_2^{p-2}t_3)$. Then, $g_2 = \partial_2 f = t_2^{p-1}(p + (2p-2)t_2^{p-2}t_3) = t_2^{p-1}\varphi$ and $\partial_3 f = t_2^{2p-2}$. Now h_2 is determined by $\partial_3 f = h_2(\partial_2 f)^2$ and is $h_2 = \varphi^{-2}$.

Now all statements in the Remarks 5.6 except those on the Euler field are shown. The terms in the square brackets in (4.41)–(4.43) vanish. Therefore, we really have an F-manifold. The multiplication is as in (5.16), because

$$\begin{aligned}
\partial_2^{\circ 2} &= h_2^{-1}\partial_3 = \varphi^2\partial_3,\\
\partial_2^{\circ 3} &= g_2\partial_2^{\circ 2} = t_2^{p-1}\varphi\partial_2^{\circ 2},\\
\partial_2 \circ \partial_3 &= h_2\partial_2^{\circ 3} = h_2 g_2\partial_2^{\circ 2} = g_2\partial_3 = t_2^{p-1}\varphi\partial_3,\\
\partial_3^{\circ 2} &= h_2^2\partial_2^{\circ 4} = h_2^2 g_2^2\partial_2^{\circ 2} = h_2 g_2^2\partial_3 = t_2^{2p-2}\partial_3.
\end{aligned}$$

It remains to show the shape (5.17) of the Euler field $E = \varepsilon_1\partial_1 + \varepsilon_2\partial_2 + \varepsilon_3\partial_3$. One has to study the explicit version (5.5) of the condition $\mathrm{Lie}_E(\circ) = 1 \cdot \circ$. The case $(i,j) = (1,1)$ just gives $[e, E] = e$ and thus $\varepsilon_j \in \delta_{1j} + \mathbb{C}\{t_2, t_3\}$. The cases $(i,j) \in \{(2,1),(3,1)\}$ give nothing. The case $(i,j) = (3,3)$ leads to $\partial_3(\varepsilon_1) = 0$, $\partial_3\varepsilon_2 = 0$ and $\varepsilon_2 = (2p-2)^{-1}t_2(1 - \partial_3\varepsilon_3)$. The cases $(i,j) \in \{(2,2),(2,3)\}$ lead to $\partial_2\varepsilon_1 = 0$ and to equations which allow to relate $\varepsilon_{3,0}$ and $\varepsilon_{3,1} \in \mathbb{C}\{t_2\}$ in $\varepsilon_3 = \varepsilon_{3,0} + t_3\varepsilon_{3,1}$. At the end one obtains (5.17). We leave the details to the reader.

□

We do not have a classification of the irreducible germs $(M, 0)$ of F-manifolds with $T_0M \cong Q^{(1)}$ and $T_tM \cong Q^{(3)}$ for generic $t \in M$. The family of examples in the next lemma shows that such germs exist.

Lemma 5.7 *Fix a number $p \in \mathbb{Z}_{\geq 2}$. The manifold $M = \mathbb{C}^3$ with coordinates $t = (t_1, t_2, t_3)$ and with the multiplication on TM given by $e = \partial_1$ and*

$$\partial_2^{\circ 2} = pt_2^{p-1} \cdot \partial_2,\ \partial_2 \circ \partial_3 = 0,\ \partial_3^{\circ 2} = 0 \tag{5.25}$$

is an F-manifold with caustic

$$\begin{aligned}
\mathcal{K} &= \{t \in M \mid t_2 = 0\} = \{t \in M \mid T_tM \cong Q^{(1)}\} \text{ and}\\
M - \mathcal{K} &= \{t \in M \mid T_tM \cong Q^{(3)}\}.
\end{aligned} \tag{5.26}$$

A vector field E is an Euler field if and only if

$$E = (t_1 + c_1)\partial_1 + \frac{1}{p}t_2\partial_2 + \varepsilon_3\partial_3, \quad \text{with } c_1 \in \mathbb{C},\ \varepsilon_3 \in \mathbb{C}\{t_3\}. \tag{5.27}$$

Proof In the Notations 4.1, $b_2 = b_3 = a_3 = c_2 = c_3 = 0$, $a_2 = t_2^{p-1}$. Therefore, $A_2 = A_2^{dual} = A_3 = 0$ in Lemma 4.4, and we have an F-manifold. The statement on $\mathcal{K}$ is clear. The analytic spectrum is

$$L_M = \left\{(y, t) \in T^*M \mid y_1 = 1, y_2(y_2 - pt_2^{p-1}) = y_2 y_3 = y_3^2 = 0\right\}. \tag{5.28}$$

The set which underlies L_M is $L_M^{red} = \{(y, t) \in T^*M \mid (y_1, y_2, y_3) = (1, 0, 0)\} \cup \{(y, t) \in T^*M \mid (y_1, y_2, y_3) = (1, pt_2^{p-1}, 0)\}$, so it has two components which meet over $\mathcal{K}$. Therefore, $T_t M \cong Q^{(3)}$ for $t \in M - \mathcal{K}$. For the proof of (5.27), one has to study the explicit version (5.5) of the condition $\mathrm{Lie}_E(\circ) = 1 \cdot \circ$. We leave the details to the reader. □

6 Examples of 3-dimensional generically semisimple *F*-manifolds

A partial classification of irreducible germs $(M, 0)$ of 3-dimensional generically semisimple F-manifolds was undertaken in [15, ch. 5.5], there in Theorems 5.29 and 5.30. Theorem 5.29 in [15] gave basic facts on both cases, the case $T_0 M \cong Q^{(2)}$ and the case $T_0 M \cong Q^{(1)}$. Theorem 5.30 classified completely those germs where $T_0 M \cong Q^{(2)}$ and where the germ (L_M, λ) of the analytic spectrum has 3 components.

Below we first describe in the Remarks 6.1 the strategy of the classification results in this section and the next section. The Examples 6.2 rewrite the three distinguished F-manifolds A_3, B_3 and H_3. Theorem 6.3 is Theorem 5.29 from [15]. Lemma 6.4 and Lemma 6.5 give examples $(M, 0)$ of generically semisimple F-manifolds with $T_0 M \cong Q^{(1)}$. We do not have a classification of all such germs. Lemma 6.4 is Theorem 5.32 from [15].

Remarks 6.1 (i) Let $(M, 0)$ be an irreducible germ of a 3-dimensional generically semisimple F-manifold with analytic spectrum (L_M, λ). Here and in the following, we choose coordinates $t = (t_1, t_2, t_3)$ on $(M, 0)$ such that $(M, 0) \cong (\mathbb{C}^3, 0)$ and $e = \partial_1$. Then, (y_1, y_2, y_3) are the fiber coordinates on T^*M which correspond to $\partial_1, \partial_2, \partial_3$, and $\alpha = \sum_{i=1}^3 y_i \mathrm{d}t_i$ is the canonical 1-form. In the following, M denotes a suitable (small) representative of $(M, 0)$.

It turns out that often the best way to arrive at a normal form for $((M, 0), \circ, e)$ is to control the function $F : (L_M, \lambda) \to (\mathbb{C}, 0)$ from Theorem 2.15 (b). It is holomorphic on L_M^{reg} and continuous on L, and it satisfies

$$\mathrm{d}F|_{L_M^{reg}} = \alpha|_{L_M^{reg}}. \tag{6.1}$$

We consider it as a 3-valued holomorphic function on M which is branched precisely over the caustic $\mathcal{K} \subset M$. Locally on $M - \mathcal{K}$ it splits into three holomorphic functions $F^{(1)}, F^{(2)}, F^{(3)}$. We will use this notation without specifying a simply connected region in $M - \mathcal{K}$. This is imprecise, but not in a harmful way. With this notation, F determines L_M as follows (this rewrites (6.1)), locally on $M-\mathcal{K}$,

$$L_M=\bigcup_{j=1}^{3}\{(y,t)\in T^*M\mid y_i=\partial_i F^{(j)}\text{ for }i\in\{1,2,3\}\}. \tag{6.2}$$

Let $M^{(r)}$ be a suitable neighborhood of 0 in the (t_2,t_3)-plane $\mathbb{C}^2$. It can be identified with the set of e-orbits of M. The condition $\mathrm{Lie}_e(\circ)=0$ implies that the multiplication, the caustic $\mathcal{K}$ and the analytic spectrum L_M are invariant under the flow of e. As $e=\partial_1$, $\mathcal{K}$ induces a hypersurface $\mathcal{K}^{(r)}\subset M^{(r)}$, and $F=t_1+f$ where f is a 3-valued holomorphic function on $M^{(r)}$ which is branched along $\mathcal{K}^{(r)}$. Locally on $M^{(r)}-\mathcal{K}^{(r)}$, f splits into three holomorphic functions $f^{(1)}$, $f^{(2)}$, $f^{(3)}$. The coefficients of the polynomials $\prod_{j=1}^3(x-f^{(j)})$ and $\prod_{j=1}^3(x-\partial_2 f^{(j)})$ and $\prod_{j=1}^3(x-\partial_3 f^{(j)})$ are univalued holomorphic functions on $M^{(r)}$, i.e., they are in $\mathbb{C}\{t_2,t_3\}$. The last two polynomials are the characteristic polynomials of $\partial_2\circ$ and $\partial_3\circ$.

(ii) The Euler field $E=\varepsilon_1\partial_1+\varepsilon_2\partial_2+\varepsilon_3\partial_3$ on $M-\mathcal{K}$, which corresponds to F by Theorem 2.15, is given by $F^{(j)}=E(F^{(j)})$, i.e., by

$$\varepsilon_1=t_1,\ \ f^{(j)}=\varepsilon_2\cdot\partial_2 f^{(j)}+\varepsilon_3\cdot\partial_3 f^{(j)}. \tag{6.3}$$

ε_2 and ε_3 depend only on (t_2,t_3), but often are meromorphic along $\mathcal{K}^{(r)}$. If they are in $\mathbb{C}\{t_2,t_3\}$, then E extends from $M-\mathcal{K}$ to M.

(iii) Now consider the case $T_0M\cong Q^{(2)}$. Denote by $\mathbf{m}\subset T_0M$ the maximal ideal in T_0M. We can and will choose the coordinates t such that

$$\begin{aligned}\mathbb{C}\cdot\partial_2|_0\oplus\mathbb{C}\cdot\partial_2^{\circ 2}|_0&=\mathbf{m}\subset T_0M,\\ \mathbb{C}\cdot\partial_3|_0=\mathbb{C}\cdot\partial_2^{\circ 2}|_0&=\mathbf{m}^2\subset T_0M.\end{aligned} \tag{6.4}$$

Then

$$\partial_3=h_2\cdot\partial_2^{\circ 2}+h_1\cdot\partial_2+h_0\cdot\partial_1 \tag{6.5}$$

for suitable coefficients $h_2,h_1,h_0\in\mathbb{C}\{t_2,t_3\}$ with $h_2(0)\neq 0$, $h_1(0)=h_0(0)=0$. These coefficients are determined by

$$\partial_3 f^{(j)}=h_2\cdot(\partial_2 f^{(j)})^2+h_1\cdot\partial_2 f^{(j)}+h_0. \tag{6.6}$$

Also write $\prod_{j=1}^3(x-\partial_2 f^{(j)})=x^3-g_2x^2-g_1x-g_0$ with $g_2,g_1,g_0\in\mathbb{C}\{t_2,t_3\}$. Then,

$$\begin{aligned}L_M=\{(y,t)\in T^*M\mid y_1=1,\ &y_2^3=g_2y_2^2+g_1y_2+g_0,\\ &y_3=h_2y_2^2+h_1y_2+h_0\}.\end{aligned} \tag{6.7}$$

Examples 6.2 Here, the F-manifolds A_3, B_3, H_3 from Theorem 5.22 (i) in [15] will be rewritten with the notions from the Remarks 6.1. They arise as complex orbit spaces of the corresponding Coxeter groups. Their discriminants had been studied especially by O.P. Shcherbak [24], and their Lagrange maps (which correspond to the F-manifold structures by Theorem 3.16 in [15]) had been studied by Givental [13].

They are simple F-manifolds with Euler fields with positive weights. Their germs $(M, 0)$ at 0 are the only simple 3-dimensional germs of F-manifolds with $T_0M \cong Q^{(2)}$, see Theorem 6.3 (b).

We use the notations from the Remarks 6.1. Though here we have F-manifolds $M = \mathbb{C}^3$, not just germs. The following table gives for each of the three cases the following data:

(i) a 3-valued function ξ on $\mathbb{C}^2 = M^{(r)}$ (with coordinates (t_2, t_3)) which is branched along $\mathcal{K}^{(r)}$. It is given by the equation of degree 3 which it satisfies. The equation is denoted $\xi^3 - g_2\xi^2 - g_2\xi - g_0 = 0$ with $g_2, g_1, g_0 \in \mathbb{C}[t_2, t_3]$.
(ii) A weight system $(w_1, w_2, w_3) \in \mathbb{Q}^3_{>0}$.
(iii) The components of $\mathcal{K}^{(r)}$, and which germs of an F-manifold are at generic points of each component.

	A_3	B_3	H_3
ξ	$\xi^3 + 2\xi t_3 + t_2$	$\xi(\xi^2 + 2\xi t_3 + t_2)$	$\xi^3 - (2\xi t_3 + t_2)^2$
(w_1, w_2, w_3)	$(1, \frac{3}{4}, \frac{1}{2})$	$(1, \frac{2}{3}, \frac{1}{3})$	$(1, \frac{3}{5}, \frac{1}{5})$
$\mathcal{K}^{(r)}:\ A_2A_1$	$27t_2^2 + 32t_3^3 = 0$	$t_2 - t_3^2$	$27t_2 + 32t_3^3$
$\mathcal{K}^{(r)}$ 2nd comp.	$-$	$t_2 = 0:\ I_2(4)A_1$	$t_2 = 0:\ I_2(5)A_1$

The 3-valued function f on $M^{(r)}$ with $F = t_1 + f$ is

$$f = w_2\xi t_2 + w_3\xi^2 t_3. \tag{6.8}$$

The following identities are crucial. They will be proved below.

$$\partial_2 f = \xi, \quad \partial_3 f = \xi^2. \tag{6.9}$$

Because of them, the Euler field is $E = t_1\partial_1 + w_2t_2\partial_2 + w_3t_3\partial_3$, and the multiplication and the analytic spectrum are given as follows,

$$L_M = \{(y, t) \in T^*M \mid y_1 = 1, y_2^3 = g_2y_2^2 + g_1y_2 + g_0, y_3 = y_2^2\}. \tag{6.10}$$

One sees $L \cong \mathbb{C}^2 \times C$, where C is a plane curve, and C is smooth in the case A_3, C has two smooth components which intersect transversely in the case B_3, and C has one ordinary cusp in the case H_3. We will prove now (6.9) and the claims on the caustic.

Proof of (6.9): It is equivalent to $\mathrm{d}f = \xi\mathrm{d}t_2 + \xi^2\mathrm{d}t_3$. And this is equivalent to the claim that L_M^{reg} is Lagrange, i.e., $\alpha|_{L_M^{reg}}$ is closed. And then $\int_{L_M}\alpha = t_1 + f$ as a 3-valued function on M. In all three cases

$$\mathrm{d}f = w_2\xi \mathrm{d}t_2 + w_3\xi^2\mathrm{d}t_3 + (w_2t_2 + 2w_3\xi t_3)\mathrm{d}\xi,$$

so in all three cases

$$(w_2t_2 + 2w_3\xi t_3)\mathrm{d}\xi = (1 - w_2)\xi \mathrm{d}t_2 + (1 - w_3)\xi^2\mathrm{d}t_3 \tag{6.11}$$

has to be shown.

The case A_3: Use $0 = \xi^3 + 2\xi t_3 + t_2$ to calculate

$$\begin{aligned} 0 &= \xi \mathrm{d}(\xi^3 + 2\xi t_3 + t_2) = (3\xi^3 + 2\xi t_3)\mathrm{d}\xi + 2\xi^2\mathrm{d}t_3 + \xi \mathrm{d}t_2 \\ &= (-4\xi t_3 - 3t_2)\mathrm{d}\xi + 2\xi^2\mathrm{d}t_3 + \xi \mathrm{d}t_2, \end{aligned}$$

which shows (6.11).

The case B_3: Use $0 = \xi(\xi^2 + 2\xi t_3 + t_2)$ to calculate

$$\begin{aligned} 0 &= \xi \mathrm{d}(\xi(\xi^2 + 2\xi t_3 + t_2)) = (3\xi^3 + 4\xi^2 t_3 + \xi t_2)\mathrm{d}\xi + 2\xi^3\mathrm{d}t_3 + \xi^2\mathrm{d}t_2 \\ &= (-2\xi^2 t_3 - 2\xi t_2)\mathrm{d}\xi + 2\xi^3\mathrm{d}t_3 + \xi^2\mathrm{d}t_2 \\ &= 2\xi \cdot \eta \quad \text{with} \quad \eta := (-\xi t_3 - t_2)\mathrm{d}\xi + \xi^2\mathrm{d}t_3 + \frac{1}{2}\xi \mathrm{d}t_2. \end{aligned}$$

For (6.11), we need $\eta = 0$. Here, ξ consists of one holomorphic function $\xi^{(1)} = 0$ and a 2-valued function $\xi^{(2\&3)} \neq 0$. For $\xi^{(1)}$, $\eta = 0$ is trivial. For $\xi^{(2\&3)}$, $\eta = 0$ follows from $0 = 2\xi^{(2\&3)} \cdot \eta$, as then we may divide by $2\xi^{(2\&3)}$.

The case H_3: Use $0 = \xi^3 - (2\xi t_3 + t_2)^2$ to calculate

$$\begin{aligned} 0 &= \xi \mathrm{d}(\xi^3 - (2\xi t_3 + t_2)^2) = 3\xi^3\mathrm{d}\xi - 2\xi(2\xi t_3 + t_2)\mathrm{d}(2\xi t_3 + t_2) \\ &= (2\xi t_3 + t_2) \cdot \Big(3(2\xi t_3 + t_2)\mathrm{d}\xi - 2\xi \mathrm{d}(2\xi t_3 + t_2)\Big), \quad \text{thus} \\ 0 &= 3(2\xi t_3 + t_2)\mathrm{d}\xi - 2\xi \mathrm{d}(2\xi t_3 + t_2) \\ &= (2\xi t_3 + 3t_2)\mathrm{d}\xi - 4\xi^2\mathrm{d}t_3 - 2\xi \mathrm{d}t_2, \end{aligned}$$

which shows (6.11). □

Proof of the statements on the caustic: The case A_3: The discriminant of $x^3 + 2t_3x + t_2$ is $4(2t_3)^3 + 27t_2^2 = 32t_3^3 + 27t_2^2$. Over generic points of the caustic (all except $(t_1, 0, 0)$ for $t_1 \in \mathbb{C}$) the multigerm of L_M has 2 smooth components, so there we have the germ A_2A_1.

The case B_3: The two components of L_M meet over points with $t_2 = 0$. Over generic points of this component of $\mathcal{K}$, we have the germ $I_2(4)A_1$. The discriminant of $x^2 + 2t_3x + t_2$ is $(2t_3)^2 - 4t_2 = 4(t_3^2 - t_2)$. Over generic components of this component, the multigerm of L_M has 2 smooth components, so there we have the germ A_2A_1.

The case H_3: The discriminant of $x^3 - (2t_3x + t_2)^2$ is $t_2^3(32t_3^3 + 27t_2)$. The cusp surface of L_M lies over the component with $t_2 = 0$ of $\mathcal{K}$. So there we have the germ

$I_2(5)A_1$. Over generic points of the component with $32t_3^3 + 27t_2 = 0$, the multigerm L_M has 2 smooth components, so there we have the germ A_2A_1. □

The following theorem is Theorem 5.29 from [15]. It gives basic facts on irreducible germs of 3-dimensional F-manifolds with generically semisimple multiplication.

Theorem 6.3 [15, Theorem 5.29] *Let $(M, 0)$ be an irreducible germ of a 3-dimensional generically semisimple F-manifold with analytic spectrum $(L_M, \lambda) \subset T^*M$.*

(a) Suppose $T_0M \cong Q^{(2)}$. Then, (L_M, λ) has embedding dimension 3 or 4 and $(L_M, \lambda) \cong (\mathbb{C}^2, 0) \times (C, 0)$ for a plane curve $(C, 0) \subset (\mathbb{C}^2, 0)$ with $\mathrm{mult}(C, 0) \le 3$. The Euler field E_0 from Theorem 2.15 (c) on $M - \mathcal{K}$ extends holomorphically to M if and only if $(C, 0)$ is quasihomogeneous.
(b) Suppose $T_0M \cong Q^{(2)}$ and $(L_M, \lambda) \cong (\mathbb{C}^2, 0) \times (C, 0)$ with $\mathrm{mult}(C, 0) < 3$. Then, $((M, 0), \circ, e)$ is one of the germs A_3, B_3, H_3.
(c) Suppose $T_0M \cong Q^{(2)}$ and $(L_M, \lambda) \cong (\mathbb{C}^2, 0) \times (C, 0)$ with $\mathrm{mult}(C, 0) = 3$. Then, the caustic $\mathcal{K}$ is a smooth surface and coincides with the μ-constant stratum. That means, $T_qM \cong Q^{(2)}$ for each $q \in \mathcal{K}$. The modality is $\mathrm{mod}_\mu(M, 0) = 1$ (the maximal possible) (recall Definition 2.9 (a)).
(d) Suppose $T_0M \cong Q^{(1)}$. Then, (L_M, λ) has embedding dimension 5 and $(L_M, \lambda) \cong (\mathbb{C}, 0) \times (L^{(r)}, 0)$. Here, $(L^{(r)}, 0)$ is a Lagrange surface with embedding dimension 4. Its ring $\mathcal{O}_{L^{(r)},0}$ is a Cohen–Macaulay ring, but not a Gorenstein ring.

Sketch of the proof: (a) One chooses the coordinates (t_1, t_2, t_3) as in (6.4). Then L_M is as in (6.7). Because of the equations $y_1 = 1$ and $y_3 = \sum_{i=0}^{2} h_i y_2^i$, (L_M, λ) has embedding dimension ≤ 4. Theorem 2.14 applies and gives $(L_M, \lambda) \cong (\mathbb{C}^2, 0) \times (C, 0)$ for a plane curve C. The germ $(C, 0)$ has multiplicity ≤ 3 because the projection $\pi_L : L_M \to M$ is a branched covering of degree 3. The Euler field E_0 from Theorem 2.15 (c) on $M - \mathcal{K}$ extends to M if and only if $(C, 0)$ is quasihomogeneous because of Theorem 2.16 (e) and Theorem 2.15 (b)+(c).

(b) $\mathrm{mult}(C, 0) \le 2$ means that $(C, 0)$ is either smooth or a double point or a cusp. In the first two cases, one can apply the correspondence between F-manifolds and hypersurface or boundary singularities [15, Theorem 5.6 and Theorem 5.14] and the fact that A_3, B_3 and C_3 are the only hypersurface or boundary singularities with Milnor number 3. In the case of a cusp, results of Givental [13] are used, see the proof of [15, Theorem 5.29].

(c) $(C, 0)$ has multiplicity 3. And the projection $\pi_L : L_M \to M$ is a branched covering of degree 3. Together these facts imply that $\pi_L : L_M \to M$ is precisely branched at the points of L_M which correspond to $(\mathbb{C}^2, 0) \times \{0\}$ in $(\mathbb{C}^2, 0) \times (C, 0) \cong (L_M, \lambda)$. This implies all the statements.

(d) If the embedding dimension of (L_M, λ) were ≤ 4, then by Theorem 2.14 $(L_M, \lambda) \cong (\mathbb{C}^2) \times (C, 0)$ for a plane curve, so then (L_M, λ) were a complete intersection, and thus $T_0M \cong Q^{(2)}$, a contradiction. Therefore, the embedding dimension is 5 and by Theorem 2.14 $(L_M, \lambda) \cong (\mathbb{C}, 0) \times (L^{(r)}, 0)$ where $(L^{(r)}, 0)$ has embedding dimension 4. The ring $\mathcal{O}_{L^{(r)},0}$ is Cohen–Macaulay because $\pi^{(r)} : L^{(r)} \to M^{(r)}$ is finite and flat. It is not Gorenstein, because $T_0M \cong Q^{(1)}$ is not Gorenstein. □

The classification of germs $(M, 0)$ of 3-dimensional generically semisimple F-manifolds with $T_0M \cong Q^{(1)}$ is not treated in this paper. Only one family of examples from [15] and one other interesting example will be given now with all details, and a family of examples in [18, 6.2–6.6] will be described without details.

Lemma 6.4 [15, Theorem 5.32] *Fix two numbers $p_2, p_3 \in \mathbb{Z}_{\geq 2}$. The manifold $M = \mathbb{C}^3$ with coordinates $t = (t_1, t_2, t_3)$ and with the multiplication on TM given by $e = \partial_1$ and*

$$\partial_2^{\circ 2} = p_2 t_2^{p_2-1} \cdot \partial_2,\ \partial_2 \circ \partial_3 = 0,\ \partial_3^{\circ 2} = p_3 t_3^{p_3-1} \partial_3 \tag{6.12}$$

is a simple (and thus generically semisimple) F-manifold with $T_0M \cong Q^{(1)}$. Its caustic $\mathcal{K}$ has two components $\mathcal{K}^{(1)} = \{t \in M \mid t_2 = 0\}$ and $\mathcal{K}^{(2)} = \{t \in M \mid t_3 = 0\}$. The germ (M, t) is of type $A_1 I_2(2p_2)$ for $t \in \mathcal{K}^{(1)} - \mathbb{C} \times \{0\}$ and of type $A_1 I_2(2p_3)$ for $t \in \mathcal{K}^{(2)} - \mathbb{C} \times \{0\}$. A vector field E is an Euler field if and only if

$$E = (t_1 + c_1)\partial_1 + \frac{1}{p_2} t_2 \partial_2 + \frac{1}{p_3} t_3 \partial_3 \quad \text{with } c_1 \in \mathbb{C}. \tag{6.13}$$

Proof The analytic spectrum is

$$\begin{aligned} L_M = \{(y, t) \in T^*M \mid y_1 = 1,\, y_2(y_2 - p_2 t_2^{p_2-1}) = y_2 y_3 = 0, \\ y_3(y_3 - p_3 t_3^{p_3-1}) = 0\}. \end{aligned} \tag{6.14}$$

The set which underlies L_M has three components $L^{(1)}, L^{(2)}, L^{(3)}$ with

$$\begin{aligned} L^{(1)} &= \{(y, t) \in T^*M \mid (y_1, y_2, y_3) = (1, 0, 0)\}, \\ L^{(2)} &= \{(y, t) \in T^*M \mid (y_1, y_2, y_3) = (1, p_2 t_2^{p_2-1}, 0)\}, \\ L^{(3)} &= \{(y, t) \in T^*M \mid (y_1, y_2, y_3) = (1, 0, p_3 t_3^{p_3-1})\}. \end{aligned}$$

The functions $f^{(1)} := 0$, $f^{(2)} := t_2^{p_2}$, $f^{(3)} := t_3^{p_3}$ on M satisfy

$$f^{(j)} = \frac{1}{p_2} t_2 \partial_2 f^{(j)} + \frac{1}{p_3} t_3 \partial_3 f^{(j)}.$$

If one lifts $f^{(j)}$ to $L^{(j)}$, the resulting function on L_M is $\frac{1}{p_2} t_2 y_2 + \frac{1}{p_3} t_3 y_3$, so a holomorphic function on L_M. $F := t_1 + f$ satisfies all properties in the Remarks 6.1 (i)+(ii). Therefore, $(M, \circ, e)$ is an F-manifold, and the Euler field is as claimed.

$L^{(2)}$ and $L^{(3)}$ meet only over $t = 0$, $L^{(1)}$ and $L^{(2)}$ meet over $\mathcal{K}^{(1)}$, and $L^{(1)}$ and $L^{(3)}$ meet over $\mathcal{K}^{(2)}$. From their intersection multiplicities or from the coefficients of the Euler field, one concludes that the germs of F-manifolds have the types $A_1 I_2(2p_2)$, respectively, $A_2 I_2(2p_3)$ at points of $\mathcal{K}^{(1)}$, respectively, $\mathcal{K}^{(2)}$ not equal to $\mathbb{C} \times \{0\} \subset M$. The stratification of M by the types of the germs of the F-manifolds shows that the F-manifold is simple. □

Also the following example is a simple F-manifold $M=\mathbb{C}^3$ with $T_0M\cong Q^{(1)}$. Its analytic spectrum is irreducible, and it is singular only in codimension 2.

Lemma 6.5 *The manifold $M=\mathbb{C}^3$ with coordinates $t=(t_1,t_2,t_3)$ and with the multiplication on TM given by $e=\partial_1$ and*

$$\begin{aligned}
(\partial_2-\frac{1}{2}t_3\partial_1)^{\circ 2} &= \frac{9}{4}t_2^2\partial_1-\frac{3}{2}t_3(\partial_2-\frac{1}{2}t_3\partial_1)-\frac{3}{2}t_2(\partial_3+\frac{1}{2}t_2\partial_1),\\
(\partial_2-\frac{1}{2}t_3\partial_1)\circ(\partial_3+\frac{1}{2}t_2\partial_1) &= \frac{3}{4}t_2t_3\partial_1,\\
(\partial_3+\frac{1}{2}t_2\partial_1)^{\circ 2} &= -\frac{3}{4}t_3^2\partial_1-\frac{1}{2}t_3(\partial_2-\frac{1}{2}t_3\partial_1)+\frac{3}{2}t_2(\partial_3+\frac{1}{2}t_2\partial_1),
\end{aligned}\tag{6.15}$$

is a simple (and thus generically semisimple) F-manifold with $T_0M\cong Q^{(1)}$. The Lagrange surface $L^{(r)}$ in $T^*M^{(r)}$ with $L_M\cong\mathbb{C}\times L^{(r)}$ is smooth outside 0. The caustic $\mathcal{K}$ has 4 components. The corresponding 4 components of $\mathcal{K}^{(r)}\subset M^{(r)}$ are the 4 lines through 0 which are together given by

$$0=t_3^4+6t_2^2t_3^2-3t_2^4. \tag{6.16}$$

The germ (M,t) is of type A_1A_2 for $t\in\mathcal{K}-\mathbb{C}\times\{0\}$. A vector field E is an Euler field if and only if

$$E=(t_1+c_1)\partial_1+\frac{1}{2}t_2\partial_2+\frac{1}{2}t_3\partial_3 \quad\text{with } c_1\in\mathbb{C}. \tag{6.17}$$

Proof In the Notations 4.1 in (4.4)–(4.6),

$$(a_2,a_3,b_2,b_3,c_2,c_3)=\left(\frac{-3}{2}t_3,\frac{-3}{2}t_2,\frac{-1}{2}t_2,\frac{1}{2}t_3,\frac{-1}{2}t_3,\frac{3}{2}t_2\right), \tag{6.18}$$

$$(a_1,b_1,c_1)=\left(\frac{9}{4}t_2^2,\frac{3}{4}t_2t_3,\frac{-3}{4}t_3^2\right)=(-a_3c_3,a_3c_2,-a_2c_2). \tag{6.19}$$

Therefore the multiplication is associative. One checks easily that A_2, A_2^{dual}, and A_3 in Lemma 4.4 vanish. Therefore $(M,\circ,e)$ is an F-manifold. The function $9R_3^2-4R_1R_2$ in Lemma 4.3 is here $9R_3^2-4R_1R_2=\frac{9}{4}(t_3^4+6t_2^2t_3^2-3t_2^4)$. Therefore the F-manifold is generically semisimple, and the caustic is given by (6.16). One checks also easily that the explicit version (5.5) for $\mathrm{Lie}_E(\circ)=1\cdot\circ$ is satisfied for E as in (6.17). Therefore, E is one Euler field. By Theorem 2.15 (b)+(c) and the irreducibility of the germ $(M,0)$, the vector fields $E+c_1e$ for $c_1\in\mathbb{C}$ are the only Euler fields on M.

The smoothness of $L^{(r)}$ outside 0 would imply together with the classification of the 2-dimensional germs of F-manifolds that the germ (M,t) for $t\in\mathcal{K}-\mathbb{C}\times\{0\}$ is of type A_1A_2. It remains to show that $L^{(r)}$ is smooth. For this, we reveal how the F-manifold was constructed. Consider the coordinate change on $T^*M^{(r)}$ with new coordinates $x=(x_1,x_2,x_3,x_4)$,

$$t_2=x_1+x_3,\ t_3=x_2-x_4,\ y_2=x_2+x_4,\ y_3=-x_1+x_3.$$

In the new coordinates, the functions Y_{22}, Y_{23} and Y_{33} in Lemma 4.4, which define $L^{(r)}$, become

$$\begin{aligned} Y_{22} &= (x_2^2 - 3x_1x_3) - 3(x_1^2 - x_2x_4), \\ Y_{33} &= (x_2^2 - 3x_1x_3) + (x_1^2 - x_2x_4), \\ Y_{23} &= -x_1x_2 + 3x_3x_4. \end{aligned}$$

In the new coordinates on $T^*M^{(r)} \cong \mathbb{C}^4$, $L^{(r)}$ is the cone over the curve in $\mathbb{P}^3$ which is defined in homogeneous coordinates by the vanishing of $x_1^2 - x_2x_4$, $x_2^2 - 3x_1x_3$ and $x_1x_2 - 3x_3x_4$. On the affine chart of $\mathbb{P}^3$ with $x_4 = 1$, this is the twisted cubic $(x_1 \mapsto (x_1, x_1^2, \frac{1}{3}x_1^3))$, which is smooth. On the affine chart of $\mathbb{P}^3$ with $x_3 = 1$, this is the smooth curve $(x_2 \mapsto (\frac{1}{3}x_2^2, x_2, \frac{1}{9}x_2^3))$ which is also a twisted cubic. The curve has no points with $x_3 = x_4 = 0$. Therefore it is smooth. □

Remark 6.6 The second author is grateful to Paul Seidel who showed him in 2000 this curve in $\mathbb{P}^3$ and explained that it is not a complete intersection, that it is a Legendre curve with respect to the 1-form $x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1 - \mathrm{d}x_3$ (in the affine chart with $x_4 = 1$) and that the cone over it in $\mathbb{C}^4$ is smooth outside 0 and is Lagrange.

Examples 6.7 Kawakami, Mano and Sekiguchi [18, section 6] found many flat F-manifold structures on $M = \mathbb{C}^3$ which are generically semisimple and which have Euler fields with positive weights. Because the Euler fields have positive weights, they are simple F-manifolds. They satisfy either $T_0M \cong Q^{(2)}$ or $T_0M \cong Q^{(1)}$.

They are related to algebraic solutions of the Painlevé VI equations. Some of them are also given by Arsie and Lorenzoni [3, 5.2–5.4]. They are polynomial on $M = \mathbb{C}^3$ with flat coordinates t_1, t_2, t_3 with unit field $e = \partial_1$ and Euler field

$$\begin{aligned} E &= t_1\partial_1 + w_2t_2\partial_2 + w_3t_3\partial_3 \\ \text{with} \quad & w_2, w_3 \in \mathbb{Q},\ 1 > w_2 \geq w_3 > 0,\ 1 + w_3 \geq 2w_2. \end{aligned} \tag{6.20}$$

If we write $\mathbf{w} = (1, w_2, w_3)$ and

$$\partial_i \circ \partial_j = \sum_{k=1}^{3} a_{ij}^k \partial_k, \tag{6.21}$$

then $a_{ij}^k \in \mathbb{C}[t_1, t_2, t_3]$ with $a_{1j}^k = a_{j1}^k = \delta_{jk}$, and a coefficient a_{ij}^k with $i, j \in \{2, 3\}$ has the weighted degree $\deg_{\mathbf{w}} a_{ij}^k = 1 + w_k - w_i - w_j \geq 0$. In the cases with $1 + w_3 > 2w_2$, this implies $a_{ij}^k(0) = 0$ for $i, j \in \{2, 3\}$ and $k \in \{1, 2, 3\}$. Therefore, then $T_0M \cong Q^{(1)}$. These cases comprise the cases in 6.2 and 6.3 in [18] and 5.2 and 5.4 in [3], which are related to the complex reflection groups G_{24} and G_{27}, and the cases in 6.4–6.6 in [18], which are related to the free divisors F_{B_6}, F_{H_2} and $F_{E_{14}}$ in $\mathbb{C}^3$ which are defined there. The authors are grateful to a referee for pointing to these examples.

7 Partial classification of 3-dimensional generically semisimple *F*-manifolds

The long Theorem 7.1 is the main result of this section. It gives normal forms for all germs of generically semisimple F-manifolds with $T_0M \cong Q^{(2)}$, except A_3, B_3 and H_3. Part (a) of it is essentially Theorem 5.30 in [15], but with some change in the normal form. The parts (b)–(e) are new. Corollary 7.2 distinguishes those germs of F-manifolds in Theorem 7.1 which have an Euler field. The germs of F-manifolds in Theorem 7.1 are closely related to the germs of plane curves with multiplicity 3. The Remarks 7.3 comment on this and make the cases in Theorem 7.1 more transparent.

Theorem 7.1 *In the following, normal forms for all irreducible germs* $(M,0)$*of 3-dimensional generically semisimple* F*-manifolds with* $T_0M \cong Q^{(2)}$ *except* A_3, B_3 *and* H_3 *are listed by their data in the Remarks 6.1. Each isomorphism class of such a germ is represented by a finite positive number of normal forms. The normal forms split into 5 families with discrete and holomorphic parameters, with*

family in	(a)	(b)	(c)	(d)	(e)
number of components of (L_M, λ)	3	2	2	1	1
discrete parameters	p, q	p	p, q	p	p

with $p, q \in \mathbb{Z}_{\geq 2}$ *and* $q \geq p$. *There are always* $p-1$ *holomorphic parameters* $(\gamma_0, ..., \gamma_{p-2}) \in \mathbb{C}^{p-1}$ *or in an open subset. We use the notations in the Remarks 6.1, especially* $(M,0) = (\mathbb{C}^3, 0)$ *with coordinates* $t = (t_1, t_2, t_3)$. *In all cases the caustic is* $\mathcal{K} = \{t \in M \mid t_2 = 0\}$. *It coincides with the* μ*-constant stratum. For* $t \in \mathcal{K}$ $T_tM \cong Q^{(2)}$. *Locally on* $M - \mathcal{K}$, *the analytic spectrum is*

$$L_M = \bigcup_{j=1}^{3} \{(y,t) \in T^*M \mid y_1 = 1,\ y_2 = \partial_2 f^{(j)},\ y_3 = h_2 y_2^2 + h_1 y_2 + h_0\} \quad (7.1)$$

with $h_2, h_1, h_0 \in \mathbb{C}\{t_2, t_3\}$ *as below, with* $h_2(0) \neq 0$, $h_1(0) = h_0(0) = 0$. *The Euler field on* $M - \mathcal{K}$ *is* $E = (t_1 + c_1)\partial_1 + \varepsilon_2\partial_2 + \varepsilon_3\partial_3$ *with* $c_1 \in \mathbb{C}$ *and* $\varepsilon_2, \varepsilon_3$ *as below. Most often,* ε_3 *and* E *are meromorphic along* $\mathcal{K}$ *(see Corollary 7.2 for the cases when they are holomorphic on* M*). The function*

$$\rho := t_2^{p-2} t_3 + \sum_{i=0}^{p-2} \gamma_i t_2^i \in \mathbb{C}\{t_2, t_3\} \quad (7.2)$$

will always turn up in some $f^{(j)}$.

(a) $(\gamma_0, ..., \gamma_{p-2}) \in \mathbb{C}^* \times \mathbb{C}^{p-2}$ *with* $\gamma_0 \neq 1$ *if* $p = q$,
then $\rho \in \mathbb{C}\{t_2, t_3\}^*$, *i.e., it is a unit in* $\mathbb{C}\{t_2, t_3\}$, *because* $\gamma_0 \neq 0$,
$f^{(1)} = 0,\ f^{(2)} = t_2^p,\ f^{(3)} = t_2^q \cdot \rho$,

$L_M=\bigcup_{j=1}^3 L^{(j)}$ *has 3 smooth components,*
$h_2=\Big((q+t_2\partial_2)(\rho)\Big)^{-1}\Big(((q+t_2\partial_2)(\rho))t_2^{q-p}-p\Big)^{-1}\in\mathbb{C}\{t_2,t_3\}^*,$
$h_2^{-1}h_1=-pt_2^{p-1}$, $h_0=0$,
Euler field: $\varepsilon_2=\frac{1}{p}t_2$, $\varepsilon_3=-\frac{1}{p}t_2^{2-p}((q-p+t_2\partial_2)(\rho))$.

(b) $(\gamma_0,...,\gamma_{p-2})\in\mathbb{C}^{p-1}$, $\rho\in\mathbb{C}\{t_2,t_3\}$,
$f^{(1)}=0$, $f^{(2\&3)}=t_2^{\frac{1}{2}+p}+t_2^{1+p}\cdot\rho$,
$L_M=L^{(1)}\cup L^{(2\&3)}$ *has 1 smooth component* $L^{(1)}$ *and 1 singular component* $L^{(2\&3)}$,
$h_2=\Big((\frac{1}{2}+p)^2-t_2((1+p+t_2\partial_2)(\rho))^2\Big)^{-1}\in\mathbb{C}\{t_2,t_3\}^*,$
$h_2^{-1}h_1=-2t_2^p((1+p+t_2\partial_2)(\rho))$, $h_0=0$,
Euler field: $\varepsilon_2=\frac{1}{\frac{1}{2}+p}t_2$, $\varepsilon_3=-\frac{1}{\frac{1}{2}+p}t_2^{2-p}((\frac{1}{2}+t_2\partial_2)(\rho))$.

(c) $(\gamma_0,...,\gamma_{p-2})\in\mathbb{C}^*\times\mathbb{C}^{p-2}$, *and thus* $\rho\in\mathbb{C}\{t_2,t_3\}^*$,
$f^{(1)}=0$, $f^{(2\&3)}=t_2^{\frac{1}{2}+q}\cdot\rho+t_2^p$,
$L_M=L^{(1)}\cup L^{(2\&3)}$ *has 1 smooth component* $L^{(1)}$ *and 1 singular component* $L^{(2\&3)}$,
$h_2=\Big((\frac{1}{2}+q+t_2\partial_2)(\rho)\Big)^{-1}\Big(p-\frac{1}{p}t_2^{1+2(q-p)}((\frac{1}{2}+q+t_2\partial_2)(\rho))^2\Big)^{-1}\in\mathbb{C}\{t_2,t_3\}^*,$
$h_2^{-1}h_1=-pt_2^{p-1}-\frac{1}{p}t_2^{2q-p}((\frac{1}{2}+q+t_2\partial_2)(\rho))^2$, $h_0=0$,
Euler field: $\varepsilon_2=\frac{1}{p}t_2$, $\varepsilon_3=-\frac{1}{p}t_2^{2-p}((\frac{1}{2}+q-p+t_2\partial_2)(\rho))$.

(d) $(\gamma_0,...,\gamma_{p-2})\in\mathbb{C}^{p-1}$, $\rho\in\mathbb{C}\{t_2,t_3\}$,
$f=f^{(1\&2\&3)}=t_2^{\frac{1}{3}+p}+t_2^{\frac{2}{3}+p}\cdot\rho$,
L_M *is irreducible,*
$h_2=\Big((\frac{1}{3}+p)^2-t_2(\frac{1}{3}+p)^{-1}((\frac{2}{3}+p+t_2\partial_2)(\rho))^3\Big)^{-1}\in\mathbb{C}\{t_2,t_3\}^*,$
$h_2^{-1}h_1=-t_2^p(\frac{1}{3}+p)^{-1}((\frac{2}{3}+p+t_2\partial_2)(\rho))^2$,
$h_2^{-1}h_0=-2t_2^{2p-1}(\frac{1}{3}+p)((\frac{2}{3}+p+t_2\partial_2)(\rho))$,
Euler field: $\varepsilon_2=\frac{1}{\frac{1}{3}+p}t_2$, $\varepsilon_3=-\frac{1}{\frac{1}{3}+p}t_2^{2-p}((\frac{1}{3}+t_2\partial_2)(\rho))$.

(e) $(\gamma_0,...,\gamma_{p-2})\in\mathbb{C}^{p-1}$, $\rho\in\mathbb{C}\{t_2,t_3\}$,
$f=f^{(1\&2\&3)}=t_2^{\frac{4}{3}+p}\cdot\rho+t_2^{\frac{2}{3}+p}$,
L_M *is irreducible,*
$h_2=\Big((\frac{2}{3}+p)^2-t_2^2(\frac{2}{3}+p)^{-1}((\frac{4}{3}+p+t_2\partial_2)(\rho))^3\Big)^{-1}\in\mathbb{C}\{t_2,t_3\}^*,$
$h_2^{-1}h_1=-t_2^{p+1}(\frac{2}{3}+p)^{-1}((\frac{4}{3}+p+t_2\partial_2)(\rho))^2$,
$h_2^{-1}h_0=-2t_2^{2p}(\frac{2}{3}+p)((\frac{4}{3}+p+t_2\partial_2)(\rho))$,
Euler field: $\varepsilon_2=\frac{1}{\frac{2}{3}+p}t_2$, $\varepsilon_3=-\frac{1}{\frac{2}{3}+p}t_2^{2-p}((\frac{2}{3}+t_2\partial_2)(\rho))$.

Proof The proofs of the parts (a)–(e) are similar. Part (a) is essentially proved in [15], though here we chose a different normal form than in [15]. The first steps in the following proof hold for (a)–(e). We give all the details for part (b) and part (d). We discuss differences and similarities for the parts (c), (e) and (a).

We consider an irreducible germ $(M,0)$ of a 3-dimensional generically semisimple F-manifold with $T_0M\cong Q^{(2)}$, which is not A_3, B_3 or H_3. Theorem 6.3 says that then $(L_M,\lambda)\cong(\mathbb{C}^2,0)\times(C,0)$ where $(C,0)\subset(\mathbb{C}^2,0)$ is the germ of a plane curve with multiplicity 3. And the caustic $\mathcal{K}$ is isomorphic to the image in M of the part of L_M which is isomorphic to $(\mathbb{C}^2,0)\times\{0\}$, and $\mathcal{K}$ is a smooth surface in M.

The coordinates $t=(t_1,t_2,t_3)$ can and will be chosen such that $\mathcal{K}=\{t\in M\mid t_2=0\}$.

$(C,0)$ has multiplicity 3, and therefore, it has either 3 smooth components or 1 smooth and 1 singular component or only 1 singular component. The corresponding components of L_M are called $L^{(j)}$, $j\in\{1,2,3\}$, in the first case, $L^{(1)}$ and $L^{(2\&3)}$ in the second case and $L_M=L^{(1\&2\&3)}$ in the third case. The parts of the multivalued function f which correspond to these components are called accordingly $f^{(j)}$, $f^{(2\&3)}$ or $f^{(1\&2\&3)}$.

Recall $F=t_1+f$ from the Notations 6.1. The coordinate t_1 can and will be chosen (by a coordinate change as in (4.9)) such that

$$f^{(1)}=0\quad\text{in the cases with 3 or 2 components,}\tag{7.3}$$
$$f^{(1)}+f^{(2)}+f^{(3)}=0\quad\text{in the cases with 1 component.}\tag{7.4}$$

In fact, in all cases $f^{(1)}+f^{(2)}+f^{(3)}$ is univalued, and t_1 can be chosen such that (7.4) holds. Then, also $\partial_2(f^{(1)}+f^{(2)}+f^{(3)})=0$ and $\partial_3(f^{(1)}+f^{(2)}+f^{(3)})=0$. We see that this choice was already discussed in Remark 4.5. In the generically semisimple case, the function F gives an alternative starting point for understanding this choice of the coordinate t_1.

In the cases with 1 component, we use this choice in (7.4), and there it gives

$$\prod_{j=1}^{3}(x-\partial_2 f^{(j)})=x^3+g_1x+g_0,\qquad\text{so } g_2=0.\tag{7.5}$$

In the cases with 3 or 2 components, we prefer the choice in (7.3), as it makes there the calculations easier. Then in the cases with 3 or 2 components, (6.6) for $j=1$ gives $h_0=0$ and

$$\prod_{j=1}^{3}(x-\partial_2 f^{(j)})=x^3+g_2x^2+g_1x,\quad\text{with}$$
$$g_2=-\partial_2 f^{(2)}-\partial_2 f^{(3)},\ g_1=\partial_2 f^{(2)}\cdot\partial_2 f^{(3)}.\tag{7.6}$$

(b) and (c) Now we turn to the cases where L_M has 2 components, the smooth component $L^{(1)}$ and the singular component $L^{(2\&3)}$. We have $f^{(1)}=0$ and

$$f^{(2\&3)}=t_2^{1/2+p_1}\rho_1+t_2^{p_2}\rho_2,\tag{7.7}$$

with $\rho_1 \in \mathbb{C}\{t_2, t_3\} - t_2\mathbb{C}\{t_2, t_3\}$ and $\rho_2 \in \mathbb{C}\{t_2, t_3\} - (t_2\mathbb{C}\{t_2, t_3\} - \{0\})$. Here ρ_1, ρ_2 and $p_1 \in \mathbb{Z}_{\geq 0}$ are unique, and $p_2 \in \mathbb{Z}_{\geq 0}$ is unique if $\rho_2 \neq 0$. If $\rho_2 = 0$, we put $p_2 := \infty$.

The branched covering $\pi_L : L_M \to M$ is branched only over $\mathcal{K} = \{t \in M \mid t_2 = 0\}$. This implies two facts: First, $L^{(1)}$ and $L^{(2\&3)}$ intersect only over $\mathcal{K}$, and second, the branched covering $\pi_L : L^{(2\&3)} \to M$ is branched only over $\mathcal{K}$. The second fact tells $\rho_1 \in \mathbb{C}\{t_2, t_3\}^*$, i.e., ρ_1 is a unit, i.e., $\rho_1(0) \neq 0$. If $p_1 < p_2$, this is sufficient also for the first fact. Then, we are in the cases in (b). If $p_1 \geq p_2$, the first fact tells $\rho_2 \in \mathbb{C}\{t_2, t_3\}^*$. Then, we are in the cases in (c).

(b) Now we turn to the cases in (b), i.e., $f^{(1)} = 0$ and $f^{(2\&3)}$ as in (7.7) with $p_1 < p_2$. Rename $p := p_1$. Then, t_2 can and will be chosen such that $t_2^{\frac{1}{2}+p}\rho_1 = t_2^{\frac{1}{2}+p}$. Then, we write

$$f^{(2\&3)} = t_2^{\frac{1}{2}+p} + t_2^{1+p}\rho \tag{7.8}$$

for some $\rho \in \mathbb{C}\{t_2, t_3\}$. Next we will exploit (6.6) together with $h_2(0) \neq 0$ in order to put ρ into a normal form by a good choice of t_3, and to calculate h_2 and h_1 (recall $h_0 = 0$ because of (6.6) for $f^{(1)} = 0$). (6.6) gives

$$\begin{aligned} t_2^{1+p}\partial_3\rho &= \partial_3 f^{(2\&3)} = h_2 \cdot \partial_2 f^{(2\&3)} \cdot \Big(\partial_2 f^{(2\&3)} + h_2^{-1}h_1\Big) && (7.9)\\ &= h_2\Big(\big[(\partial_2 t_2^{\frac{1}{2}+p})^2 + \partial_2(t_2^{1+p}\rho)(\partial_2(t_2^{1+p}\rho) + h_2^{-1}h_1)\big] \\ &\quad + \big[\partial_2 t_2^{\frac{1}{2}+p}(2\partial_2(t_2^{1+p}\rho) + h_2^{-1}h_1)\big]\Big). && (7.10) \end{aligned}$$

The term in square brackets in the line (7.10) must vanish because of the half-integer exponent of t_2. This allows to calculate $h_2^{-1}h_1 = -2\partial_2(t_2^{1+p}\rho)$, see the formula in part (b) in the theorem. And it simplifies the other summand,

$$\begin{aligned} \partial_3\rho &= h_2 t_2^{-1-p}\Big((\partial_2 t_2^{\frac{1}{2}+p})^2 - (\partial_2(t_2^{1+p}\rho))^2\Big) \\ &= h_2 t_2^{p-2}\Big((\frac{1}{2}+p)^2 - t_2((1+p+t_2\partial_2)(\rho))^2\Big), \end{aligned} \tag{7.11}$$

so $\partial_3\rho = t_2^{p-2} \cdot$ (a unit in $\mathbb{C}\{t_2, t_3\}$). This implies $p \in \mathbb{Z}_{\geq 2}$. And we can and will choose t_3 such that ρ is as in (7.2). Then, h_2 is determined by (7.11) with $\partial_3\rho = t_2^{p-2}$.

The coefficients ε_2 and ε_3 of the Euler field are determined by (6.3) for $f^{(2\&3)}$ as in (7.8) and (7.2):

$$\begin{aligned} t_2^{\frac{1}{2}+p} + t_2^{1+p}\rho &= f^{(2\&3)} = \varepsilon_2\partial_2 f^{(2\&3)} + \varepsilon_3\partial_3 f^{(2\&3)} \\ &= \varepsilon_2\Big((\frac{1}{2}+p)t_2^{-\frac{1}{2}+p} + ((1+p+t_2\partial_2)(\rho))t_2^p\Big) + \varepsilon_3 t_2^{2p-1}. \end{aligned} \tag{7.12}$$

Comparison of the terms with half-integer exponents gives $\varepsilon_2 = (\frac{1}{2} + p)^{-1} t_2$. Then, comparison of the terms with integer exponents gives ε_3.

Almost all steps in this reduction process to a normal form were unique. (7.7) was the general ansatz. The choice of t_2 with $t_2^{\frac{1}{2}+p}\rho_1 = t_2^{\frac{1}{2}+p}$ was unique up to a unit root of order $1 + 2p$. The choice of t_3 was unique. Therefore, the isomorphism class of $(M, 0)$ is represented by up to $1 + 2p$ normal forms.

(c) Now we turn to the cases in (c), i.e., $f^{(1)} = 0$ and $f^{(2\&3)}$ as in (7.7) with $p_1 \geq p_2$. Rename $q := p_1$ and $p := p_2$. Above we showed that ρ_1 and ρ_2 are units in $\mathbb{C}\{t_2, t_3\}$. We can and will choose t_2 such that $t_2^p \rho_2 = t_2^p$. Then, we write

$$f^{(2\&3)} = t_2^{\frac{1}{2}+q}\rho + t_2^p \tag{7.13}$$

for some $\rho \in \mathbb{C}\{t_2, t_3\}^*$. As in the proof of part (b), the next step is to exploit (6.6) together with $h_2(0) \neq 0$ in order to put ρ into a normal form by a good choice of t_3, and to calculate h_2 and h_1. The calculation is similar to the calculation of (7.9) above. It leads to $\partial_3 \rho = t_2^{p-2} \cdot$ (a unit in $\mathbb{C}\{t_2, t_3\}$). This implies $p \in \mathbb{Z}_{\geq 2}$. And it allows to choose t_3 such that ρ is as in (7.2). We skip the details of the calculations. The results are written in part (c) in the theorem. The fact that here ρ is a unit, implies $\gamma_0 \in \mathbb{C}^*$. Also the calculation of the coefficients ε_2 and ε_3 of the Euler field is similar to the calculation (7.12) above. Again we skip the details. The results are written in part (c) in the theorem. The choice of t_2 was unique up to a unit root of order p. The choice of t_3 was unique. Therefore, the isomorphism class of $(M, 0)$ is represented by up to p normal forms.

(d) and (e) Now we turn to the cases where L_M is irreducible. A priori we have

$$f = t_2^{\frac{1}{3}+p_1}\rho_1 + t_2^{\frac{2}{3}+p_2}\rho_2 + t_2^{1+p_3}\rho_3, \tag{7.14}$$

with $\rho_1, \rho_2, \rho_3 \in \mathbb{C}\{t_2, t_3\} - (t_2\mathbb{C}\{t_2, t_3\} - \{0\})$ and $(\rho_1, \rho_2) \neq (0, 0)$. Now $\sum_{j=1}^3 f^{(j)} = 0$ tells $\rho_3 = 0$. If $\rho_1 \neq 0$ then ρ_1 and $p_1 \in \mathbb{Z}_{\geq 0}$ are unique, else $p_1 := \infty$. If $\rho_2 \neq 0$ then ρ_2 and $p_2 \in \mathbb{Z}_{\geq 0}$ are unique, else $p_2 := \infty$.

The branched covering $\pi_L : L_M \to M$ is branched only over $\mathcal{K} = \{t \in M \mid t_2 = 0\}$. If $p_1 \leq p_2$, this implies $\rho_1 \in \mathbb{C}\{t_2, t_3\}^*$, and then we are in the cases in (d). If $p_1 > p_2$, this implies $\rho_2 \in \mathbb{C}\{t_2, t_3\}^*$, and then we are in the cases in (e).

(d) Now we turn to the cases in (d), i.e., f is as in (7.14) with $\rho_3 = 0$ and $p_1 \leq p_2$ and $\rho_1 \in \mathbb{C}\{t_2, t_3\}^*$. Rename $p := p_1$. Then, t_2 can and will be chosen such that $t_2^{\frac{1}{3}+p}\rho_1 = t_2^{\frac{1}{3}+p}$. Then, we write

$$f = t_2^{\frac{1}{3}+p} + t_2^{\frac{2}{3}+p}\rho, \tag{7.15}$$

for some $\rho \in \mathbb{C}\{t_2, t_3\}$. As in the proofs of the parts (b) and (c), the next step is to exploit (6.6) together with $h_2(0) \neq 0$ in order to put ρ into a normal form by a good choice of t_3, and to calculate h_2, h_1 and h_0. The calculation is as follows.

$$\begin{aligned} t_2^{\frac{2}{3}+p}\partial_3\rho &= \partial_3 f = h_2\cdot\Bigl((\partial_2 f)^2 + h_2^{-1}h_1\partial_2 f + h_2^{-1}h_0\Bigr) && (7.16)\\ &= h_2\Bigl(\bigl[(\partial_2 t_2^{\frac{1}{3}+p})^2 + h_2^{-1}h_1\partial_2(t_2^{\frac{2}{3}+p}\rho)\bigr] && (7.17)\\ &\quad + \bigl[(\partial_2(t_2^{\frac{2}{3}+p}\rho))^2 + h_2^{-1}h_1\partial_2 t_2^{\frac{1}{3}+p}\bigr] && (7.18)\\ &\quad + \bigl[2\partial_2 t_2^{\frac{1}{3}+p}\partial_2(t_2^{\frac{2}{3}+p}\rho) + h_2^{-1}h_0\bigr]\Bigr) && (7.19)\end{aligned}$$

The terms in square brackets in the lines (7.18) and (7.19) must vanish because of the exponents in $\frac{1}{3}+\mathbb{Z}$ and $\mathbb{Z}$ of t_2. This allows to calculate $h_2^{-1}h_0$ and $h_2^{-1}h_1 = -(\partial_2 t_2^{\frac{1}{3}+p})^{-1}(\partial_2(t_2^{\frac{2}{3}+p}\rho))^2$, see the formulas in part (d) in the theorem. And it simplifies the term in square brackets in the line (7.17),

$$\begin{aligned} \partial_3\rho &= h_2 t_2^{-\frac{2}{3}-p}\Bigl((\partial_2 t_2^{\frac{1}{3}+p})^2 - (\partial_2 t_2^{\frac{1}{3}+p})^{-1}(\partial_2(t_2^{\frac{2}{3}+p}\rho))^3\Bigr)\\ &= h_2 t_2^{p-2}\Bigl((\frac{1}{3}+p)^2 - t_2(\frac{1}{3}+p)^{-1}((\frac{2}{3}+p+t_2\partial_2)(\rho))^3\Bigr), \qquad (7.20)\end{aligned}$$

so $\partial_3\rho = t_2^{p-2}\cdot$ (a unit in $\mathbb{C}\{t_2,t_3\}$). This implies $p\in\mathbb{Z}_{\geq 2}$. And we can and will choose t_3 such that ρ is as in (7.2). Then, h_2 is determined by (7.20) with $\partial_3\rho = t_2^{p-2}$.

The coefficients ε_2 and ε_3 of the Euler field are determined by (6.3) for f as in (7.15) and (7.2):

$$\begin{aligned} &t_2^{\frac{1}{3}+p} + t_2^{\frac{2}{3}+p}\rho = f = \varepsilon_2\partial_2 f + \varepsilon_3\partial_3 f\\ &= \varepsilon_2\Bigl((\frac{1}{3}+p)t_2^{-\frac{2}{3}+p} + ((\frac{2}{3}+p+t_2\partial_2)(\rho))t_2^{-\frac{1}{3}+p}\Bigr) + \varepsilon_3 t_2^{-\frac{4}{3}+2p}. \qquad (7.21)\end{aligned}$$

Comparison of the terms with exponents in $\frac{1}{3}+\mathbb{Z}$ gives $\varepsilon_2 = (\frac{1}{3}+p)^{-1}t_2$. Then, comparison of the terms with exponents in $\frac{2}{3}+\mathbb{Z}$ gives ε_3. The choice of t_2 with $t_2^{\frac{1}{3}+p}\rho_1 = t_2^{\frac{1}{3}+p}$ was unique up to a unit root of order $1+3p$. The choice of t_3 was unique. Therefore, the isomorphism class of $(M,0)$ is represented by up to $1+3p$ normal forms.

(e) Now we turn to the cases in (e), i.e., f as in (7.15) with $p_1 > p_2$ and $\rho_2\in\mathbb{C}\{t_2,t_3\}^*$. Rename $p := p_2$. Then, t_2 can and will be chosen such that $t_2^{\frac{2}{3}+p}\rho_2 = t_2^{\frac{2}{3}+p}$. Then, we write

$$f = t_2^{\frac{4}{3}+p}\rho + t_2^{\frac{2}{3}+p}, \qquad (7.22)$$

for some $\rho\in\mathbb{C}\{t_2,t_3\}$. As in the proofs of the parts (b), (c) and (d), the next step is to exploit (6.6) together with $h_2(0)\neq 0$ in order to put ρ into a normal form by a good choice of t_3, and to calculate h_2, h_1 and h_0. The calculation is similar to the calculation of (7.16) above. It leads to $\partial_3\rho = t_2^{p-2}\cdot$ (a unit in $\mathbb{C}\{t_2,t_3\}$). This implies $p\in\mathbb{Z}_{\geq 2}$. Furthermore, it allows to choose t_3 such that ρ is as in (7.2). We skip the details of the calculations. The results are written in part (e) in the theorem. Also the calculation of the coefficients ε_2 and ε_3 of the Euler field is similar to the calculation (7.21) above. Again we skip the details. The results are written in part (e) in the theorem. The choice of t_2 was unique up to a unit root of order $2+3p$. The choice of t_3 was unique. Therefore, the isomorphism class of $(M,0)$ is represented by up to $2+3p$ normal forms.

(a) Now we turn to the cases in (a), the cases where L has 3 components. They were treated in Theorem 5.30 in [15]. However, here we choose the normal forms a bit differently.

The plane curve germs

$$(C^{(j)},0) := \{(y_2,t_2)\in(\mathbb{C}^2,0)\,|\,y_2 = \partial_2 f^{(j)}(t_2,0)\} \tag{7.23}$$

in the (y_2,t_2)-plane satisfy $(L,\lambda)\cong(\mathbb{C}^2,0)\times\bigcup_{j=1}^3(C^{(j)},0)$ by the proof of Theorem 6.3 (a). We choose their numbering such that the pair $(C^{(1)},C^{(3)})$ has the highest intersection number, which we call $q-1$ for some $q\in\mathbb{Z}_{\geq 2}$. Then, the pairs $(C^{(1)},C^{(2)})$ and $(C^{(2)},C^{(3)})$ have the same intersection number $p-1$ for some $p\in\mathbb{Z}_{\geq 2}$ with $p\leq q$.

We have $f^{(1)}=0$ by (7.3) and

$$f^{(2)} = t_2^{p_1}\rho_1,\quad f^{(3)} = t_2^{p_2}\rho_2, \tag{7.24}$$

with $\rho_1,\rho_2\in\mathbb{C}\{t_2,t_3\}-t_2\mathbb{C}\{t_2,t_3\}$ and $p_1,p_2\in\mathbb{N}$.

The branched covering $\pi_L : L_M\to M$ is branched only over $\mathcal{K}=\{t\in M\,|\,t_2=0\}$, so the components $L^{(i)}$ and $L^{(j)}$ of L_M intersect only over $\mathcal{K}$. This and $(L,\lambda)\cong(\mathbb{C}^2,0)\times\bigcup_{j=1}^3(C^{(j)},0)$ shows $\rho_1,\rho_2\in\mathbb{C}\{t_2,t_3\}^*$, $p_1=p$, $p_2=q$, and in the case $p=q$ additionally $\rho_1(0)\neq\rho_2(0)$.

t_2 can and will be chosen such that $t_2^p\rho_1 = t_2^p$. Then, we have

$$f^{(1)}=0,\quad f^{(2)}=t_2^p,\quad f^{(3)}=t_2^q\cdot\rho, \tag{7.25}$$

for some $\rho\in\mathbb{C}\{t_2,t_3\}^*$ with $\rho(0)\neq 1$ if $p=q$. As in the proofs of the parts (b)–(e), the next step is to exploit (6.6) together with $h_2(0)\neq 0$ in order to put ρ into a normal form by a good choice of t_3, and to calculate h_2 and h_1. The calculation is similar to the calculations in (b)–(e), and, in fact, easier. It allows to choose t_3 such that ρ is as in (7.2). Then, h_2 and h_1 are as in the theorem. Also the calculation of the Euler field is similar to the calculations in (b)–(e). The numbering of $L^{(1)}$, $L^{(2)}$ and $L^{(3)}$ was unique up to a permutation of $L^{(1)}$ and $L^{(3)}$ if $p<q$ and arbitrary if $p=q$. The choice of t_2 such that $t_2^p\rho_1 = t_2^p$ was unique up to a unit root of order p. The choice of t_3 was unique. Therefore, the isomorphism class of $(M,0)$ is represented by up to $2p$ or $6p$ normal forms. □

Probably the most interesting of the F-manifolds in Theorem 7.1 are those where the Euler field is holomorphic on M. The next corollary makes them explicit.

Corollary 7.2 *Each irreducible germ $(\widetilde{M},0)$ of a 3-dimensional generically semisimple F-manifold with $T_0\widetilde{M}\cong Q^{(2)}$ and with (holomorphic) Euler field is isomorphic to one of the germs A_3, B_3 or H_3 or to a germ $(M,(t_1,0,t_3))$ for suitable $t_1\in\mathbb{C}$ and $t_3\in\mathbb{C}$ (or $\mathbb{C}^*$ or $\mathbb{C}-\{0;1\}$) of one of the F-manifolds which are listed below. We use the same notations as in Theorem 7.1. There are 7 families of F-manifolds. The family in (a)(i) has no parameter, so there is a single F-manifold. The family in (a)(iii) has one discrete and one holomorphic parameter γ_0. The other families have one discrete parameter and no holomorphic parameter. The Euler field is $E=t_1\partial_1+\varepsilon_2\partial_2+\varepsilon_3\partial_3$ with $\varepsilon_2,\varepsilon_3$ as below.*

family in	(a)(i)	(a)(ii)	(a)(iii)	(b)	(c)	(d)	(e)
number of comp. of (L_M,λ)	3	3	3	2	2	1	1
discrete parameter	–	q	p	p	q	p	p

(a) (i) $M=\mathbb{C}^2\times(\mathbb{C}-\{0;1\})$, $(p=q=2,)$
$f^{(1)}=0,\ f^{(2)}=t_2^2,\ f^{(3)}=t_2^2t_3,$
$h_2=(4t_3(t_3-1))^{-1},\ h_2^{-1}h_1=-2t_2,\ h_0=0,$
Euler field: $\varepsilon_2=\frac{1}{2}t_2,\ \varepsilon_3=0.$

(a) (ii) $M=\mathbb{C}^2\times\mathbb{C}^*$, $(p=2,)\ q\in\mathbb{Z}_{\geq 3}$,
$f^{(1)}=0,\ f^{(2)}=t_2^2,\ f^{(3)}=t_2^qt_3,$
$h_2=(qt_3)^{-1}(qt_3t_2^{q-2}-2)^{-1},\ h_2^{-1}h_1=-2t_2,\ h_0=0,$
Euler field: $\varepsilon_2=\frac{1}{2}t_2,\ \varepsilon_3=-\frac{q-2}{2}t_3.$

(a) (iii) $M=\mathbb{C}^3$, $(p=q,)\ p\in\mathbb{Z}_{\geq 3},\ \gamma_0\in\mathbb{C}-\{0;1\}$,
$f^{(1)}=0,\ f^{(2)}=t_2^p,\ f^{(3)}=t_2^p(\gamma_0+t_2^{p-2}t_3),$
$h_2=(p\gamma_0+(p-2)t_2^{p-2}t_3)^{-1}(p(\gamma_0-1)+(p-2)t_2^{p-2}t_3)^{-1},$
$h_2^{-1}h_1=-pt_2^{p-1},\ h_0=0,$
Euler field: $\varepsilon_2=\frac{1}{p}t_2,\ \varepsilon_3=-\frac{p-2}{p}t_3.$

(b) $M=\mathbb{C}^3,\ p\in\mathbb{Z}_{\geq 2}$,
$f^{(1)}=0,\ f^{(2\&3)}=t_2^{\frac{1}{2}+p}+t_2^{2p-1}t_3,$
$h_2=((\frac{1}{2}+p)^2-(2p-1)^2t_2^{2p-3}t_3^2)^{-1},$
$h_2^{-1}h_1=-2(2p-1)t_2^{2p-2}t_3,\ h_0=0,$
Euler field: $\varepsilon_2=\frac{1}{\frac{1}{2}+p}t_2,\ \varepsilon_3=-\frac{p-\frac{3}{2}}{\frac{1}{2}+p}t_3.$

(c) $M=\mathbb{C}^2\times\mathbb{C}^*$, $(p=2,)\ q\in\mathbb{Z}_{\geq 2}$,
$f^{(1)}=0,\ f^{(2\&3)}=t_2^{\frac{1}{2}+q}t_3+t_2^2,$
$h_2=((\frac{1}{2}+q)t_3)^{-1}\Big(2-\frac{1}{2}(\frac{1}{2}+q)^2t_2^{2q-3}t_3^2\Big)^{-1},$
$h_2^{-1}h_1=-2t_2-\frac{1}{2}(\frac{1}{2}+q)^2t_2^{2q-2}t_3^2,\ h_0=0,$
Euler field: $\varepsilon_2=\frac{1}{2}t_2,\ \varepsilon_3=-\frac{1}{2}(q-\frac{3}{2})t_3.$

(d) $M = \mathbb{C}^3$, $p \in \mathbb{Z}_{\geq 2}$,
$f = f^{(1\&2\&3)} = t_2^{\frac{1}{3}+p} + t_2^{2p-\frac{4}{3}} t_3$,
$h_2 = \Big((\frac{1}{3}+p)^2 - (\frac{1}{3}+p)^{-1}((2p-\frac{4}{3})^3 t_2^{3p-5} t_3^3 \Big)^{-1}$,
$h_2^{-1} h_1 = -(\frac{1}{3}+p)^{-1}(2p-\frac{4}{3})^2 t_2^{3p-4} t_3^2$,
$h_2^{-1} h_0 = -2(\frac{1}{3}+p)(2p-\frac{4}{3}) t_2^{3p-3} t_3$,
Euler field: $\varepsilon_2 = \frac{1}{\frac{1}{3}+p} t_2$, $\varepsilon_3 = -\frac{p-\frac{5}{3}}{\frac{1}{3}+p} t_3$.

(e) $M = \mathbb{C}^3$, $p \in \mathbb{Z}_{\geq 2}$,
$f = f^{(1\&2\&3)} = t_2^{2p-\frac{2}{3}} t_3 + t_2^{\frac{2}{3}+p}$,
$h_2 = \Big((\frac{2}{3}+p)^2 - (\frac{2}{3}+p)^{-1}(2p-\frac{2}{3})^3 t_2^{3p-4} t_3^3 \Big)^{-1}$,
$h_2^{-1} h_1 = -(\frac{2}{3}+p)^{-1}(2p-\frac{2}{3})^2 t_2^{3p-3} t_3^2$,
$h_2^{-1} h_0 = -2(\frac{2}{3}+p)(2p-\frac{2}{3}) t_2^{3p-2} t_3$,
Euler field: $\varepsilon_2 = \frac{1}{\frac{2}{3}+p} t_2$, $\varepsilon_3 = -\frac{p-\frac{4}{3}}{\frac{2}{3}+p} t_3$.

Proof The shape of the Euler field in Theorem 6.3 tells precisely under which conditions it is holomorphic. The conditions are as follows.

(a) $p = 2$ or ($p = q$ and $\gamma_1 = ... = \gamma_{p-3} = 0$).
(b) ($\gamma_0 = ... = \gamma_{p-3} = 0$.
(c) $p = 2$.
(d) $\gamma_0 = ... = \gamma_{p-3} = 0$.
(e) $\gamma_0 = ... = \gamma_{p-3} = 0$.

In all cases, we consider germs also at points with $t_3 \neq 0$, and therefore we can replace $\gamma_{p-2} + t_3$ by t_3 in ρ. A condition on γ_{p-2} (to be in $\mathbb{C}^*$ or $\mathbb{C} - \{0; 1\}$) translates into a condition on t_3. This gives all statements in the corollary. □

Remarks 7.3 (i) The classification in Theorem 7.1 of 3-dimensional germs $(M, 0)$ of generically semisimple F-manifolds with $T_0 M \cong Q^{(2)}$ which are different from A_3, B_3, H_3 is precise, but not so transparent. It becomes more transparent if one takes a closer look at the reduced plane curve germs $(C, 0)$ with $(L_M, \lambda) \cong (\mathbb{C}^2, 0) \times (C, 0)$. By Theorem 6.3, they have multiplicity 3. And by Corollary 4.7, all reduced plane curve germs $(C, 0)$ with multiplicity 3 appear.

(ii) Each reduced plane curve germ has a topological type. See [14, 3.4] for its definition. An old result of Brauner and Zariski (see, e.g., [14, Lemma 3.31 + Proposition 3.41 + Theorem 3.42]) is that the topological type of a reduced plane curve germ is determined by the topological types of the irreducible components and by their intersection numbers. And the topological type of an irreducible plane curve germ is determined by its Puiseux pairs (see, e.g., [14, 3.4] for their definition). The topological types of reduced plane curve germs of multiplicity 3 can be described and listed as follows. In all cases, the number $p \in \mathbb{Z}_{\geq 2}$ and, if it exists, also the number $q \in \mathbb{Z}_{\geq 2}$ are topological invariants.

(a) 3 smooth curve germs $C^{(1)}$, $C^{(2)}$, $C^{(3)}$ with intersection multiplicities $i(C^{(1)}, C^{(2)}) = p-1$, $i(C^{(1)}, C^{(3)}) = q-1$, $i(C^{(2)}, C^{(3)}) = p-1$ for $p, q \in \mathbb{Z}_{\geq 2}$ with $q \geq p$.

(b) 1 smooth germ $C^{(1)}$ and one germ $C^{(2\&3)}$ of type A_{2p-2} (namely with normal form $x_1^{2p-1} + x_2^2$) with the maximal possible intersection number $2p-1 = i(C^{(1)}, C^{2\&3)})$.

(c) 1 smooth germ $C^{(1)}$ and one germ $C^{(2\&3)}$ of type A_{2q-2} with an even intersection number $i(C^{(1)}, C^{(2\&3)}) = 2p-2$ for $q, p \in \mathbb{Z}_{\geq 2}$ with $q \geq p$.

(d) 1 irreducible germ with the only Puiseux pair $(3p-2, 3)$, so with a parametrization $(x \mapsto (x, \sum_{n\geq 3p-2} a_n x^{n/3}))$ with $a_n \in \mathbb{C}$ and $a_{3p-2} \neq 0$.

(e) 1 irreducible germ with the only Puiseux pair $(3p-1, 3)$, so with a parametrization $(x \mapsto (x, \sum_{n\geq 3p-1} a_n x^{n/3}))$ with $a_n \in \mathbb{C}$ and $a_{3p-1} \neq 0$.

One sees that the cases (a)–(e) correspond precisely to the cases (a)–(e) in Theorem 7.1. There the curve $(C, 0)$ is the zero set of the polynomial $\prod_{j=1}^{3}(y_2 - \partial_2 f^{(j)}|_{t_3=0}) \in \mathbb{C}[y_2, t_2]$.

(iii) The following topological types contain quasihomogeneous plane curve germs $(C, 0)$: all in (b), (d) and (e); those in (a) with $p = 2$ or $p = q$; those in (c) with $p = 2$. This fits to the cases in Corollary 7.2. The topological types in (a) with $p = q \geq 3$ contain a 1-parameter family of quasihomogeneous curves (up to coordinate changes). This gives the holomorphic parameter γ_0 in Corollary 7.2 (a) (iii).

(iv) In all cases in Theorem 7.1, there are $p-1$ holomorphic parameters $(\gamma_0, \gamma_1, ..., \gamma_{p-3}, \gamma_{p-2} + t_3)$ for the germs of F-manifolds. Here, the last parameter $\gamma_{p-2} + t_3$ is an *internal parameter*, it is the parameter of the 1-dimensional μ-constant stratum. The other parameters $(\gamma_0, ..., \gamma_{p-3})$ (for $p \geq 3$; no other parameter for $p = 2$) catch the isomorphism class of the plane curve germ $(C, 0)$ and the choice of a symplectic structure on the germ $(\mathbb{C}^2, 0)$ of the (y_2, t_2)-plane. This is the choice of a volume form, i.e., a form $u\,dy_2 dt_2$ with $u \in \mathbb{C}\{y_2, t_2\}^*$. In [15, Remark 5.31], the 3 types of parameters are rephrased as follows.

(α) moduli for the complex structure of the germ $(C, 0)$,
(β) moduli for the Lagrange structure of $(C, 0)$ or, equivalently, for the symplectic structure of $(\mathbb{C}^2, 0) \supset (C, 0)$,
(γ) moduli for the Lagrange fibration.

Here we have only one parameter of type (γ), the internal parameter $\gamma_{p-2} + t_3$. Remarkably, the sum of the numbers of parameters of types (α) and (β) is constant, it is $p-2$. This is remarkable, as the number of parameters of type (α) depends on the plane curve germ $(C, 0)$ with which one starts. It has the shape $\tau(C, 0) - b^{top}$, where $b^{top} \in \mathbb{N}$ is a topological invariant and the *Tjurina number* $\tau(C, 0)$ was defined in Theorem 2.16 (b) and is not a topological invariant. Though by Theorem 2.16 (b)+(c), the number of parameters of type (β) compensates this, as it is precisely $\mu - \tau(C, 0) = \dim H^1_{Giv}(\mathbb{C}^2, C, 0)$. So the sum of the numbers of parameters of types (α) and (β) is $\mu - b^{top}$. Here this number is $p-2$.

(v) The same number $\mu - b^{top}$ is also the number of parameters of right equivalence classes of plane curve germs with fixed topological type. Here, the right equivalence class is the class up to holomorphic coordinate changes. This follows from the fact that $\mu-\tau(C,0)$ is also the difference of the dimensions of the base space of a universal unfolding of a function germ for $(C,0)$ and of a semiuniversal deformation of $(C,0)$. However, for a given reduced plane curve germ $(C,0)$, there is no canonical relation between the choice of a volume form on $(\mathbb{C}^2,0)$ and the choice of a function germ $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ with $(f^{-1}(0),0)=(C,0)$.

(vi) Here, the normal forms in Theorem 7.1 are misleading. In all topological types which contain quasihomogeneous curves (up to coordinate changes), the following holds (and probably it holds also for the other topological types in (a) and (c)): The parameters $(\gamma_0,\ldots,\gamma_{p-3})$ in $\prod_{j=1}^3(y_2-\partial_2 f^{(j)}|_{t_3=0})$ are also the parameters for the right equivalence classes. Furthermore, if they are fixed, the internal parameter $\gamma_{p-2}+t_3$ does not change the right equivalence class. Both statements follow by inspection of the curves and a description of the μ-constant stratum in a universal unfolding of a quasihomogeneous singularity in [25].

(vii) The property in (vi), that the internal parameter $\gamma_{p-2}+t_3$ does not change the right equivalence class, is a lucky coincidence of the chosen normal forms. It is easy to construct a concrete description of a germ of an F-manifold in Theorem 7.1 where this does not hold. Start with a plane curve germ $(C,0)$ which is not quasihomogeneous (up to coordinate changes) and choose function germs $g_2^{(0)}:=0$ and $g_1^{(0)},g_0^{(0)}\in\mathbb{C}\{t_2\}$ such that $(C,0)\cong\{(y_2,t_2)\in(\mathbb{C}^2,0)\,|\,y_2^3-\sum_{i=0}^2 g_i^{(0)}t_2^i=0\}$. The system of partial differential equations

$$\partial_3\begin{pmatrix}g_1\\g_0\end{pmatrix}=\begin{pmatrix}2g_{02}+g_{12}t_2+2g_1\\g_{02}t_2+3g_0+\frac{2}{3}g_1g_{12}\end{pmatrix}.$$

is obtained from (4.45) by inserting $(g_2,h_2,h_1,h_0)=(0,1,t_2,-\frac{2}{3}g_1)$. By the theorem of Cauchy–Kowalevski (cited in the proof of Corollary 4.7), it has a unique solution with initial values $(g_1,g_0)|_{t_3=0}=(g_1^{(0)},g_0^{(0)})$. By construction, $(g_2,g_1,g_0,h_2,h_1,h_0)=(0,g_1,g_0,1,t_2,-\frac{2}{3}g_1)$ solve (4.45). We obtain a germ of an F-manifold with $g_2=0$. It is isomorphic to a germ in Theorem 7.1. Now Lemma 4.6 gives

$$H_{Z_3}(Z_2)=Z_2\cdot[2g_{22}h_2+(3y_2+g_2)h_{22}+3h_{12}]=Z_2\cdot 3. \tag{7.26}$$

The plane curve germs $(C(t_3^0),0):=(Z_2|_{t_3=t_3^0})^{-1}(0)$ are isomorphic for all t_3^0, but the function germs $Z_2|_{t_3=t_3^0}$ are not right equivalent for different t_3^0, because $Z_2|_{t_3=0}$ is not quasihomogeneous (up to coordinate changes) and because of (7.26).

Remark 7.4 In [6], 3-dimensional Frobenius manifolds with Euler fields $E=t_1\partial_1+\frac{1}{2}t_2\partial_2$ were constructed which enrich the following three F-manifolds with Euler fields:

(i) The F-manifold $M=\mathbb{C}^3$ in Theorem 5.3 (a) with $T_tM\cong Q^{(2)}$ for all $t\in M$ and with Euler field E as in (5.7) with $\varepsilon_2=\frac{1}{2}$ and $\varepsilon_{3,0}=0$.

(ii) The F-manifold $M = \mathbb{C}^3$ in Theorem 5.5 for $p = 2$ (so the first one in the series) with $T_0M \cong Q^{(2)}$ and $T_tM \cong Q^{(3)}$ for generic $t \in M$ and with Euler field as in (5.10) with $\varepsilon_{3,0} = 0$.

(iii) The F-manifold $\widetilde{M} = \mathbb{C}^2 \times \mathbb{H}$ which is the universal covering of the F-manifold $\mathbb{C}^2 \times (\mathbb{C} - \{0; 1\})$ in Corollary 7.2 (a)(i) (so the one with $p = q = 2$ in Theorem 7.1 (a)) with the Euler field as above (which is here unique up to adding a multiple of e).

Natural questions are now which other F-manifolds in this paper can be enriched to Frobenius manifolds or flat F-manifolds, and with which Euler fields, and in how many ways.

Funding Open Access funding enabled and organized by Projekt DEAL.

Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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