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On M-functions for the value-distributions of L-functions

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Dedicated to Professors Antanas Laurinčikas and Eugenijus Manstavičius
on the occasion of their 70th birthdays

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Abstract. Bohr and Jessen proved the existence of a certain limit value regarded as the probability that values of the Riemann zeta function belong to a given region in the complex plane. They also studied the density of the probability, which has been called the M-function since the studies of Ihara and Matsumoto. In this paper, we construct M-functions for the value-distributions of L-functions in a class containing many kinds of zeta and L-functions. Moreover, we improve the estimate on the rate of the convergence of the limit studied by Bohr and Jessen.

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1 Introduction

Let $s = \sigma + \mathrm{i}t$ be a complex variable, and let $\zeta(s)$ be the Riemann zeta function. The study of the value-distribution of $\zeta(s)$ is a classical topic in number theory. For $\sigma > 1/2$ and $T > 0$, we define

$$\mathcal{V}_\sigma(T, R) = \mu_1\bigl\{t \in [-T, T] \bigm| \log\zeta(\sigma + \mathrm{i}t) \in R\bigr\},$$

where R is any rectangle in the complex plane whose edges are parallel to the coordinate axes, and μ_k denotes the k-dimensional Lebesgue measure. The choice of the branch of logarithm is described in Section 2. Bohr and Jessen [1] proved the existence of the limit

$$\mathcal{W}_\sigma(R) = \lim_{T\to\infty} \frac{1}{2T}\mathcal{V}_\sigma(T, R) \tag{1.1}$$

for any $\sigma > 1/2$. This is called the Bohr–Jessen limit theorem, and the method for the proof was improved by Jessen and Wintner [7], Borchsenius and Jessen [2], Nikishin [17], and Laurinčikas [9]. The limit $\mathcal{W}_\sigma(R)$ is regarded as the probability that the values of $\log\zeta(\sigma + \mathrm{i}t)$ belong to R. Bohr and Jessen [1] also studied the density of the probability $\mathcal{W}_\sigma(R)$. They proved the existence of a continuous and nonnegative real-valued

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function $\mathcal{M}_\sigma(z)$ such that

$$\mathcal{W}_\sigma(R) = \int_R \mathcal{M}_\sigma(z)\,|\mathrm{d}z| \tag{1.2}$$

with $|\mathrm{d}z| = (2\pi)^{-1}\,\mathrm{d}x\,\mathrm{d}y$. Ihara [6] named $\mathcal{M}_\sigma(z)$ the "M-function" for the value-distribution of $\zeta(s)$. The estimate on the rate of convergence of limit (1.1) was firstly given by Matsumoto [10], which was improved by Harman and Matsumoto [4]. They showed that, for an arbitrarily small $\epsilon > 0$,

$$\frac{1}{2T}\mathcal{V}_\sigma(T,R) = \mathcal{W}_\sigma(R) + O\big(\big(\mu_2(R)+1\big)(\log T)^{-A(\sigma)+\epsilon}\big) \tag{1.3}$$

as $T \to \infty$, where the implied constant depends only on σ and ϵ, and

$$A(x) = \begin{cases} (x-1)/(3+2x) & \text{if } x > 1, \\ (4x-2)/(21+8x) & \text{if } 1/2 < x \leqslant 1. \end{cases}$$

The Bohr–Jessen limit theorem is generalized to several zeta and L-functions. Matsumoto [11] introduced a certain class $\mathscr{M}$ of L-functions with polynomial Euler products and proved the Bohr–Jessen limit theorem for the functions in $\mathscr{M}$. On the other hand, M-functions for the value-distributions of L-functions in $\mathscr{M}$ are only known in a few cases. All cases for which analogues of (1.2) were already proved are Dedekind zeta functions of finite Galois extensions of $\mathbb{Q}$ [12], automorphic L-functions of normalized holomorphic Hecke-eigen cusp forms with respect to congruence subgroups [14], and their symmetric power L-functions [15]. Furthermore, as for formula (1.3), we know its analogue only in the case of Dedekind zeta functions of arbitrary number fields [13].

Recently, the author [16] proved results analogous to (1.2) and (1.3) in a subclass of $\mathscr{M}$ by considering not the logarithms $\log F(s)$ but rather the logarithmic derivatives $(F'/F)(s)$. In this paper, we prove (1.2) and (1.3) for $\log F(s)$ by suitably modifying the method in [16].

2 Statement of the result

DEFINITION 1. The class $\mathcal{S}_I$ is the set of all functions $F(s)$ representable as the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}$$

in some half-plane and satisfying the following Axioms 1–5:

1. *Ramanujan hypothesis.* The Dirichlet coefficients $a_F(n)$ satisfy $a_F(n) \ll_\epsilon n^\epsilon$ for every $\epsilon > 0$.
2. *Analytic continuation.* There exists an integer $m \geqslant 0$ such that $(s-1)^m F(s)$ is an entire function of finite order.
3. *Functional equation.* $F(s)$ satisfies a functional equation of the form

$$\Lambda_F(s) = \omega\overline{\Lambda_F(1-\overline{s})},$$

where

$$|\omega| = 1, \qquad \Lambda_F(s) = F(s)Q^s\prod_{j=1}^{r}\Gamma(\lambda_j s + \mu_j), \quad Q > 0,\ \lambda_j > 0,\ \Re(\mu_j) \geqslant 0.$$

4. *Polynomial Euler product.* For $\sigma > 1$, $F(s)$ is expressed as the infinite product

$$F(s) = \prod_{p\colon \text{prime}} \prod_{j=1}^{g} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1},$$

where g is a positive integer, and $\alpha_j(p) \in \mathbb{C}$ for $j = 1, \dots, g$.

5. *Prime mean square.* There exists a positive constant κ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{\substack{p\colon \text{prime} \\ p \leqslant x}} \left|a_F(p)\right|^2 = \kappa,$$

where $\pi(x)$ is the number of prime numbers less than or equal to x.

The class $\mathcal{S}_{II}$ is the set of all $F(s)$ satisfying Axioms 1–5 and the following one:

6. *Zero density estimate.* There exist positive constants c and A such that

$$N_F(T, \sigma) \ll T^{1-c(\sigma-1/2)} (\log T)^A$$

as $T \to \infty$, uniformly for $\sigma \geqslant 1/2$, where $N_F(\sigma, T)$ denotes the number of zeros $\rho = \beta + \mathrm{i}\gamma$ of $F(s)$ with $\beta > \sigma$ and $0 < \gamma < T$.

We note that $\mathcal{S}_I$ is equal to the intersection of the Selberg class $\mathcal{S}$ [18] and the Steuding class $\tilde{\mathcal{S}}$ [19]. Furthermore, we have the inclusions

$$\mathcal{S}_{II} \subset \mathcal{S}_I = \mathcal{S} \cap \tilde{\mathcal{S}} \subset \tilde{\mathcal{S}} \subset \mathscr{M}$$

by the definitions of these classes. The Riemann zeta function $\zeta(s)$ is a typical example of a member of the class $\mathcal{S}_I$. In addition to $\zeta(s)$, Dirichlet L-functions $L(s, \chi)$ of primitive characters χ, Dedekind zeta functions, and automorphic L-functions $L(s, f)$ of normalized holomorphic Hecke-eigen cusp forms f with respect to $SL_2(\mathbb{Z})$ also belong to $\mathcal{S}_I$. Moreover, $\zeta(s)$, $L(s, \chi)$, $L(s, f)$ belong to the class $\mathcal{S}_{II}$; see [16].

In this paper, we construct the M-function for the value-distribution of $\log F(s)$ with $F \in \mathcal{S}_I$ or $F \in \mathcal{S}_{II}$ and prove analogues of (1.2) and (1.3). We define $\log F(s)$ for $F \in \mathcal{S}_I$ as follows. By Axiom 4 we define

$$\log F(s) = -\sum_p \sum_{j=1}^{g} \mathrm{Log}\left(1 - \frac{\alpha_j(p)}{p^s}\right)$$

for $\sigma > 1$, where Log denotes the principal branch of logarithm. Let

$$G_F = \left\{ s = \sigma + \mathrm{i}t \;\middle|\; \sigma > \frac{1}{2} \right\} \setminus \bigcup_{\rho=\beta+\mathrm{i}\gamma} \left\{ s = \sigma + \mathrm{i}\gamma \;\middle|\; \frac{1}{2} < \sigma \leqslant \beta \right\},$$

where ρ runs through all zeros and poles of $F(s)$ with $\beta > 1/2$. We define $\log F(s)$ for $s \in G_F$ by analytic continuation along the horizontal line from right. Then, for $\sigma > 1/2$ and $T > 0$, we define

$$\mathcal{V}_\sigma(T, R; F) = \mu_1\left\{ t \in [0, T] \;\middle|\; \log F(\sigma + \mathrm{i}t) \in R \right\},$$

where R is a rectangle with edges parallel to the axes. We recall another estimate on the zero density of $F(s)$ to state the first result. Kaczorowski and Perelli [8] proved that, for any $F \in \mathcal{S}$, there exists a constant $\delta > 0$

such that, for an arbitrarily small $\epsilon > 0$,

$$N_F(T,\sigma) \ll_\epsilon T^{\delta(1-\sigma)+\epsilon} \tag{2.1}$$

as $T \to \infty$, uniformly for $\sigma \geqslant 1/2$. From the proof in [8], estimate (2.1) holds with $\delta = 4(d_F+3)$, where d_F is the degree of F defined by $d_F = 2(\lambda_1 + \cdots + \lambda_r)$ with λ_j in Axiom 3.

Theorem 1. *Let $F \in \mathcal{S}_I$, and let $\sigma > 1 - \delta^{-1}$ be a fixed real number, where δ is the constant in* (2.1). *Then there exists a continuous and nonnegative real-valued function $\mathcal{M}_\sigma(z;F)$ such that, for an arbitrarily small $\epsilon > 0$,*

$$\frac{1}{T}\mathcal{V}_\sigma(T,R;F) = \int_R \mathcal{M}_\sigma(z;F)\,|\mathrm{d}z| + O\bigl(\bigl(\mu_2(R)+1\bigr)(\log T)^{-1/2+\epsilon}\bigr) \tag{2.2}$$

as $T \to \infty$. The implied constant depends only on F, σ, and ϵ.

Theorem 2. *Let $F \in \mathcal{S}_{II}$. Then formula* (2.2) *holds for any fixed real number $\sigma > 1/2$.*

Since the Riemann zeta function $\zeta(s)$ belongs to the class $\mathcal{S}_{II}$ and $A(\sigma)$ in (1.3) is less than $1/2$, Theorem 2 is an improvement of Harman and Matsumoto's result (1.3).

3 Certain mean values of $\log F(\sigma + \mathrm{i}t)$

In this section, we prove Theorem 1. Before the proof, we define the Fourier transform with respect to the additive character $\psi_w(z) = \exp(\mathrm{i}\Re(z\overline{w}))$, $z, w \in \mathbb{C}$. Throughout this paper, we identify a function $f(z)$ on $\mathbb{C}$ with $f(x,y)$ on $\mathbb{R}^2$ by the bijection $z = x + \mathrm{i}y \mapsto (x,y)$. Then, for an absolutely integrable function $f(z)$, we define its Fourier transform and Fourier inverse transform as

$$f^\wedge(z) = \int_{\mathbb{C}} f(w)\psi_z(w)\,|\mathrm{d}w| \quad \text{and} \quad f^\vee(z) = \int_{\mathbb{C}} f(w)\psi_{-z}(w)\,|\mathrm{d}w|,$$

respectively. We further define the class

$$\Lambda = \bigl\{ f \in L^1 \bigm| f, f^\wedge \in L^1 \cap L^\infty \text{ and } \bigl(f^\wedge\bigr)^\vee = f \bigr\}.$$

Proposition 1. *Let $F \in \mathcal{S}_I$. Let η, θ be real numbers with $2\eta + 3\theta < 1$. Let $\sigma > 1 - \delta^{-1}$ be a fixed real number. Let $\Phi \in \Lambda$. Then there exists a constant $T_0 = T_0(F,\eta,\theta,\sigma) > 0$ such that*

$$\frac{1}{T}\int_0^T \Phi\bigl(\log F(\sigma + \mathrm{i}t)\bigr)\,\mathrm{d}t = \int_{\mathbb{C}} \Phi(z)\mathcal{M}_\sigma(z;F)\,|\mathrm{d}z| + E$$

for all $T \geqslant T_0$, where the error term E is estimated as

$$E \ll \exp\bigl(-c(F,\sigma)(\log T)^\theta\bigr)\int_\Omega \bigl|\Phi^\wedge(z)\bigr|\,|\mathrm{d}z| + \int_{\mathbb{C}\setminus\Omega} \bigl|\Phi^\wedge(z)\bigr|\,|\mathrm{d}z|.$$

Here $c(F,\sigma)$ is a positive constant, the region Ω is the rectangle

$$\Omega = \bigl\{ z = x + \mathrm{i}y \in \mathbb{C} \bigm| -(\log T)^\eta \leqslant x, y \leqslant (\log T)^\eta \bigr\}, \tag{3.1}$$

and the implied constant depends only on F and σ.

Proposition 1 is an analogue of the result of Guo [3, Thm. 1.1.1] for $\log F(s)$. Theorem 1 is deduced from it in the same way as in [16], but we omit the details. Therefore the proof of Theorem 1 is completed if Proposition 1 is established. Moreover, Proposition 1 is deduced from the following Propositions 2 and 3 by a simple Fourier analysis; see the argument in [3, Proof of Thm. 1.1.1].

Proposition 2. *Let $F \in \mathcal{S}_I$. Let $\eta, \theta > 0$ be real numbers with $2\eta + 3\theta < 1$. Let $\sigma > 1 - \delta^{-1}$ be a fixed real number. Then there exists a constant $T_0 = T_0(F, \eta, \theta, \sigma) > 0$ such that*

$$\frac{1}{T}\int_0^T \psi_z\bigl(\log F(\sigma + \mathrm{i}t)\bigr)\,\mathrm{d}t = \widetilde{\mathcal{M}}_\sigma(z;F) + O\bigl(\exp\bigl(-c(F,\sigma)(\log T)^\theta\bigr)\bigr) \tag{3.2}$$

for all $T \geqslant T_0$ and $z \in \Omega$, where $\widetilde{\mathcal{M}}_\sigma(z;F)$ is a function determined from F and $\sigma > 1/2$, $c(F,\sigma)$ is a positive constant, and Ω is the rectangle given in (3.1)*. The implied constant depends only on F and σ.*

Proposition 3. *Let $F \in \mathcal{S}_I$, and let $\sigma > 1/2$ be a fixed real number. Then the function $\widetilde{\mathcal{M}}_\sigma(z;F)$ in Proposition 1 belongs to the class Λ.*

3.1 Proof of Proposition 2

For $F \in \mathcal{S}_I$, let $\Lambda_F(n)$ be the arithmetic function given by

$$\Lambda_F(n) = \begin{cases} (\alpha_1(p)^m + \cdots + \alpha_g(p)^m)\log p & \text{if } n = p^m, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_j(p)$ are parameters in Axiom 4. If $F(s) = \zeta(s)$, then $\Lambda_F(n)$ is equal to the usual von Mangoldt function $\Lambda(n)$. By Axiom 4 we have

$$\log F(s) = \sum_{n=1}^{\infty} \frac{\Lambda_F(n)}{\log n} n^{-s}$$

for $\sigma > 1$. In this section, we approximate $\log F(\sigma + \mathrm{i}t)$ in the integral in (3.2) by several Dirichlet polynomials. First, we approximate $\log F(s)$ by the function

$$F_X(s;F) = \sum_{n \leqslant X^2} \frac{\Lambda_F(n) w_X(n)}{\log n} n^{-s},$$

where

$$w_X(n) = \begin{cases} 1 & \text{if } 1 \leqslant n \leqslant X, \\ \log(X^2/n)/\log X & \text{if } X \leqslant n \leqslant X^2. \end{cases}$$

Lemma 1. *Let $F \in \mathcal{S}_I$, and let $\sigma > 1 - \delta^{-1}$ be a fixed real number. Then there exists an absolute constant $T_0 > 0$ such that*

$$\frac{1}{T}\int_0^T \psi_z\bigl(\log F(\sigma + \mathrm{i}t)\bigr)\,\mathrm{d}t = \frac{1}{T}\int_0^T \psi_z\bigl(F_X(\sigma + \mathrm{i}t;F)\bigr)\,\mathrm{d}t + E_1$$

for all $T \geqslant T_0$ and $z \in \mathbb{C}$. The error term E_1 is estimated as

$$E_1 \ll \frac{1}{T} + YT^{-\delta(\sigma-(1-\delta^{-1}))/4} + \frac{|z|}{\log X}\left(\frac{X \log Y \log T}{Y} + \frac{X^{-(\sigma-(1-\delta^{-1}))/2}\log T}{\{\sigma-(1-\delta^{-1})\}^2} + \frac{X}{T} + X^{-1/2}\log^2 T\right)$$

for any $X, Y > 1$, where the implied constant depends only on F and σ.

Proof. According to the proof of [3, Lemma 2.1.4], let $\mathscr{B}_Y(\sigma, T; F)$ be the set of all $t \in [0, T]$ for which there exists a zero $\rho = \beta + \mathrm{i}\gamma$ of $F(s)$ with $\beta \geqslant (\sigma + 1 - \delta^{-1})/2$ and $|\gamma - t| \leqslant Y$. Then we have

$$E_1 \ll \frac{1}{T} + \frac{\mu_1(\mathscr{B}_Y(\sigma, T; F))}{T} + \frac{|z|}{T} \int\limits_{[2,T]\cap\mathscr{B}_Y(\sigma,T;F)^c} \left|\log F(\sigma + \mathrm{i}t) - F_X(t, \sigma; F)\right| \mathrm{d}t \tag{3.3}$$

by the inequalities $|\psi_z(w)| \leqslant 1$ and $|\psi_z(w) - \psi_z(w')| \leqslant |z||w - w'|$. The second term of the right-hand side in (3.3) is estimated as

$$\frac{\mu_1(\mathscr{B}_Y(\sigma, T; F))}{T} \leqslant \frac{2Y}{T} N_F\left(\frac{\sigma + 1 - \delta^{-1}}{2}, T\right) \ll_\epsilon YT^{-1+\delta(1-(\sigma+1-\delta^{-1})/2)+\epsilon},$$

due to estimate (2.1). Therefore, taking $\epsilon = \delta(\sigma - (1 - \delta^{-1}))/4 > 0$, the term we consider is

$$\ll_\sigma YT^{-\delta(\sigma-(1-\delta^{-1}))/4}.$$

Then we estimate the third term. Let $\sigma_0 = \max\{\sigma, 2\}$. If $\sigma + \mathrm{i}t \in G_F$, then we see that

$$\log F(\sigma + \mathrm{i}t) - F_X(\sigma + \mathrm{i}t; F) = \log F(\sigma_0 + \mathrm{i}t) - F_X(\sigma_0 + \mathrm{i}t; F) - \int\limits_\sigma^{\sigma_0} \left(\frac{F'}{F}(u + \mathrm{i}t) - f_X(u + \mathrm{i}t; F)\right) \mathrm{d}u \tag{3.4}$$

by the definition of $\log F(s)$, where

$$f_X(s; F) = -\sum_{n \leqslant X^2} \Lambda_F(n) w_X(n) n^{-s}.$$

Since $\sigma_0 \geqslant 2$, we have

$$\log F(\sigma_0 + \mathrm{i}t) - F_X(\sigma_0 + \mathrm{i}t; F) \ll \sum_{n \geqslant X} \frac{|\Lambda_F(n)|}{\log n} n^{-2} \ll X^{-1}.$$

Here we used the fact that $|\Lambda_F(n)|$ is less than or equal to $g\Lambda(n)$, where g is the constant in Axiom 4. To estimate the integral in (3.4), we use the following formula obtained in [16]:

$$\frac{F'}{F}(s) = f_X(s; F) - \frac{m_1}{\log X} \frac{X^{1-s} - X^{2(1-s)}}{(1-s)^2} + \frac{m_0}{\log X} \frac{X^{-s} - X^{-2s}}{s^2} + \frac{1}{\log X} \sum_\rho \frac{X^{\rho-s} - X^{2(\rho-s)}}{(\rho - s)^2} + O\left(\frac{X^{-\sigma}}{\log X} \log^2 T\right) \tag{3.5}$$

for $\sigma > 1/2$ and $t \in [1, T]$, where $m_1, m_0 \geqslant 0$ are the orders of a possible pole of $F(s)$ at $s = 1$ and a possible zero at $s = 0$, respectively, and ρ runs through nontrivial zeros of $F(s)$. Following the argument in [3, Lemma 2.1.3], we deduce from (3.5) that, for all $u > 1/2$ and $t \in [2, T] \cap \mathscr{B}_Y(\sigma, T; F)^c$,

$$\begin{aligned}&\frac{F'}{F}(u+\mathrm{i}t) - f_X(u+\mathrm{i}t;F)\\&\quad\ll \frac{X}{t^2\log X} + \frac{X\log Y\log T}{Y\log X} + \frac{X^{-(u-(1-\delta^{-1}))/2}\log T}{\{u-(1-\delta^{-1})\}^2\log X} + \frac{X^{-u}}{\log X}\log^2 T.\end{aligned}$$

Hence by (3.4) we obtain

$$\begin{aligned}&\log F(\sigma+\mathrm{i}t) - F_X(\sigma+\mathrm{i}t;F)\\&\quad\ll_\sigma \frac{1}{X} + \frac{1}{\log X}\left\{\frac{X}{t^2} + \frac{X\log Y\log T}{Y} + \frac{X^{-(\sigma-(1-\delta^{-1}))/2}\log T}{\{\sigma-(1-\delta^{-1})\}^2} + X^{-\sigma}\log^2 T\right\}\end{aligned}$$

for all $\sigma + \mathrm{i}t \in G_F$ with $t \in [2, T] \cap \mathscr{B}_Y(\sigma, T; F)^c$. Therefore, the third term of (3.3) is estimated as

$$\begin{aligned}&\frac{|z|}{T}\int\limits_{[2,T]\cap\mathscr{B}_Y(\sigma,T;F)^c} \big|\log F(\sigma+\mathrm{i}t) - F_X(\sigma+\mathrm{i}t;F)\big|\,\mathrm{d}t\\&\quad\ll_\sigma \frac{|z|}{\log X}\left\{\frac{X}{T} + \frac{X\log Y\log T}{Y} + \frac{X^{-(\sigma-(1-\delta^{-1}))/2}\log T}{\{\sigma-(1-\delta^{-1})\}^2} + X^{-1/2}\log^2 T\right\}.\end{aligned}$$

Hence Lemma 1 follows. □

Next, we use the following functions:

$$G_X(s;F) = \sum_{n\leqslant X^2}\frac{\Lambda_F(n)}{\log n}n^{-s}, \qquad H_X(s;F) = \sum_{p\leqslant X^2}\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{\log p^m}p^{-ms}.$$

Lemma 2. *Let $F \in \mathcal{S}_I$, and let $\sigma > 1/2$ be a fixed real number. Then there exists an absolute constant $T_0 > 0$ such that*

$$\frac{1}{T}\int_0^T \psi_z\big(f_X(\sigma+\mathrm{i}t;F)\big)\,\mathrm{d}t = \frac{1}{T}\int_0^T \psi_z\big(g_X(\sigma+\mathrm{i}t;F)\big)\,\mathrm{d}t + E_2$$

for all $T \geqslant T_0$ and for all $z \in \mathbb{C}$. The error term E_2 is estimated as

$$E_2 \ll \frac{g|z|\log X}{(2\sigma-1)^{1/2}}\left(1+\frac{X^2}{T}\right)^{1/2} X^{1/2-\sigma}$$

for any $X > 1$, with an absolute implied constant.

Lemma 3. *Let $F \in \mathcal{S}_I$, and let $\sigma > 1/2$ be a fixed real number. Then there exists an absolute constant $T_0 > 0$ such that*

$$\frac{1}{T}\int_0^T \psi_z\big(G_X(\sigma+\mathrm{i}t;F)\big)\,\mathrm{d}t = \frac{1}{U}\int_0^U \psi_z\big(G_X(\sigma+\mathrm{i}u;F)\big)\,\mathrm{d}u + E_3$$

for all $U \geqslant T \geqslant T_0$ *and* $z \in \mathbb{C}$*. The error term* E_3 *is estimated as*

$$E_3 \ll \frac{(2g)^N X^{5N}}{T}\left(1+|z|^2\right)^{N/2} + \frac{(16g|z|)^N}{N!}\left(1+\frac{X^N}{T}\right)\left\{\left(\zeta(2\sigma)^{1/2}\log X\right)^N\left(\frac{N}{2}\right)! + \zeta'(2\sigma)^N\right\}$$

for any $X > 1$ *and for any large even integer* N*. The implied constant is absolute.*

Lemma 4. *Let* $F \in \mathcal{S}_I$*, and let* $\sigma > 1/2$ *be a fixed real number. Then there exists an absolute constant* $T_0 > 0$ *such that*

$$\frac{1}{U}\int_0^U \psi_z\left(G_X(\sigma+\mathrm{i}u;F)\right)\mathrm{d}u = \frac{1}{U}\int_0^U \psi_z\left(H_X(\sigma+\mathrm{i}u;F)\right)\mathrm{d}u + E_4$$

for all $U \geqslant T \geqslant T_0$ *and* $z \in \mathbb{C}$*. The error term* E_4 *is estimated as*

$$E_4 \ll \frac{g|z|\log X}{2\sigma-1}X^{1-2\sigma}$$

for any $X > 1$*, with an absolute implied constant.*

These three lemmas correspond to [3, Lemmas 2.1.5, 2.1.6, and 2.1.10]. The only difference is that we use $\Lambda_F(n)/\log n$ instead of $\Lambda(n)$ as the coefficients of the Dirichlet polynomials. Then, applying the inequality $|\Lambda_F(n)/\log n| \leqslant 2g\Lambda(n)$, we obtain Lemmas 2, 3, and 4 in the way similar to the proofs of lemmas in [3].

Let $\eta, \theta > 0$ be real numbers with $2\eta + 3\theta < 1$. Assume that $z = x + \mathrm{i}y$ satisfies $|x|, |y| \leqslant (\log T)^\eta$. We take X, Y, and N as the functions

$$X = \exp\left((\log T)^\theta\right), \quad Y = \exp\left((\log T)^{\theta_1}\right), \quad \text{and} \quad N = 2\left\lfloor(\log T)^{\theta_2}\right\rfloor$$

with constants θ_1, θ_2 satisfying $\theta < \theta_1 < 1$ and $2(\eta+\theta) < \theta_2 < 1-\theta$. Then, combining Lemmas 1–4, we obtain

$$\frac{1}{T}\int_0^T \psi_z\left(\log F(\sigma+\mathrm{i}t)\right)\mathrm{d}t = \frac{1}{U}\int_0^U \psi_z\left(H_X(\sigma+\mathrm{i}u;F)\right)\mathrm{d}u + O\left(\exp\left(-c(F,\sigma)(\log T)^\theta\right)\right) \tag{3.6}$$

for all $U \geqslant T \geqslant T_0$ with some $T_0 = T_0(F,\eta,\theta,\sigma) > 0$ and $c(F,\sigma) > 0$. Next, by applying [5, Lemma 2] we find that

$$\begin{aligned}\frac{1}{U}\int_0^U \psi_z\left(H_X(\sigma+\mathrm{i}u;F)\right)\mathrm{d}u &= \frac{1}{U}\prod_{p\leqslant X^2}\int_0^U \psi_z\left(\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{\log p^m}p^{-m\sigma}\exp\left(2\pi\mathrm{i}m\frac{\log p}{2\pi}u\right)\right)\mathrm{d}u \\ &\to \prod_{p\leqslant X^2}\int_0^1 \psi_z\left(\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{\log p^m}p^{-m\sigma}\exp(2\pi\mathrm{i}m\theta)\right)\mathrm{d}\theta \quad \text{as } U\to\infty.\end{aligned}$$

Here we used the linear independence of $\{\log p\}_p$ over the rational numbers. Therefore by (3.6) we have

$$\frac{1}{T}\int_0^T \psi_z\left(\log F(\sigma+\mathrm{i}t)\right)\mathrm{d}t = \prod_{p\leqslant X^2}\widetilde{\mathcal{M}}_{\sigma,p}(z;F) + O\left(\exp\left(-c(F,\sigma)(\log T)^\theta\right)\right), \tag{3.7}$$

where

$$\widetilde{\mathcal{M}}_{\sigma,p}(z;F)=\int_0^1 \psi_z\left(\sum_{m=1}^{\infty}\frac{\Lambda_F(p^m)}{\log p^m}p^{-m\sigma}\exp(2\pi\mathrm{i}m\theta)\right)\mathrm{d}\theta.$$

In Section 3.2, we prove the following result.

Lemma 5. *Let $F\in\mathcal{S}_I$. Let $\eta,\theta>0$ be real numbers with $2\eta+3\theta<1$. Let $\sigma>1/2$ be a fixed real number. Then there exists a constant $T_0=T_0(F,\eta,\theta,\sigma)>0$ such that*

$$\prod_{p>X^2}\widetilde{\mathcal{M}}_{\sigma,p}(z;F)=1+O\big(\exp\big(-c(\sigma)(\log T)^{\theta}\big)\big)$$

for all $T\geqslant T_0$ and $z\in\Omega$, where $X=\exp((\log T)^{\theta})$, $c(\sigma)$ is a positive constant, and Ω is the rectangle in (3.1)*. The implied constant depends only on F.*

We have $|\widetilde{\mathcal{M}}_{\sigma,p}(z;F)|\leqslant 1$ for every p from the definition of $\widetilde{\mathcal{M}}_{\sigma,p}(z;F)$, and hence Lemma 5 implies

$$\prod_{p>X^2}\widetilde{\mathcal{M}}_{\sigma,p}(z;F)=\prod_{p}\widetilde{\mathcal{M}}_{\sigma,p}(z;F)+O\big(\exp\big(-c(\sigma)(\log T)^{\theta}\big)\big). \tag{3.8}$$

Therefore, by formulas (3.7) and (3.8), Proposition 2 holds for the function

$$\widetilde{\mathcal{M}}_{\sigma}(z;F)=\prod_{p}\widetilde{\mathcal{M}}_{\sigma,p}(z;F). \tag{3.9}$$

3.2 Proof of Proposition 3

In this section, we study the function $\widetilde{\mathcal{M}}_{\sigma}(z;F)$ more precisely. To begin with, we prove the convergence of the infinite product in definition (3.9). More generally, let

$$\widetilde{\mathcal{M}}_{p}(s,z_1,z_2;F)=\int_0^1\exp\big(\mathrm{i}z_1A_p(\theta,s;F)+\mathrm{i}z_2B_p(\theta,s;F)\big)\,\mathrm{d}\theta \tag{3.10}$$

for $s,z_1,z_2\in\mathbb{C}$ with $\sigma>1/2$, where

$$A_p(\theta,s;F)=\sum_{m=1}^{\infty}\frac{1}{p^{ms}}\left\{\frac{\Re\Lambda_F(p^m)}{\log p^m}\cos(2\pi m\theta)-\frac{\Im\Lambda_F(p^m)}{\log p^m}\sin(2\pi m\theta)\right\},$$
$$B_p(\theta,s;F)=\sum_{m=1}^{\infty}\frac{1}{p^{ms}}\left\{\frac{\Re\Lambda_F(p^m)}{\log p^m}\sin(2\pi m\theta)+\frac{\Im\Lambda_F(p^m)}{\log p^m}\cos(2\pi m\theta)\right\}.$$

We note that if $\sigma,x,y\in\mathbb{R}$, then $\widetilde{\mathcal{M}}_{p}(\sigma,x,y;F)=\widetilde{\mathcal{M}}_{\sigma,p}(x+\mathrm{i}y;F)$.

Lemma 6. *Let $F\in\mathcal{S}_I$. Let K be any compact subset on the half-plane $\sigma>1/2$, and let K_1,K_2 be any compact subsets on $\mathbb{C}$. Then the infinite product*

$$\widetilde{\mathcal{M}}(s,z_1,z_2;F)=\prod_{p}\widetilde{\mathcal{M}}_{p}(s,z_1,z_2;F)$$

uniformly converges on $K\times K_1\times K_2$.

Proof. Expanding the exponential in the right-hand side in (3.10), we have

$$\widetilde{\mathcal{M}}_p(s, z_1, z_2; F) = 1 - m_p + r_p, \tag{3.11}$$

where

$$m_p = m_p(s, z_1, z_2; F) = \frac{z_1^2 + z_2^2}{4} \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{\log^2 p^m} p^{-2ms}$$

and

$$r_p = r_p(s, z_1, z_2; F) = \int_0^1 \sum_{k=3}^{\infty} \frac{\mathrm{i}^k}{k!} \bigl\{ z_1 A_p(\theta, s; F) + z_2 B_p(\theta, s; F) \bigr\}^k d\theta.$$

Then we have $m_p \ll g^2(|z_1|^2 + |z_2|^2)p^{-2\sigma}$ and $r_p \ll g^3(|z_1|^3 + |z_2|^3)p^{-3\sigma}$ with the absolute implied constants. Hence we see that

$$\operatorname{Log} \widetilde{\mathcal{M}}_p(s, z_1, z_2; F) = -m_p + r_p + O\bigl(|m_p|^2 + |r_p|^2\bigr)$$

for sufficiently large p if (s, z_1, z_2) varies in $K \times K_1 \times K_2$. Let σ_0 be the smallest real part of $s \in K$. Then $\operatorname{Log} \widetilde{\mathcal{M}}_p(s, z_1, z_2; F)$ is estimated as $\ll p^{-2\sigma_0}$, where the implied constant depends only on K_1, K_2, and F. Since the series $\sum_p p^{-2\sigma_0}$ converges, we have the desired result. □

We know that $\widetilde{\mathcal{M}}_\sigma(z; F) = \widetilde{\mathcal{M}}(\sigma, x, y; F)$ for $z = x + \mathrm{i}y \in \mathbb{C}$. Hence by Lemma 6 the infinite product (3.9) converges. Next, we prove the following estimate.

Lemma 7. *Let $F \in \mathcal{S}_I$, and let $\sigma > 1/2$ be a fixed real number. Then, for each $\epsilon > 0$, there exist constants $c = c(\sigma, \epsilon; F) > 0$ and $Z = Z(\sigma; F) \geqslant 1$ such that, for all $x, y \in \mathbb{R}$ with $|x| + |y| \geqslant Z$,*

$$\bigl|\widetilde{\mathcal{M}}(\sigma, z_1, z_2; F)\bigr| \leqslant \exp\bigl(-c\bigl(|x| + |y|\bigr)^{1/\sigma - \epsilon}\bigr)$$

for any $z_1, z_2 \in \mathbb{C}$ with $|z_1 - x| < 1/2$, $|z_2 - y| < 1/2$.

Proof. Let $0 < c_0 < 1$ be a constant to be chosen later, and let

$$P_0 = \left(\frac{g(|x| + |y|)}{c_0}\right)^{1/\sigma}.$$

Due to $|z_1 - x| < 1/2$ and $|z_2 - y| < 1/2$, we have

$$\frac{g(|z_1| + |z_2|)}{p^\sigma} \leqslant \frac{2g(|x| + |y|)}{P_0^\sigma} = 2c_0$$

for $p \geqslant P_0$. We use formula (3.11) again. Since $m_p, r_p \ll \{g(|z_1| + |z_2|)p^{-\sigma}\}^2$, we have $|m_p|, |r_p| < 1/2$ if c_0 is suitably small. Then we obtain

$$\operatorname{Log} \widetilde{\mathcal{M}}_p(s, z_1, z_2; F) = -m_p + r_p + O\bigl(|m_p|^2 + |r_p|^2\bigr) \tag{3.12}$$

for $p \geqslant P_0$. Let

$$n_p = \frac{x^2 + y^2}{4} \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{\log^2 p^m} p^{-2m\sigma}.$$

Then we have

$$|m_p - n_p| \leqslant \big(|x| + |y|\big) \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{\log^2 p^m} p^{-2m\sigma}$$

by the assumptions $|z_1 - x| < 1/2$ and $|z_2 - y| < 1/2$. The remainder terms in (3.12) are estimated as

$$r_p, m_p^2, \quad \text{and} \quad r_p^2 \ll \frac{g^3(|x|+|y|)^3}{p^{3\sigma}} \leqslant 2c_0 \frac{g^2(|x|+|y|)^2}{p^{2\sigma}},$$

where the implied constants are absolute. Thus the estimate

$$\big|\mathrm{Log}\, \widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F) + n_p\big| \leqslant \big(|x| + |y|\big) \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{\log^2 p^m} p^{-2m\sigma} + Bc_0 g^2 \big(|x| + |y|\big)^2 p^{-2\sigma}$$

with an absolute constant $B > 0$ follows from (3.12) . Hence, if $|x| + |y|$ is sufficiently large, then we have

$$\begin{aligned} \log \big|\widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)\big| &\leqslant -A\big(|x| + |y|\big)^2 \sum_{m=1}^{\infty} \frac{|\Lambda_F(p^m)|^2}{\log^2 p^m} p^{-2m\sigma} + Bc_0 g^2 \big(|x| + |y|\big)^2 p^{-2\sigma} \\ &\leqslant -A\big(|x| + |y|\big)^2 \big|a_F(p)\big|^2 p^{-2\sigma} + Bc_0 g^2 \big(|x| + |y|\big)^2 p^{-2\sigma}, \end{aligned} \tag{3.13}$$

where A is an absolute constant. For the last inequality, we used the formula

$$\Lambda_F(p) = \big(\alpha_1(p) + \cdots + \alpha_g(p)\big) \log p = a_F(p) \log p$$

deduced from Axiom 4. Then, using Axiom 5, we obtain that there exists a positive constant $X_0(\sigma; F)$ such that, for any $X \geqslant X_0(\sigma; F)$,

$$\sum_{p \geqslant X} \big|a_F(p)\big|^2 p^{-2\sigma} \geqslant c_1(\sigma)\kappa \frac{X^{1-2\sigma}}{\log X} \quad \text{and} \quad \sum_{p \geqslant X} p^{-2\sigma} \leqslant c_2(\sigma) \frac{X^{1-2\sigma}}{\log X},$$

where $c_1(\sigma)$ and $c_2(\sigma)$ are positive constants. Therefore, if $|x| + |y| \geqslant Z$ with $Z = Z(\sigma; F)$ large enough to satisfy $P_0 \geqslant X_0(\sigma; F)$, then by (3.13) we have

$$\begin{aligned} \Bigg|\prod_{p \geqslant P_0} \widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)\Bigg| &\leqslant \exp\Bigg(-A(|x| + |y|)^2 \sum_{p \geqslant P_0} \big|a_F(p)\big|^2 p^{-2\sigma} \\ &\qquad + Bc_0 g^2 \big(|x| + |y|\big)^2 \sum_{p \geqslant P_0} p^{-2\sigma}\Bigg) \\ &\leqslant \exp\Bigg(\big(-Ac_1(\sigma)\kappa + Bc_0 c_2(\sigma)\big)\big(|x| + |y|\big)^2 \frac{P_0^{1-2\sigma}}{\log P_0}\Bigg). \end{aligned} \tag{3.14}$$

If we take the constant c_0 such that $Ac_1(\sigma)\kappa > Bc_0 c_2(\sigma)$, then (3.14) is

$$\leqslant \exp\Bigg(-c_3\big(|x| + |y|\big)^2 \frac{P_0^{1-2\sigma}}{\log P_0}\Bigg)$$

with positive constant $c_3 = c_3(\sigma; F)$. Since $(|x|+|y|)^2 P_0^{1-2\sigma}(\log P_0)^{-1} \gg (|x|+|y|)^{1/\sigma-\epsilon}$ with the implied constant depending only on σ, ϵ, and F, we obtain

$$\Bigl|\prod_{p\geqslant P_0} \widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)\Bigr| \leqslant \exp\bigl(-c\bigl(|x|+|y|\bigr)^{1/\sigma-\epsilon}\bigr) \tag{3.15}$$

with some constant $c = c(\sigma, \epsilon; F) > 0$.

The estimate of $\widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)$ for $p < P_0$ is as follows. From the definition we have

$$\bigl|\widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)\bigr| \leqslant \int_0^1 \exp\bigl(-\Im(z_1)A_p(\theta, \sigma; F) - \Im(z_2)B_p(\theta, \sigma; F)\bigr)\,\mathrm{d}\theta.$$

Note that $|\Im(z_1)| < 1/2$ and $|\Im(z_2)| < 1/2$ due to $|z_1 - x| < 1/2$ and $|z_2 - y| < 1/2$. Hence we estimate $|\widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)|$ as

$$\bigl|\widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)\bigr| \leqslant \int_0^1 \exp\biggl(\frac{1}{2}\bigl(\bigl|A_p(\theta, \sigma; F)\bigr| + \bigl|B_p(\theta, \sigma; F)\bigr|\bigr)\biggr)\,\mathrm{d}\theta \leqslant \exp\bigl(4gp^{-1/2}\bigr),$$

and we have

$$\Bigl|\prod_{p>P_0} \widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)\Bigr| \leqslant \exp\biggl(4g\sum_{p>P_0} p^{-1/2}\biggr) \leqslant \exp\bigl(8gP_0^{1/2}\bigr).$$

Therefore, since $P_0^{1/2} \ll (|x|+|y|)^{1/(2\sigma)}$ with the implied constant depending only on F and σ, we obtain

$$\Bigl|\prod_{p>P_0} \widetilde{\mathcal{M}}_p(\sigma, z_1, z_2; F)\Bigr| \leqslant \exp\bigl(c'\bigl(|x|+|y|\bigr)^{1/(2\sigma)}\bigr) \tag{3.16}$$

with positive constant $c' = c'(\sigma; F)$. By estimates (3.15) and (3.16) we have the result. □

We assume that (s, z_1, z_2) varies in $\{\Re s > 1/2\} \times \mathbb{C} \times \mathbb{C}$. If we fix two of the variables, every local parts $\tilde{M}_p(s, z_1, z_2; F)$ are holomorphic with respect to the reminder variable. Hence, by Lemma 6, the function $\tilde{M}(s, z_1, z_2; F)$ is also holomorphic. Therefore Cauchy's integral formula can be applied, and we deduce from Lemma 7 that

$$\frac{\partial^{m+n}}{\partial x^m \partial y^n}\widetilde{\mathcal{M}}_\sigma(x + \mathrm{i}y; F) \ll_{m,n} \exp\bigl(-c\bigl(|x|+|y|\bigr)^{1/\sigma-\epsilon}\bigr)$$

for any $m, n \geqslant 0$. This implies that the function $\widetilde{\mathcal{M}}_\sigma(z; F)$ is a Schwartz function on $\mathbb{C}$, and hence it belongs to the class Λ, as desired in Proposition 3.

Proof of Lemma 5. We use formula (3.11). For $p \geqslant X^2$, we see that

$$m_p \ll g^2\bigl(x^2 + y^2\bigr)p^{-2\sigma} \ll \bigl\{g\bigl(|x|+|y|\bigr)X^{-2\sigma}\bigr\}^2.$$

Since $|x|, |y| \leqslant (\log T)^\eta$ and $X = \exp((\log T)^\theta)$, we have

$$\bigl\{g\bigl(|x|+|y|\bigr)X^{-2\sigma}\bigr\}^2 \leqslant \exp\bigl(-c(\sigma)(\log T)^\theta\bigr) \to 0$$

as $T \to \infty$. Similarly, we have $r_p \to 0$ as $T \to \infty$. Hence we obtain

$$\operatorname{Log} \widetilde{\mathcal{M}}_{\sigma,p}(z;F) \ll \big(x^2+y^2\big)p^{-2\sigma}$$

for sufficiently large T, where the implied constant depends only on F. Therefore

$$\begin{aligned}\prod_{p \geqslant X^2} \widetilde{\mathcal{M}}_{\sigma,p}(z;F) &= \exp\bigg(\sum_{p \geqslant X^2} \operatorname{Log} \widetilde{\mathcal{M}}_{\sigma,p}(z;F)\bigg) = 1 + O\bigg(\big(x^2+y^2\big)\sum_{p \geqslant X^2} p^{-2\sigma}\bigg)\\ &= 1 + O\big(\exp\big(-c(\sigma)(\log T)^{\theta}\big)\big)\end{aligned}$$

with a positive constant $c(\sigma)$, and the result follows. □

4 Application of the zero density estimate

We explain how to use Axiom 6 for the proof of Theorem 2. From the previous arguments we see that all we have to do is to modify the proof of Lemma 1 in Section 3.1.

Lemma 8. *Let $F \in \mathcal{S}_{II}$, and let $\sigma > 1/2$ be a fixed real number. Then there exists an absolute constant $T_0 > 0$ such that*

$$\frac{1}{T}\int_0^T \psi_z\big(\log F(\sigma+\mathrm{i}t)\big)\,\mathrm{d}t = \frac{1}{T}\int_0^T \psi_z\big(F_X(\sigma+\mathrm{i}t;F)\big)\,\mathrm{d}t + E_1$$

for all $T \geqslant T_0$ and $z \in \mathbb{C}$. The error term E_1 is estimated as

$$\begin{aligned}E_1 \ll{} & \frac{1}{T} + YT^{-c(\sigma-1/2)/2}(\log T)^A\\ &+ \frac{|z|}{\log X}\bigg(\frac{X\log Y\log T}{Y} + \frac{X^{-(\sigma-1/2)/2}\log T}{(\sigma-\frac{1}{2})^2} + \frac{X}{T} + X^{-1/2}\log^2 T\bigg)\end{aligned}$$

for any $X, Y > 1$, where the implied constant depends only on F and σ.

Proof. We define $\mathscr{B}_Y(\sigma,T;F)$ as the set of all $t \in [0,T]$ such that there exists a zero $\rho = \beta + \mathrm{i}\gamma$ of $F(s)$ with $\beta \geqslant (\sigma+1/2)/2$ and $|\gamma - t| \leqslant Y$. Then we obtain

$$E_1 \ll \frac{1}{T} + \frac{\mu_1(\mathscr{B}_Y(\sigma,T;F))}{T} + \frac{|z|}{T}\int_{[2,T]\cap\mathscr{B}_Y(\sigma,T;F)^c} \big|\log F(\sigma+\mathrm{i}t) - F_X(t,\sigma;F)\big|\,\mathrm{d}t.$$

We use Axiom 6 to estimate the second term:

$$\frac{\mu_1(\mathscr{B}_Y(\sigma,T;F))}{T} \leqslant \frac{2Y}{T}N_F\bigg(\frac{\sigma+\frac{1}{2}}{2},T\bigg) \ll YT^{-c(\sigma-1/2)/2}(\log T)^A.$$

The remainder estimates are established by the method quite similar to Lemma 1. □

We need no modification for all lemmas in Section 3 other than Lemma 1. Combining them, we can prove an analogue of Proposition 1 for $F \in \mathcal{S}_{II}$ as follows.

Proposition 4. *Let $F \in \mathcal{S}_{II}$. Let η, θ be real numbers with $2\eta + 3\theta < 1$. Let $\sigma > 1/2$ be a fixed real number. Let $\Phi \in \Lambda$. Then there exists a constant $T_0 = T_0(F, \eta, \theta, \sigma) > 0$ such that*

$$\frac{1}{T}\int_0^T \Phi\bigl(\log F(\sigma + \mathrm{i}t)\bigr)\,\mathrm{d}t = \int_{\mathbb{C}} \Phi(z)\mathcal{M}_\sigma(z; F)\,|\mathrm{d}z| + E$$

for all $T \geqslant T_0$, where the error term E is estimated as

$$E \ll \exp\bigl(-c(F,\sigma)(\log T)^\theta\bigr)\int_{\Omega} \bigl|\Phi^\wedge(z)\bigr|\,|\mathrm{d}z| + \int_{\mathbb{C}\setminus\Omega} \bigl|\Phi^\wedge(z)\bigr|\,|\mathrm{d}z|.$$

Here $c(F,\sigma)$ is a positive constant, the region Ω is the rectangle in (3.1)*, and the implied constant depends only on F and σ.*

Therefore we obtain Theorem 2 in the same way as Theorem 1.

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