



Transformations and singularities of polarized curves

Andreas Fuchs¹ 

Received: 29 June 2018 / Accepted: 9 November 2018 / Published online: 19 November 2018
© The Author(s) 2018

Abstract

We study the limiting behaviour of Darboux and Calapso transforms of polarized curves in the conformal n -dimensional sphere when the polarization has a pole of first or second order at some point. We prove that for a pole of first order, as the singularity is approached, all Darboux transforms converge to the original curve and all Calapso transforms converge. For a pole of second order, a generic Darboux transform converges to the original curve while a Calapso transform has a limit point or a limit circle, depending on the value of the transformation parameter. In particular, our results apply to Darboux and Calapso transforms of isothermic surfaces when a singular umbilic with index $\frac{1}{2}$ or 1 is approached along a curvature line.

Keywords Isothermic surfaces · Singular umbilics · Polarized curve · Darboux transformation · Calapso transformation

Mathematics Subject Classification 53A30 · 37K35 · 37K40 · 58K45

1 Introduction

Transformations of surfaces play a central role in our present understanding of smooth and discrete differential geometry. Not only do they allow the construction of new surfaces of a given class from existing ones, but their existence reveals a great deal about the underlying (integrable) structure of the corresponding classes of surfaces, c.f. [5]. For example, given a pseudospherical surface ($K = -1$), the Bäcklund transformation yields a two-parameter family of new pseudospherical surfaces by solving an integrable, first-order partial differential equation (see [11, Sect. 120]). In [22], these Bäcklund transformations are shown to be generators of an infinite-dimensional transformation group of the space of pseudospherical surfaces and the relations among these generators are given by a permutability theorem discovered by Bianchi [1, Sect. 257] in the nineteenth century. In particular, it is shown that pseudospherical surfaces fall into the class of integrable systems. Similar developments have been achieved for other classes of surfaces such as minimal surfaces, constant mean curvature

✉ Andreas Fuchs
afuchs@geometrie.tuwien.ac.at

¹ Vienna, Austria

surfaces, constant Gaussian curvature surfaces (c.f. [2]) and curved flats in symmetric spaces such as Darboux pairs of isothermic surfaces, c.f. [7,9,12].

In the discrete theory, the importance of transformations becomes even more apparent, as articulated in [3] (see also [4]): “In this setting, discrete surfaces appear as two-dimensional layers of multidimensional discrete nets, and their transformations correspond to shifts in the transversal lattice directions. A characteristic feature of the theory is that all lattice directions are on equal footing with respect to the defining geometric properties. Due to this symmetry, discrete surfaces and their transformations become indistinguishable.”

The interplay between aspects of discrete and smooth differential geometry was explored in [6] with the study of semi-discrete isothermic surfaces, introduced in [18]. In analogy to the transformation theory of smooth isothermic surfaces (see [15, Chap. 8.6], [5] or [8]), the authors develop a notion of Christoffel, Darboux and Calapso transformations of polarized curves, that is, smooth curves equipped with a nowhere zero quadratic differential, called a polarization. They then show that semi-discrete isothermic surfaces are sequences of Darboux transforms of polarized curves. In line with the ideas of discrete differential geometry, permutability theorems of the transformations of polarized curves are shown to yield a corresponding transformation theory for semi-discrete isothermic surfaces.

We are interested in the class of smooth isothermic surfaces, classically characterized by the local existence of conformal curvature line coordinates, away from umbilics. For these surfaces, the transformation theory is defined only locally, that is, on simply connected surface patches on which regular nets of conformal curvature line coordinates exist (c.f. [5,15,20]). A global transformation theory is still missing. In particular, it seems necessary to reconsider the definition of an isothermic surface. The classical definition makes no restriction whatsoever on umbilic points. But certain configurations of umbilics are an obstacle for a global definition of the transformations (see [15, Sect. 5.2.20]). One candidate for an alternative definition of an isothermic surface is to require the existence of a globally defined holomorphic quadratic differential whose trajectories (see [21, Sect. 5.5]) agree with the curvature lines of the surface on the complement of its umbilic set. But due to the Poincaré–Hopf theorem [16], this would exclude all surfaces homeomorphic to a sphere, for example the ellipsoid, c.f. [20]. If we merely require the existence of a meromorphic quadratic differential, that includes topological spheres. Moreover, according to the local Carathéodory conjecture, the poles of such a meromorphic quadratic differential are at most of order two (see [14]).

This paper makes a first step towards a global transformation theory of isothermic surfaces for which an underlying meromorphic quadratic differential exists. A curvature line of an isothermic surface together with the restriction of such a meromorphic quadratic differential yields a polarized curve in the sense of [6, Sect. 2]. Moreover, the transformations for isothermic surfaces descend to the corresponding transformations of the polarized curvature lines. Thus, in order to understand how the transforms of an isothermic surface behave when a pole of the underlying meromorphic quadratic differential is approached along a curvature line, we investigate how Darboux and Calapso transforms of polarized curves with singular polarizations behave when the singularity is approached. In particular, we investigate the limiting behaviour of Darboux and Calapso transforms of polarized curves where the polarization has a pole of first or second order. According to the local Carathéodory conjecture, these are the only poles that can occur on a curvature line of an isothermic surface.

In Sect. 2, we give Möbius geometric definitions of the Darboux and Calapso transformations of polarized curves in the conformal n -dimensional sphere S^n . Our definitions are formulated with a projective model of Möbius geometry: we identify S^n with the projectivization $\mathbb{P}(\mathcal{L}^{n+1})$ of the light cone \mathcal{L}^{n+1} in $(n+2)$ -dimensional Minkowski space. The Möbius group can then be identified with the projective Lorentz group $\mathbb{P}O\left(\mathbb{R}^{n+2}\right)$. Curves

are described as maps into the projective light cone. For computations and in order to prove that our definitions agree with those of [6], we relate the projective model to its linearization based on the action of $O\left(\mathbb{R}_1^{n+2}\right)$ on \mathcal{L}^{n+1} , where curves are described via their light cone lifts. We then give a definition of a polarization with a pole of first or second order.

The central object in our definition of the transformations (Definition 1) is the primitives $\Gamma_p(\lambda\omega)$ of a family $(\lambda\omega)_{\lambda\in\mathbb{R}}$ of 1-forms associated with a polarized curve. Here, the primitive of a 1-form ψ with values in the Lie algebra $\mathfrak{po}\left(\mathbb{R}_1^{n+2}\right)$ of $\mathbb{P}O\left(\mathbb{R}_1^{n+2}\right)$ is the unique map $t \mapsto \Gamma_p^t(\psi)$ into $\mathbb{P}O\left(\mathbb{R}_1^{n+2}\right)$ which satisfies

$$d\Gamma_p(\psi) = \Gamma_p(\psi)\psi, \quad \Gamma_p^p(\psi) = \text{id},$$

c.f. [19]. When the polarization has a pole of first or second order at some point, the 1-forms $\lambda\omega$ associated with the polarized curve also have a pole at that point and hence the primitives are not defined there. Nevertheless, as we prove in Corollary 2 and Proposition 2 in Sect. 3, the primitives of a certain class of $\mathfrak{po}\left(\mathbb{R}_1^{n+2}\right)$ -valued 1-forms with a pole of first order do have limits in $\mathbb{P}End\left(\mathbb{R}_1^{n+2}\right) \supset \mathbb{P}O\left(\mathbb{R}_1^{n+2}\right)$ at the singular point.

In Sect. 4, we use these results to show that when the polarization of a polarized curve has a pole of first order, every Calapso transform converges to some point and all Darboux transforms converge to the original curve.

In Sect. 5, we consider the case of a polarization with a pole of second order. We cannot apply the results of Sect. 3 here directly because the 1-forms $\lambda\omega$ associated with the polarized curve have poles of second order. We first have to do a singular gauge transformation to transform the 1-forms with poles of second order to 1-forms with a pole of first order. The behaviour of the Darboux and Calapso transforms in this case is more diverse. A generic Darboux transform still converges to the original curve as the singularity is approached, but there are Darboux transforms which do not converge. Moreover, the Calapso transform has a limit point or a limit circle, depending on the value of the transformation parameter λ .

2 Darboux and Calapso transforms of polarized curves

In this section, we define Darboux and Calapso transforms of polarized curves, show that our definitions agree with those of [6] and specify the goal of this paper: the study of the limiting behaviour of Darboux and Calapso transforms at points where the polarization has a pole of first or second order.

We use the projective model of Möbius geometry (c.f. [5, Sect. 1.1.] and [15, Chap. 1]) and identify the conformal n -sphere S^n with the projectivization $\mathbb{P}(\mathcal{L}^{n+1})$ of the light cone in $(n+2)$ -dimensional Minkowski space. In this way, the action of the Möbius group on S^n can be identified with the action of the projective Lorentz group $\mathbb{P}O\left(\mathbb{R}_1^{n+2}\right)$ on $\mathbb{P}(\mathcal{L}^{n+1})$. We view $\mathbb{P}O\left(\mathbb{R}_1^{n+2}\right)$ as a subgroup of the group of projective transformations of $\mathbb{P}\left(\mathbb{R}_1^{n+2}\right)$, which in turn is a submanifold of $\mathbb{P}End\left(\mathbb{R}_1^{n+2}\right)$. Taking the linear span $\langle \cdot \rangle$ of elements of $End\left(\mathbb{R}_1^{n+2}\right)$ then provides a diffeomorphism from the group $O^+\left(\mathbb{R}_1^{n+2}\right) \subset End\left(\mathbb{R}_1^{n+2}\right)$ of orthochronous Lorentz transformations to $\mathbb{P}O\left(\mathbb{R}_1^{n+2}\right)$. Similarly, we view the Lie algebra $\mathfrak{po}\left(\mathbb{R}_1^{n+2}\right)$ as a subspace of the tangent space $End\left(\mathbb{R}_1^{n+2}\right)/\mathbb{R}\text{id}$ of $\mathbb{P}End\left(\mathbb{R}_1^{n+2}\right)$ at the

identity. The differential of $\langle \cdot \rangle$ at the identity then restricts to an isomorphism of Lie algebras

$$\begin{aligned} \text{d}_{\text{id}} \langle \cdot \rangle : \text{End} \left(\mathbb{R}_1^{n+2} \right) \supset \mathfrak{o} \left(\mathbb{R}_1^{n+2} \right) &\rightarrow \mathfrak{po} \left(\mathbb{R}_1^{n+2} \right) \subset \text{End} \left(\mathbb{R}_1^{n+2} \right) / \mathbb{R} \text{id}, \\ v \wedge w &\mapsto v \wedge w + \mathbb{R} \text{id}, \end{aligned} \tag{1}$$

where we further identify $\Lambda^2 \left(\mathbb{R}_1^{n+2} \right)$ with $\mathfrak{o} \left(\mathbb{R}_1^{n+2} \right)$ via

$$(v \wedge w)(x) = \langle\langle v, x \rangle\rangle w - \langle\langle w, x \rangle\rangle v$$

for $v, w, x \in \mathbb{R}_1^{n+2}$. Here, $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the Minkowski inner product on \mathbb{R}_1^{n+2} .

A regular curve $\langle c \rangle$ in $S^n = \mathbb{P}(\mathcal{L}^{n+1})$ is an immersion of some interval (a, b) into $\mathbb{P}(\mathcal{L}^{n+1})$, where the notation $\langle c \rangle$ indicates that such a map may also be described by pointwise taking the linear span of a light cone lift $c : (a, b) \rightarrow \mathcal{L}^{n+1}$ of $\langle c \rangle$. For simplicity, we always assume $\langle c \rangle$ to be smooth. A polarized curve $(\langle c \rangle, Q)$ in S^n is a regular curve $\langle c \rangle$ in S^n together with a nowhere zero quadratic differential Q on (a, b) . To a polarized curve $(\langle c \rangle, Q)$, we associate the $\mathfrak{po} \left(\mathbb{R}_1^{n+2} \right)$ -valued 1-form ω given by

$$\omega = \Omega c \wedge \text{d}c + \mathbb{R} \text{id}, \tag{2}$$

where c is any light cone lift of $\langle c \rangle$ and the function Ω is related to Q via that lift c by

$$Q = \Omega \| \text{d}c \|^2.$$

The 1-form ω is independent of the choice of lift c of $\langle c \rangle$.

Let G be a Lie group with Lie algebra \mathfrak{g} and ψ a \mathfrak{g} -valued 1-form on (a, b) . Then, for any $p \in (a, b)$, denote by

$$\Gamma_p(\psi) : (a, b) \rightarrow G, \quad t \mapsto \Gamma_p^t(\psi),$$

the unique primitive that satisfies (c.f. [19, Chap. 3, Theorem 6.1])

$$\text{d} \Gamma_p(\psi) = \Gamma_p(\psi) \psi, \quad \Gamma_p^p(\psi) = \text{id}. \tag{3}$$

From the defining properties (3) of $\Gamma_p(\psi)$ and its uniqueness, it follows readily that

$$\forall p, t \in (a, b) : \Gamma_p^t(\psi) \Gamma_t(\psi) = \Gamma_p(\psi). \tag{4}$$

Therefore, the map $\Gamma^p(\psi) : (a, b) \ni t \mapsto \Gamma_t^p(\psi) \in G$ is the composition of $\Gamma_p(\psi)$ with taking the inverse in G and thus satisfies

$$\text{d} \Gamma^p(\psi) = -\psi \Gamma^p(\psi), \quad \Gamma_p^p(\psi) = \text{id}.$$

Under a gauge transformation

$$\psi \mapsto g \times \psi := g^{-1} \psi g + g^{-1} \text{d}g$$

using a smooth map $g : (a, b) \rightarrow G$, the primitives $\Gamma_p(\psi)$ transform as

$$\Gamma_p(\psi) = g(p) \Gamma_p(g \times \psi) g^{-1}. \tag{5}$$

Definition 1 Let $(\langle c \rangle, Q)$ be a polarized curve in S^n with associated 1-form ω given by (2).

For $\lambda \in \mathbb{R} \setminus \{0\}$, $p \in (a, b)$ and $\langle \hat{c}_p \rangle \in S^n$, the curve

$$\langle \hat{c} \rangle := \Gamma^p(\lambda \omega) \langle \hat{c}_p \rangle$$

is called the λ -Darboux transform of $(\langle c \rangle, Q)$ with initial point $\langle \hat{c}(p) \rangle = \langle \hat{c}_p \rangle$.

For $\lambda \in \mathbb{R}$ and $p \in (a, b)$, the curve

$$\langle c_{\lambda,p} \rangle := \Gamma_p(\lambda\omega)\langle c \rangle$$

is called the λ -Calapso transform of $(\langle c \rangle, Q)$ normalized at p .

Any λ -Calapso transform of a polarized curve $(\langle c \rangle, Q)$ is an immersion. Similarly, any λ -Darboux transform with initial point $\langle \hat{c}_p \rangle$ not lying on the image of $\langle c_{\lambda,p} \rangle$ immerses.¹ One may turn the immersed transforms of $(\langle c \rangle, Q)$ into polarized curves themselves by equipping them with some polarization, for example again Q . This is useful in the study of repeated transforms of polarized curves, as it is done in [6]. However, in this work we do not study repeated transforms and thus, for simplicity, we consider the transforms merely as curves and not as polarized curves.

We remark that due to (4), two λ -Calapso transforms normalized at p and q , respectively, differ by a Möbius transformation. Moreover, the Darboux and Calapso transformations of a polarized curve $(\langle c \rangle, Q)$ in S^n are invariant under Möbius transformations of S^n and reparametrizations of $\langle c \rangle$.

This paper is devoted to the study of the limiting behaviour of the Darboux and Calapso transforms at a in the case that $\langle c \rangle$ can be extended regularly to some $(a - \epsilon, b)$, but the polarization has a pole of first or second order at a in the sense of

Definition 2 A quadratic differential $Q : (a, b) \ni t \mapsto Q_t = Q(t) dt^2$ or a Lie algebra-valued 1-form $\psi : (a, b) \ni t \mapsto \psi_t = \psi(t) dt$ on (a, b) has a pole of order $k \in \mathbb{N}$ at a if the function $t \mapsto Q(t)(a - t)^k$ or $t \mapsto \psi(t)(a - t)^k$, respectively, has a smooth extension to $(a - \epsilon, b)$ for some $\epsilon > 0$ with nonzero value at a .

If $\langle c \rangle$ has a smooth extension to $(a - \epsilon, b)$, then the polarization Q has a pole of first or second order at a if and only if the 1-form ω associated with $(\langle c \rangle, Q)$ has a pole of first or second order at a .

The above definition of a pole is invariant under those diffeomorphisms of (a, b) to (\tilde{a}, \tilde{b}) which extend to diffeomorphisms of some larger interval $(a - \epsilon, b)$ to $(\tilde{a} - \tilde{\epsilon}, \tilde{b})$ with $\epsilon, \tilde{\epsilon} > 0$, for example the translation

$$(a, b) \ni t \mapsto t - a \in (0, b - a).$$

Together with the reparametrization invariance of the Darboux and Calapso transformation, we can thus make the convenient choice $a = 0$.

Our definitions of the transformations of a polarized curve are formulated entirely in the projective model of Möbius geometry, where $\mathbb{P}O(\mathbb{R}_1^{n+2})$ acts on $\mathbb{P}(\mathcal{L}^{n+1})$ and surfaces are described as immersions into $\mathbb{P}(\mathcal{L}^{n+1})$. Especially in computations, a linearization of this model proves useful, where $O^+(\mathbb{R}_1^{n+2})$ acts on the light cone \mathcal{L}^{n+1} and surfaces are described via their light cone lifts as immersions into \mathcal{L}^{n+1} . Let ψ be a $\mathfrak{po}(\mathbb{R}_1^{n+2})$ -valued 1-form on $(0, b)$ and Ψ its orthogonal lift defined by $\Psi + \mathbb{R} \text{id} = \psi$ via the isomorphism (1). Then, the $O^+(\mathbb{R}_1^{n+2})$ -valued primitives $\Gamma_p(\Psi)$ satisfy

$$\forall p \in (0, b) : d\langle \Gamma_p(\Psi) \rangle = \langle \Gamma_p(\Psi) \rangle d\langle \cdot \rangle(\Psi) = \langle \Gamma_p(\Psi) \rangle \psi \quad \text{and} \quad \langle \Gamma_p^p(\Psi) \rangle = \text{id}.$$

¹ One can easily show that $\langle \hat{c}_p \rangle$ does not lie on the image of $\langle c_{\lambda,p} \rangle$ if and only if $\langle \hat{c}(t) \rangle$ is different from $\langle c(t) \rangle$ for all $t \in (a, b)$, which is the analogue of condition (1) of Definition 1.7 in [8] for polarized curves.

Thus, the maps $\langle \Gamma_p(\psi) \rangle$ are exactly the primitives of ψ . More generally, for any real, nowhere vanishing function h on $(0, b)$, we have

$$\forall p \in (0, b) : \Gamma_p(\psi) = \Gamma_p(\Psi + \mathbb{R} \text{id}) = \langle \Gamma_p(\Psi) \rangle = \langle h \Gamma_p(\Psi) \rangle. \tag{6}$$

In particular, this identity allows to compute the primitives of a $\mathfrak{po}(\mathbb{R}_1^{n+2})$ -valued 1-form via arbitrary nowhere zero rescalings of the primitives of its orthogonal lift.

Relation (6) also enables us to show that our Definition 1 agrees with the corresponding ones in [6]. Namely, let $\langle (c), Q \rangle$ be a polarized curve with associated 1-form ω and orthogonal lift Ω . Using (6), it follows that with $\hat{c}_p \in \langle \hat{c}_p \rangle$ the lift $\hat{c} = \Gamma^p(\lambda\Omega)\hat{c}_p$ of a λ -Darboux transform $\langle \hat{c} \rangle$ of $\langle (c), Q \rangle$ as defined in Definition 1 is a parallel section of the connection

$$\frac{D^\lambda}{dt} = \frac{d}{dt} + \lambda\Omega c \wedge c' \quad \text{with} \quad \Omega \| dc\|^2 = Q,$$

and hence yields a $(-\frac{\lambda}{2})$ -Darboux transform² of $\langle (c), Q \rangle$ in the sense of [6, Definition 2.5]. Conversely, any parallel section \hat{c} of $\frac{D^\lambda}{dt}$ yields a λ -Darboux transform $\langle \hat{c} \rangle$ according to our Definition 1.

Similarly, the λ -Calapso transform normalized at some $p \in (0, b)$ up to Möbius transformation corresponds to the $(-\frac{\lambda}{2})$ -Calapso transform defined in [6, Definition 3.4] because, up to Möbius transformation, their Calapso transformation $T^{-\lambda/2}$ is precisely $\Gamma_p(\lambda\Omega)$.

Not surprisingly, the main advantage of the linearization of the projective formalism is its linear structure. For example, integration of (3) shows that the primitives of an $\mathfrak{o}(\mathbb{R}_1^{n+2})$ -valued 1-form Ψ on $(0, b)$ satisfy the useful integral equation in $End(\mathbb{R}_1^{n+2}) \simeq \mathbb{R}^{(n+2)^2}$

$$\forall p, t \in (0, b) : \Gamma_p^t(\Psi) = \text{id} + \int_p^t \Gamma_p^\tau(\Psi)\Psi(\tau) d\tau. \tag{7}$$

Its disadvantage, on the other hand, is its reliance on the choice of lifts. As we will see in Sect. 3, when a $\mathfrak{po}(\mathbb{R}_1^{n+2})$ -valued 1-form ψ and thus also its orthogonal lift Ψ on $(0, b)$ have a pole, it may happen that the $O^+(\mathbb{R}_1^{n+2})$ -valued primitives $\Gamma_p(\Psi)$ tend towards infinity as the pole is approached, while the corresponding $\mathbb{P}O(\mathbb{R}_1^{n+2})$ -valued primitives $\Gamma_p(\psi)$ have limits in $\mathbb{P}End(\mathbb{R}_1^{n+2}) \supset \mathbb{P}O(\mathbb{R}_1^{n+2})$. If this is the case for the 1-form ω associated with a polarized curve and its orthogonal lift Ω , then the lifts $\hat{c} = \Gamma^p(\lambda\Omega)\hat{c}_p$ and $c_{\lambda,p} = \Gamma_p(\lambda\Omega)c$ diverge as the singularity is approached, while the corresponding maps $\langle \hat{c} \rangle$ and $\langle c_{\lambda,p} \rangle$ converge. In such a situation, a lift-independent formalism is clearly beneficial.

3 Main tools

In this section, we study the limiting behaviour at 0 of the primitives $\Gamma_p(\psi)$ of a certain class of $\mathfrak{po}(\mathbb{R}_1^{n+2})$ -valued 1-forms ψ on $(0, b)$ with a pole of first order at 0. In Sect. 3.1, we restrict to 1-forms of a particularly simple form, which we call *pure pole forms*. For those, the primitives and their limiting behaviour at 0 can be computed explicitly. In Sect. 3.2, we then relate the limiting behaviour of primitives of the more general pole forms to that of the

² Our parameter λ is -2 times the parameter t used in [6].

pure pole forms. In view of our definition of the transformations of a polarized curve via the primitives of multiples $\lambda\omega$ of the associated 1-form ω (Definition 1), these tools will provide the means to investigate the limiting behaviour of the transforms of a singular polarized curve in Sects. 4 and 5.

3.1 Primitives of pure pole forms

We call a $\mathfrak{po}(\mathbb{R}_1^{n+2})$ -valued 1-form ξ on $(0, b)$ a *pure pole form* if it is of the form

$$\xi_t = -v \wedge w \frac{dt}{t} + \mathbb{R} \text{ id}$$

for some $v, w \in \mathbb{R}_1^{n+2}$ with $v \wedge w \neq 0$. Pure pole forms come in three types corresponding to the three signatures which the linear span $\langle v, w \rangle \subset \mathbb{R}_1^{n+2}$ can have. Accordingly, we speak of Minkowski, degenerate and spacelike pure pole forms.

In addition to the quadratic form $\|\cdot\|^2$ induced by the indefinite Minkowski inner product $\langle \cdot, \cdot \rangle$, we introduce the standard, Euclidean, positive-definite norm $|\cdot|$ on \mathbb{R}^{n+2} and $\mathbb{R}^{(n+2)^2}$. On $\text{End}(\mathbb{R}_1^{n+2}) \simeq \mathbb{R}^{(n+2)^2}$, the norm $|\cdot|$ is submultiplicative and satisfies $|A^*| = |A|$ for A^* the adjoint of $A \in \text{End}(\mathbb{R}_1^{n+2})$ with respect to the *Minkowski* inner product on \mathbb{R}_1^{n+2} . On $\mathfrak{po}(\mathbb{R}_1^{n+2})$, we define $|\cdot|$ by the push forward of $|\cdot|$ restricted to $\mathfrak{o}(\mathbb{R}_1^{n+2}) \subset \mathbb{R}^{(n+2)^2}$ via the isomorphism (1). Furthermore, for $w \in \mathbb{R}_1^{n+2}$, we denote by w^* the linear map $w^* : v \mapsto \langle v, w \rangle$.

Proposition 1 *Let ξ be a pure pole form on $(0, b)$ and \mathfrak{E} its orthogonal lift. Then, there is a constant $B \in \mathbb{R}$ such that*

1. *if ξ is Minkowski, then*

$$\forall p \in (0, b) \forall t \in (0, p] : \left| \left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\mathfrak{E}) \right| < B, \tag{8}$$

$$\forall p \in (0, b) : \lim_{t \rightarrow 0} \left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\mathfrak{E}) = \frac{v_+ v_-^*}{\langle\langle v_+, v_- \rangle\rangle}, \tag{9}$$

where $v_\pm \in \langle v, w \rangle \cap \mathcal{L}^{n+1}$ are eigenvectors of $v \wedge w$ with eigenvalues $\pm\zeta$, $\zeta > 0$;

2. *if ξ is spacelike, then*

$$\forall p, t \in (0, b) : \left| \Gamma_p^t(\mathfrak{E}) \right| < B, \tag{10}$$

but $\Gamma_p(\xi)$ does not converge as 0 is approached as it is a rotation in the plane $\langle v, w \rangle$ with speed increasing towards infinity;

3. *if ξ is degenerate, choose $v_0 \in \langle v, w \rangle \cap \mathcal{L}^{n+1}$ and $\tilde{w} \in \langle v, w \rangle$ such that $v \wedge w = v_0 \wedge \tilde{w}$. Then,*

$$\forall p, t \in (0, b) : \left| \frac{1}{\ell(p, t)} \Gamma_p^t(\mathfrak{E}) \right| < B, \tag{11}$$

$$\forall p \in (0, b) : \lim_{t \rightarrow 0} \frac{1}{\ell(p, t)} \Gamma_p^t(\mathfrak{E}) = -\frac{\|\tilde{w}\|^2}{2} v_0 v_0^*, \tag{12}$$

where $\ell : (0, b)^2 \rightarrow \mathbb{R}$ is the symmetric function

$$\ell : (p, t) \mapsto 1 + \left(\ln \left(\frac{t}{p} \right) \right)^2. \tag{13}$$

Proof First we note that $\Gamma_p^t(\mathfrak{E})$ is given by the exponential

$$\Gamma_p^t(\mathfrak{E}) = e^{-\int_p^t \frac{d\tau}{\tau} v \wedge w} = e^{-\ln\left(\frac{t}{p}\right)v \wedge w}. \tag{14}$$

1. For $\langle v, w \rangle$ Minkowski, we may choose $v_{\pm} \in \mathcal{L}^{n+1}$ such that $\langle v_+, v_- \rangle > 0$ and $v \wedge w = v_- \wedge v_+$. Then, v_{\pm} are eigenvectors of $v \wedge w$ with eigenvalues $\pm\zeta = \pm\langle v_+, v_- \rangle$ and $\langle v_+, v_- \rangle^{\perp} = \ker(v \wedge w)$. Using that

$$(s v_- \wedge v_+)^k = (s\zeta)^k \left(\frac{v_+ v_-^*}{\zeta} \right)^k + (-s\zeta)^k \left(\frac{v_- v_+^*}{\zeta} \right)^k = (s\zeta)^k \frac{v_+ v_-^*}{\zeta} + (-s\zeta)^k \frac{v_- v_+^*}{\zeta}$$

for all $s \in \mathbb{R}$ and $k \in \mathbb{N}$, the exponential computes to

$$\Gamma_p^t(\mathfrak{E}) = e^{-\ln\left(\frac{t}{p}\right)v \wedge w} = \left(\text{id} - \frac{v_+ v_-^* + v_- v_+^*}{\zeta} \right) + \left(\frac{t}{p} \right)^{-\zeta} \frac{v_+ v_-^*}{\zeta} + \left(\frac{t}{p} \right)^{\zeta} \frac{v_- v_+^*}{\zeta}. \tag{15}$$

Therefore,

$$\left(\frac{t}{p} \right)^{\zeta} \Gamma_p^t(\mathfrak{E}) = \left(\frac{t}{p} \right)^{\zeta} \left(\text{id} - \frac{v_+ v_-^* + v_- v_+^*}{\zeta} \right) + \frac{v_+ v_-^*}{\zeta} + \left(\frac{t}{p} \right)^{2\zeta} \frac{v_- v_+^*}{\zeta}.$$

Since $\zeta > 0$, taking the limit $t \rightarrow 0$ yields (9). Since $|t/p| \leq 1$ for all $p \in (0, b)$ and $t \in (0, p]$, there is a constant $B \in \mathbb{R}$ such that (8) holds.

2. For $\langle v, w \rangle$ spacelike, $\Gamma_p^t(\mathfrak{E})$ takes values in a one-parameter subgroup $O(\mathbb{R}^2) \subset O^+(\mathbb{R}_1^{n+2})$ of Euclidean rotations. Since $O(\mathbb{R}^2)$ is compact, $\Gamma_p^t(\mathfrak{E})$ is bounded for all $p, t \in (0, b)$ such that (10) holds for some $B \in \mathbb{R}$. To see that $\Gamma_p(\xi)$ does not have a limit at 0, denote by $i\zeta$ a nonzero eigenvalue of $v \wedge w$. Under a Lie group isomorphism $O(\mathbb{R}^2) \rightarrow \mathbb{R} \bmod 2\pi$, the primitive $\Gamma_p^t(\mathfrak{E})$ gets mapped to $\pm \ln\left(\frac{t}{p}|\zeta|\right) \bmod 2\pi$ which does not have a limit for any $p \in (0, b)$ as t tends to 0. Hence, also $\Gamma_p^t(\mathfrak{E})$ does not have a limit for any $p \in (0, b)$ as t tends to 0.
3. Now let ξ be degenerate with $v_0 \in \langle v, w \rangle \cap \mathcal{L}^{n+1}$ and $\tilde{w} \in \langle v, w \rangle$ such that $v \wedge w = v_0 \wedge \tilde{w}$. Then, $(v \wedge w)^2 = -\|\tilde{w}\|^2 v_0 v_0^*$ and $(v \wedge w)^k = 0$ for all $k \geq 3$. With ℓ as in (13), we thus get

$$\frac{1}{\ell(p, t)} \Gamma_p^t(\mathfrak{E}) = \frac{\text{id} - \ln\left(\frac{t}{p}\right) v \wedge w - \frac{\left(\ln\left(\frac{t}{p}\right)\right)^2 \|\tilde{w}\|^2}{2} v_0 v_0^*}{1 + \left(\ln \left(\frac{t}{p} \right) \right)^2}. \tag{16}$$

Taking the limit $t \rightarrow 0$ yields (12). The bound (11) follows from the boundedness of $\frac{1}{1+x^2}, \frac{x}{1+x^2}, \frac{x^2}{1+x^2}$ for all $x \in \mathbb{R}$.

□

The one-parameter families $(\Gamma_p^t(\xi))_{t \in (0, b)}$ have simple Euclidean interpretations: For Minkowski ξ , stereographically project $S^n \setminus \{\langle v_+ \rangle\}$ to \mathbb{R}^n . Then, $(\Gamma_p^t(\xi))_{t \in (0, b)}$ acts as a family of similarities with centre $\langle v_- \rangle$ of \mathbb{R}^n and a t -dependent scale factor that tends towards

infinity as $t \rightarrow 0$. The limiting map $\langle v_+ v_+^* \rangle$ maps all points to the point at infinity, $\langle v_+ \rangle$, except the centre, $\langle v_- \rangle$, where $\langle v_+ v_+^* \rangle$ is not defined.

If ξ is spacelike, $(\Gamma_p^t(\xi))_{t \in (0, b)}$ takes values in a one-parameter subgroup $O(\mathbb{R}^2) \subset O^+(\mathbb{R}_1^{n+2})$ of Euclidean rotations. As t approaches 0, the rotation speed increases towards infinity.

For degenerate ξ and $v_0 \in \langle v, w \rangle \cap \mathcal{L}^{n+1}$, stereographically project $S^n \setminus \{v_0\}$ to \mathbb{R}^n . Then, $(\Gamma_p^t(\xi))_{t \in (0, b)}$ acts as a family of translations of \mathbb{R}^n by a vector of constant direction and t -dependent length which tends towards infinity as $t \rightarrow 0$. The limiting map $\langle v_0 v_0^* \rangle$ maps all points to the point at infinity, $\langle v_0 \rangle$, except $\langle v_0 \rangle$ itself, where $\langle v_0 v_0^* \rangle$ is not defined.

These limits are not elements of $\mathbb{P}O(\mathbb{R}_1^{n+2})$. Instead, they lie on the closure of $\mathbb{P}O(\mathbb{R}_1^{n+2})$ in $\mathbb{P}End(\mathbb{R}_1^{n+2})$. In fact, it can be shown (c.f. [13]) that the closure of $\mathbb{P}O(\mathbb{R}_1^{n+2})$ in $\mathbb{P}End(\mathbb{R}_1^{n+2})$ is compact and precisely the union of $\mathbb{P}O(\mathbb{R}_1^{n+2})$ and the set $\{\langle v w^* \rangle \mid v, w \in \mathcal{L}^{n+1}\} \subset \mathbb{P}End(\mathbb{R}_1^{n+2})$. In contrast, $O^+(\mathbb{R}_1^{n+2})$ is closed in $End(\mathbb{R}_1^{n+2})$, but not compact. The limits (9) and (12) can thus not be computed in $O^+(\mathbb{R}_1^{n+2})$, which demonstrates the advantage of the projective model.

3.2 Primitives of pole forms

In this section, we define pole forms and relate the limiting behaviour of their primitives at 0 to that of pure pole forms.

Definition 3 A $\mathfrak{po}(\mathbb{R}_1^{n+2})$ -valued 1-form ψ on $(0, b)$ is a *pole form* if $\psi - \xi$ is bounded with respect to the Euclidean norm $|\cdot|$ for a pure pole form ξ on $(0, b)$.

For a pole form ψ , the pure pole form ξ for which $\psi - \xi$ is bounded is clearly unique. Therefore, we may define

Definition 4 Let ψ be a pole form on $(0, b)$ with $\psi - \xi$ bounded for the pure pole form ξ . Then, ψ is called *Minkowski*, *spacelike* or *degenerate* according to whether ξ has that property. For Minkowski ψ , denote by ζ the positive eigenvalue of $v \wedge w$. If ψ is Minkowski with $\zeta < 1$, spacelike or degenerate, we say that ψ is *of the first kind*. Otherwise, it is *of the second kind*.

We remark that these properties of a pole form are invariant under those diffeomorphisms of $(0, b)$ to (\tilde{a}, \tilde{b}) which extend to diffeomorphisms of $(-\epsilon, b)$ to $(\tilde{a} - \tilde{\epsilon}, \tilde{b})$ for some $\epsilon, \tilde{\epsilon} > 0$ (c.f. [13]).

Different techniques are necessary to investigate the limiting behaviour of primitives of pole forms of the first and of the second kind. We start with those of the first kind.

Lemma 1 Let ξ be a pure pole form of the first kind on $(0, b)$, and let χ be a bounded, continuous map from $(0, b)$ to $\mathfrak{po}(\mathbb{R}_1^{n+2})$. Then, the family $(\Gamma_p(\xi)\chi\Gamma^p(\xi))_{p \in (0, b)}$ of maps from $(0, b)$ to $\mathfrak{po}(\mathbb{R}_1^{n+2})$ satisfies

$$\forall p \in (0, b) : |\Gamma_p(\xi)\chi\Gamma^p(\xi)| < \mathcal{B}_p,$$

where $(\mathcal{B}_p)_{p \in (0, b)}$ is a family of integrable functions on $(0, b)$ that satisfies

$$\forall p \in (0, b) : \lim_{t \rightarrow 0} t \mathcal{B}_p(t) = 0. \tag{17}$$

Moreover, if ξ is degenerate or spacelike, then

$$\lim_{p \rightarrow 0} \int_0^p \mathcal{B}_p(t) \, dt = 0. \tag{18}$$

Proof Let \mathfrak{E} be the orthogonal lift of ξ . If ξ is degenerate, it follows from the submultiplicativity of $|\cdot|$, Proposition 1 and the assumed boundedness of χ that there is a $C \in \mathbb{R}$ such that

$$\forall p, t \in (0, b) : |\Gamma_p^t(\xi)\chi(t)\Gamma_t^p(\xi)| \leq |\Gamma_p^t(\mathfrak{E})|^2|\chi(t)| \leq C \ell(p, t)^2,$$

where $\ell(p, t)$ is as in (13). Now set $\mathcal{B}_p(t) = C\ell(p, t)^2$. For all $i \in \mathbb{N}$, by l’Hôpital’s rule $(\ln x)^i x$ converges to zero as $x \rightarrow 0$ such that (17) holds. Moreover, for any $i \in \mathbb{N}$, the integral of $(\ln x)^i$ is a linear combination of terms of the form $x(\ln x)^k$ with $0 \leq k \leq i$. Again, since all terms $x(\ln x)^k$ converge to zero as $x \rightarrow 0$, it follows that for all $p \in (0, b)$ the function \mathcal{B}_p is integrable and that (18) holds.

If ξ is spacelike, again by Proposition 1, $\Gamma_p(\mathfrak{E})$ is bounded on $(0, b)$. Therefore, $\Gamma_p(\xi)\chi\Gamma^p(\xi)$ is bounded on $(0, b)$. Thus, we can choose \mathcal{B}_p to be the same constant for all $p \in (0, b)$. Clearly, a constant \mathcal{B}_p is integrable and satisfies (17) and (18).

Let finally ξ be Minkowski with positive eigenvalue $\zeta < 1$. Denote by v_{\pm} the eigenvectors of $v \wedge w$ with eigenvalues $\pm\zeta$. According to the decomposition

$$\begin{aligned} \mathfrak{o}(\mathbb{R}_1^{n+2}) &= \Lambda^2 \mathbb{R}_1^{n+2} = \langle v_+ \wedge v_- \rangle \oplus (\langle v_+ \rangle \wedge \langle v_+, v_- \rangle^\perp) \\ &\quad \oplus (\langle v_- \rangle \wedge \langle v_+, v_- \rangle^\perp) \oplus \Lambda^2 \langle v_+, v_- \rangle^\perp, \end{aligned}$$

write $\chi = \chi_{+-} + \chi_{+\perp} + \chi_{-\perp} + \chi_{\perp\perp} + \mathbb{R} \text{ id}$. Using (15), $\Gamma_p(\xi)\chi\Gamma^p(\xi)$ evaluated at t reads

$$\Gamma_p^t(\xi)\chi(t)\Gamma_t^p(\xi) = \chi_{+-}(t) + \left(\frac{t}{p}\right)^{-\zeta} \chi_{+\perp}(t) + \left(\frac{t}{p}\right)^\zeta \chi_{-\perp}(t) + \chi_{\perp\perp}(t) + \mathbb{R} \text{ id}.$$

Thus, there is a constant C such that

$$\forall p, t \in (0, b) : \left| \Gamma_p^t(\xi)\chi(t)\Gamma_t^p(\xi) \right| \leq C \left(\left(\frac{t}{p}\right)^{-\zeta} + \left(\frac{t}{p}\right)^\zeta \right) =: \mathcal{B}_p(t). \tag{19}$$

By assumption $\zeta < 1$. Therefore, for all $p \in (0, b)$, the function \mathcal{B}_p is integrable over $(0, b)$ and the product $t \mathcal{B}_p(t)$ converges to zero as t tends towards b . □

We remark that the bounding functions \mathcal{B}_p that we defined in (19) for the Minkowski case do not satisfy (18).

This lemma can now be used to prove that a primitive of a pole form of the first kind factorizes into a continuous map from $[0, b)$ to $\mathbb{P}O(\mathbb{R}_1^{n+2})$ and the primitive of a pure pole form, whose limiting behaviour at 0 we know from Proposition 1.

Corollary 1 *Let ψ be a pole form of the first kind so that $\psi - \xi$ is bounded for the pure pole form ξ . For all $p \in (0, b)$, define the gauge-transformed 1-form*

$$\xi \times_p \psi := \Gamma_p(\xi) \psi \Gamma^p(\xi) - \xi. \tag{20}$$

Then, the primitives $\Gamma_p(\xi \times_p \psi)$ have limits in $\mathbb{P}O(\mathbb{R}_1^{n+2})$ at 0 and

$$\Gamma_p(\psi) = \Gamma_p(\xi \times_p \psi) \Gamma_p(\xi). \tag{21}$$

Moreover, if ξ is spacelike or degenerate,

$$\lim_{p \rightarrow 0} \Gamma_p^0(\xi \times_p \psi) = \text{id}. \tag{22}$$

Proof Since ξ takes values in a fixed one-dimensional subalgebra, the orthogonal lift $\mathfrak{E} \times_p \psi$ of $\xi \times_p \psi$ can be written as

$$\mathfrak{E} \times_p \psi = \Gamma_p(\mathfrak{E}) \Psi \Gamma^p(\mathfrak{E}) - \mathfrak{E} = \Gamma_p(\mathfrak{E}) (\Psi - \mathfrak{E}) \Gamma^p(\mathfrak{E}). \tag{23}$$

By assumption, $\psi - \xi$ and hence $\Psi - \mathfrak{E}$ are bounded. Thus, by Lemma 1 and (23), $\mathfrak{E} \times_p \psi$ is bounded by an integrable function \mathcal{B}_p . By [10, Chap. 1.8], the $O^+(\mathbb{R}_1^{n+2})$ -valued primitive $\Gamma_p(\mathfrak{E} \times_p \psi)$ is continuous on $[0, b)$ and, in particular, has a limit in $O^+(\mathbb{R}_1^{n+2})$ at 0. Thus, by (6), $\Gamma_p(\xi \times_p \psi)$ has a limit in $\mathbb{P}O(\mathbb{R}_1^{n+2})$ at 0. The relation (21) follows from (5) because $\xi \times_p \psi$ is a gauge transform of ψ by the map $\Gamma^p(\xi)$. The limit (22) follows from

$$\lim_{p \rightarrow 0} \left| \Gamma_p^0(\mathfrak{E} \times_p \psi) - \text{id} \right| \leq \lim_{p \rightarrow 0} \left| e^{\int_0^p |\mathfrak{E} \times_p \psi|} - 1 \right| \leq \lim_{p \rightarrow 0} \left| e^{\int_0^p \mathcal{B}_p(t) dt} - 1 \right| = 0,$$

where the first inequality can be derived from the integral Eq. (7) and in the last step we used (18). □

From the factorization (21) and Proposition 1, we conclude that the primitives of spacelike pole forms do not have limits at the singularity, while those of degenerate and Minkowski pole forms of the first kind do have limits. The following corollary deals with these limits for the degenerate case. In Lemma 2 and Proposition 2, we then derive an analogous result for Minkowski pole forms of the first and second kind.

Corollary 2 *Let ψ be a degenerate pole form on $(0, b)$ with $\psi - \xi$ bounded for the pure pole form ξ . Choose $v_0 \in \langle v, w \rangle \cap \mathcal{L}^{n+1}$. Then, there is a continuous map $k : [0, b) \rightarrow \mathcal{L}^{n+1}$ which satisfies*

$$\forall p, t \in (0, b) : \langle k(t) \rangle = \Gamma_t^p(\psi) \langle k(p) \rangle, \quad \langle k(0) \rangle = \langle v_0 \rangle, \tag{24}$$

and is such that

$$\forall p \in (0, b) : \lim_{t \rightarrow 0} \Gamma_t^p(\psi) = \langle k(p) v_0^* \rangle, \quad \lim_{t \rightarrow 0} \Gamma_t^p(\psi) = \langle v_0 k(p)^* \rangle. \tag{25}$$

Proof For $a \in \mathbb{P}End(\mathbb{R}_1^{n+2})$ choose $A \in End(\mathbb{R}_1^{n+2})$ such that $a = \langle A \rangle$. The adjoint A^* with respect to the Minkowski inner product yields a well-defined element $a^* := \langle A^* \rangle \in \mathbb{P}End(\mathbb{R}_1^{n+2})$, which is independent of the chosen $A \in a$. If $a \in \mathbb{P}O(\mathbb{R}_1^{n+2})$, then $a^* = a^{-1}$. Now let $a : (0, b) \rightarrow \mathbb{P}O(\mathbb{R}_1^{n+2})$ be continuous with limit in $\mathbb{P}End(\mathbb{R}_1^{n+2})$ at 0. Then, since the map $a \mapsto a^*$ is continuous, we have

$$\lim_{t \rightarrow 0} a(t)^{-1} = \lim_{t \rightarrow 0} a(t)^* = (\lim_{t \rightarrow 0} a(t))^*. \tag{26}$$

Since $\Gamma_t^p(\psi) = (\Gamma_t^p(\psi))^{-1}$, one limit in (25) follows from the other due to (26). Now use Corollary 1 and Proposition 1 to find that

$$\forall p \in (0, b) : \lim_{t \rightarrow 0} \Gamma_t^p(\omega) = \Gamma_p^0(\xi \times_p \omega) \langle v_0 v_0^* \rangle.$$

Choose $k(p)$ to be a continuous lift of $\Gamma_p^0(\xi \times_p \omega) \langle v_0 \rangle$. Due to (22), $k(0) \in \langle v_0 \rangle$. The other relation in (24) follows from (25) and $\Gamma_p^q(\psi) = \Gamma_t^p(\psi) \Gamma_t^q(\psi)$. □

Lemma 1 and Corollary 1 do not hold for pole forms of the second kind. In particular, the gauge transforms (20) are in general not integrable and although the factorization (21) also exists in that case, it seems to be of little use because we do not know whether the first factor, $\Gamma_p(\xi \times_p \psi)$, has a limit at 0. Thus, we pursue a different strategy. The following lemma and proposition are based on ideas from [17].

Lemma 2 *Let Ψ be the orthogonal lift of a Minkowski pole form ψ and ξ the pure pole form such that $\psi - \xi$ is bounded. Denote by ζ the positive eigenvalue of $v \wedge w$. Then, there are a $B \in \mathbb{R}$ and a $\tilde{b} \in (0, b)$ such that*

$$\forall p \in (0, \tilde{b}) \forall t \in (0, p] : \left| \left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\Psi) \right| = \left| \left(\frac{t}{p} \right)^\zeta \Gamma_t^p(\Psi) \right| < 2B. \tag{27}$$

Proof The equality follows from $|A^{-1}| = |A^*| = |A|$ for $A \in O(\mathbb{R}^{n+2})$. We now prove the inequality. Let Ξ and Ψ be the orthogonal lifts of ξ and ψ . For $p, t \in (0, b)$, one verifies by differentiation³ that $\Gamma_p^t(\Psi)$ satisfies

$$\left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\Psi) = \int_p^t \left(\frac{\tau}{p} \right)^\zeta \Gamma_p^\tau(\Psi)(\Psi(\tau) - \Xi(\tau)) \left(\frac{t}{\tau} \right)^\zeta \Gamma_\tau^t(\Xi) d\tau + \left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\Xi). \tag{28}$$

By assumption, $\psi - \xi$ and hence $\Psi - \Xi$ are bounded by some $D \in \mathbb{R}$ w.r.t. $|\cdot|$. By (8), $\left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\Xi)$ is bounded by a constant $B \in \mathbb{R}$ for all $t \in (0, p]$. Therefore,

$$\forall p \in (0, b) \forall t \in (0, p] : \left(\frac{t}{p} \right)^\zeta |\Gamma_p^t(\Psi)| \leq DB \int_p^t \left(\frac{\tau}{p} \right)^\zeta |\Gamma_p^\tau(\Psi)| d\tau + B. \tag{29}$$

We now prove by contradiction that there is a $\tilde{b} \in (0, b)$ such that

$$\forall p \in (0, \tilde{b}) \forall t \in (0, p] : \left| \left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\Psi) \right| \neq 2B. \tag{30}$$

Namely, choose $\tilde{b} \in (0, b)$ sufficiently small such that

$$DB \int_0^{\tilde{b}} d\tau < \frac{1}{2}. \tag{31}$$

Suppose there existed a $p \in (0, \tilde{b})$ and a $t \in (0, p]$ such that $\left| \left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\Psi) \right| = 2B$. Denote by \bar{t} the smallest of those t . Then, evaluating (29) at \bar{t} and using (31) yield $2B < \frac{1}{2}2B + B$, a contradiction. Hence, indeed (30) holds.

To conclude (27), we note that for all $p \in (0, \tilde{b})$, the map $t \mapsto \left| \left(\frac{t}{p} \right)^\zeta \Gamma_p^t(\Psi) \right|$ is continuous, has the value $|id|$ at $t = p$ and, by the above, nowhere on $(0, p]$ attains the value $2B \geq 2|id| > |id|$. Thus, necessarily (27) holds. □

We can now prove the analogue of Corollary 2 for Minkowski pole forms.

³ Or by using the integral Eq. (7) for the 1-form $\xi \times_p \Psi$ and the identity $\Gamma_p(\xi \times_p \Psi) = \Gamma_p(\Psi)\Gamma^p(\Xi)$.

Proposition 2 *Let Ψ be the orthogonal lift of a Minkowski pole form ψ on $(0, b)$ with $\psi - \xi$ bounded for the pure pole form ξ . Denote by v_{\pm} eigenvectors of $v \wedge w$ with eigenvalues $\pm \zeta$, $\zeta > 0$. Then, there is a continuous map $k : [0, b) \rightarrow \mathcal{L}^{n+1}$ which satisfies*

$$\forall p \in (0, b) : k(t) = \left(\frac{t}{p}\right)^{-\zeta} \Gamma_t^p(\Psi)k(p), \quad k(0) = \frac{v_+}{\langle\langle v_+, v_- \rangle\rangle} \tag{32}$$

and is such that

$$\forall p \in (0, b) : \lim_{t \rightarrow 0} \left(\frac{t}{p}\right)^{\zeta} \Gamma_t^p(\Psi) = k(p)v_-^*, \quad \lim_{t \rightarrow 0} \left(\frac{t}{p}\right)^{\zeta} \Gamma_t^p(\Psi) = v_-k(p)^*. \tag{33}$$

In particular,

$$\forall p \in (0, b) : \lim_{t \rightarrow 0} \Gamma_t^p(\psi) = \langle k(p)v_-^* \rangle, \quad \lim_{t \rightarrow 0} \Gamma_t^p(\psi) = \langle v_-k(p)^* \rangle. \tag{34}$$

Proof First, we note that (34) follows from (33) and one limit in (33) follows from the other because $(\Gamma_t^p(\Psi))^* = \Gamma_t^p(\Psi)$ and taking the adjoint is continuous. We prove the first limit in (33).

Let Ψ and Ξ be the orthogonal lifts of ψ and ξ , respectively. Consider again the identity (28). For $t \in (0, p)$, it can be written in the form

$$\left(\frac{t}{p}\right)^{\zeta} \Gamma_t^p(\Psi) = \int_p^0 M(t, \tau, p) \, d\tau + \left(\frac{t}{p}\right)^{\zeta} \Gamma_t^p(\Xi), \tag{35}$$

where

$$M(t, \tau, p) := \Theta(\tau - t) \left(\frac{\tau}{p}\right)^{\zeta} \Gamma_{\tau}^p(\Psi)(\Psi(\tau) - \Xi(\tau)) \left(\frac{t}{\tau}\right)^{\zeta} \Gamma_{\tau}^t(\Xi)$$

$$\text{with } \Theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}.$$

In order to prove (33), we wish to take the limit $t \rightarrow 0$ of (35). To that end, we convince ourselves that we may evaluate the limit $t \rightarrow 0$ under the integral in (35) by applying the dominated convergence theorem. Let first \tilde{b} be as in Lemma 2 and fix $p \in (0, \tilde{b})$. Let $(t_i)_{i \in \mathbb{N}}$ be any sequence in $(0, p]$ converging to 0 and consider the sequence $(\tau \mapsto M(t_i, \tau, p))_{i \in \mathbb{N}}$ of functions from $(0, p]$ to $\mathbb{R}^{(n+2)^2}$. Due to (9), this sequence of functions converges pointwise. Moreover, due to Lemma 2, the assumed boundedness of $\Psi - \Xi$ and (8), the sequence of functions is uniformly bounded by some constant. Thus, the conditions of the dominated convergence theorem are satisfied and we conclude

$$\begin{aligned} \lim_{t \rightarrow 0} \left(\frac{t}{p}\right)^{\zeta} \Gamma_t^p(\Psi) &= \int_p^0 \lim_{t \rightarrow 0} M(t, \tau, p) \, d\tau + \lim_{t \rightarrow 0} \left(\frac{t}{p}\right)^{\zeta} \Gamma_t^p(\Xi) \\ &= \left[\int_p^0 \left(\frac{\tau}{p}\right)^{\zeta} \Gamma_{\tau}^p(\Psi)(\Psi(\tau) - \Xi(\tau)) \, d\tau + \text{id} \right] \frac{v_+}{\langle\langle v_+, v_- \rangle\rangle} v_-^* =: k(p)v_-^*, \end{aligned} \tag{36}$$

where we used (9). Again due to Lemma 2 and the assumed boundedness of $\Psi - \Xi$, the integrand in k is bounded uniformly in p and so the integral in k exists for all $p \in (0, \tilde{b})$ and $\lim_{p \rightarrow 0} k(p) = v_+ / \langle\langle v_+, v_- \rangle\rangle$. The first identity in (32) also follows from (4) and (36).

The map k has to take values in the light cone because for all $p \in (0, \tilde{b})$ we have

$$v_- \|k(p)\|^2 v_-^* = \lim_{t \rightarrow 0} \left(\frac{t}{p}\right)^{2\zeta} \Gamma_t^p(\Psi) \Gamma_p^t(\Psi) = \lim_{t \rightarrow 0} \left(\frac{t}{p}\right)^{2\zeta} \text{id} = 0.$$

This proves the proposition for all $p \in (0, \tilde{b})$. But using (4), it can easily be seen to hold for all $p \in (0, b)$. □

4 Pole of first order

In this section, we investigate the limiting behaviour at 0 of the Darboux and Calapso transforms of a polarized curve $(\langle c \rangle, Q)$ on $(0, b)$, where $\langle c \rangle$ has a smooth extension to $(-\epsilon, b)$ for some $\epsilon > 0$ and Q has a pole of first order at 0. For convenience, we further assume that $\langle c \rangle$ and Q have smooth extensions to some $(0, b + \tilde{\epsilon})$, $\tilde{\epsilon} > 0$.

We assume that $Q_t = Q(t) dt^2$ satisfies $Q(t) < 0$. This is convenient and no restriction of our results because replacing Q by $-Q$ has the same effect as replacing λ by $-\lambda$ and there is no restriction on the range or sign of λ . With this assumption, we can choose a smooth lift c of $\langle c \rangle$ such that $\Omega(t) = \frac{Q(t)}{\|c'(t)\|^2} = -\frac{1}{t}$ and the 1-form associated with $(\langle c \rangle, Q)$ reads

$$\omega_t = -\frac{dt}{t} c(t) \wedge c'(t) + \mathbb{R} \text{id}.$$

Now define the degenerate pure pole form

$$\xi_t := -\frac{dt}{t} c(0) \wedge c'(0) + \mathbb{R} \text{id}.$$

Then, $\lambda\omega - \lambda\xi$ has a finite limit at 0 because

$$|\omega(t) - \xi(t)| = \left| \frac{c(t) \wedge c'(t) - c(0) \wedge c'(0)}{t} \right| = \left| c(t) \wedge \frac{c'(t) - c'(0)}{t} + \frac{c(t) - c(0)}{t} \wedge c'(0) \right|$$

and $\langle c \rangle$ is assumed to be smoothly extendible to $(-\epsilon, b)$. Thus, $\lambda\omega - \lambda\xi$ is bounded on $(0, b)$ and $\lambda\omega$ is a degenerate pole form. In particular, it is a pole form of the first kind.

With the help of Corollary 1, we can now show that every Calapso transform of $(\langle c \rangle, Q)$ converges.

Theorem 1 *Let $(\langle c \rangle, Q)$ be a polarized curve on $(0, b)$ such that $\langle c \rangle$ has a regular extension to $(-\epsilon, b)$ and Q has a pole of first order at 0. Then, for all $p \in (0, b)$ and $\lambda \in \mathbb{R}$, the λ -Calapso transform $\langle c_{\lambda,p} \rangle$ normalized at p has a limit at 0.*

Proof If $\lambda = 0$, then $\langle c_{\lambda,p} \rangle = \langle c \rangle$ and the statement is trivial. So let $\lambda \neq 0$. Factorize $\Gamma_p(\lambda\omega)$ as in (21) to get

$$\langle c_{\lambda,p} \rangle = \Gamma_p(\lambda\omega) \langle c \rangle = \Gamma_p(\lambda\xi \times_p \lambda\omega) \Gamma_p(\lambda\xi) \langle c \rangle.$$

Since $\lambda\omega$ is of the first kind, from Corollary 1 we know that $\Gamma_p(\lambda\xi \times_p \lambda\omega)$ has a limit in $\mathbb{P}O\left(\mathbb{R}_1^{n+2}\right)$ at 0. To see that $\Gamma_p(\lambda\xi) \langle c \rangle$ has a limit at 0, write

$$\Gamma_p^t(\lambda\xi) = \left\langle \text{id} - \ln\left(\frac{t}{p}\right) \lambda c(0) \wedge c'(0) - \frac{\left(\ln\left(\frac{t}{p}\right)\right)^2 \lambda^2 \|c'(0)\|^2}{2} c(0)c(0)^* \right\rangle. \tag{37}$$

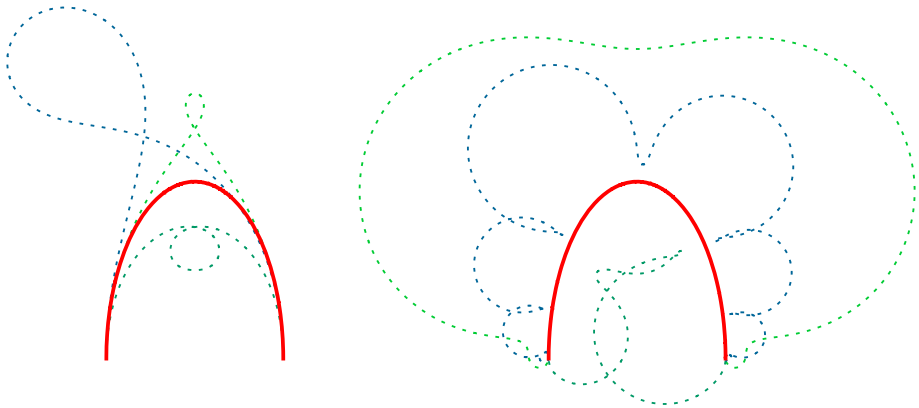


Fig. 1 Darboux transforms (dashed) of a half ellipse (solid) with respect to a polarization with a pole of first order at each end. The spectral parameter λ is positive (negative) on the right (left) side

Since both $\frac{1}{t}\langle c(0), c(t) \rangle$ and $\frac{1}{t}\langle c'(0), c(t) \rangle$ have finite limits at 0, and

$$\lim_{t \rightarrow 0} t \ln \left(\frac{t}{p} \right) = 0, \quad \lim_{t \rightarrow 0} t \left(\ln \left(\frac{t}{p} \right) \right)^2 = 0,$$

the product of (37) with $\langle c(t) \rangle$ converges to $\langle c(0) \rangle$. Thus, the limit of $\langle c_{\lambda,p} \rangle$ at 0 exists and is equal to $\Gamma_p^0(\lambda \xi \times_p \lambda \omega) \langle c(0) \rangle$. \square

The Darboux transforms of $(\langle c \rangle, Q)$ converge to $\langle c(0) \rangle$.

Theorem 2 *Let $(\langle c \rangle, Q)$ be a polarized curve on $(0, b)$ such that $\langle c \rangle$ has a regular extension to $(-\epsilon, b)$ and Q has a pole of first order at 0. Then, for all $\lambda \in \mathbb{R} \setminus \{0\}$, the limit of any λ -Darboux transform $\langle \hat{c} \rangle$ of $(\langle c \rangle, Q)$ at 0 is $\langle c(0) \rangle$ (Fig. 1),*

$$\lim_{t \rightarrow 0} \langle \hat{c}(t) \rangle = \langle c(0) \rangle.$$

Proof From Corollary 2, we know that for every $\lambda \neq 0$, there is a continuous map $k_\lambda : [0, b) \rightarrow \mathcal{L}^{n+1}$ such that

$$\lim_{t \rightarrow 0} \Gamma_t^P(\lambda \omega) = \langle c(0) k_\lambda(p)^* \rangle,$$

and k_λ satisfies (24). Therefore, if $\langle \hat{c}_p \rangle \neq \langle k_\lambda(p) \rangle$, then

$$\lim_{t \rightarrow 0} \langle \hat{c}(t) \rangle = \lim_{t \rightarrow 0} \Gamma_t^P(\lambda \omega) \langle \hat{c}_p \rangle = \langle c(0) k_\lambda(p)^* \rangle \langle \hat{c}_p \rangle = \langle c(0) \rangle.$$

If on the other hand $\langle \hat{c}_p \rangle = \langle k_\lambda(p) \rangle$, then we use the properties (24) of k_λ to find

$$\lim_{t \rightarrow 0} \langle \hat{c}(t) \rangle = \lim_{t \rightarrow 0} \Gamma_t^P(\lambda \omega) \langle k_\lambda(p) \rangle = \lim_{t \rightarrow 0} \langle k_\lambda(t) \rangle = \langle c(0) \rangle.$$

\square

5 Pole of second order

We now come to the more intricate and diverse case of a polarized curve $(\langle c \rangle, Q)$ on $(0, b)$, where $\langle c \rangle$ is smoothly extendible to $(-\epsilon, b)$ for some $\epsilon > 0$ and Q has a pole of second order at 0. In this case, also $\lambda\omega$ has a pole of second order at 0 and we cannot apply the results of Sect. 3 directly. Instead, we use a gauge transformation

$$\lambda\omega \mapsto g \times \lambda\omega = \langle g \rangle^{-1} \lambda\omega \langle g \rangle + \langle g \rangle^{-1} d\langle g \rangle$$

as in (5) to write

$$\Gamma_p(\lambda\omega) = \langle g(p) \rangle \Gamma_p(g \times \lambda\omega) \langle g \rangle^{-1}, \quad \Gamma^p(\lambda\omega) = \langle g \rangle \Gamma^p(g \times \lambda\omega) \langle g(p) \rangle^{-1}. \tag{38}$$

As we will see in Sect. 5.1, it is possible to choose $\langle g \rangle$ such that $g \times \lambda\omega$ is a pole form and, in particular, has only a pole of first order. This is achieved by a gauge transformation which is singular in the sense that the transforming map $\langle g \rangle : (0, b) \rightarrow \mathbb{P}O(\mathbb{R}_1^{n+2})$ converges to a map of the kind $\langle vw^* \rangle \in \mathbb{P}End(\mathbb{R}_1^{n+2})$ with $v, w \in \mathcal{L}^{n+1}$ as 0 is approached. In Sects. 5.2 and 5.3, we will then apply the results of Sect. 3 to the pole form $g \times \lambda\omega$ to investigate the behaviour of the Darboux and Calapso transforms of $(\langle c \rangle, Q)$ at the singular point 0.

Again, for convenience, we assume that $\langle c \rangle$ and Q have smooth extensions to some $(0, b + \tilde{\epsilon})$, $\tilde{\epsilon} > 0$.

5.1 The singular gauge transformation

We assume that the polarization Q is of the form $Q_t = Q(t) dt^2 = \frac{dt^2}{t^2}$. Again, this is no restriction of our results: We can assume $Q(t)$ to be positive because replacing Q by $-Q$ has the same effect as replacing λ by $-\lambda$, and we make no restriction on the range of λ . Furthermore, the Darboux and Calapso transformations are invariant under reparametrizations of $\langle c \rangle$. Since Q has a pole of second order (see Definition 2), there certainly is a parameter t for $\langle c \rangle$ such that $Q_t = \frac{dt^2}{t^2}$ and $\langle c \rangle$ remains smoothly extendible to $(-\epsilon, b)$.

For the gauge transformation, we use a product $g = FR$ of a frame $F : (0, b) \rightarrow O(R_1^{n+2})$, smoothly extendible to $(-\epsilon, b)$, and a particularly simple singular factor $R : (0, b) \rightarrow O(R_1^{n+2})$.

We first construct the frame F . To this end, let c be the flat lift of $\langle c \rangle$, for which $\|c'\|^2 = 1$. Then, with the above assumptions, $\Omega(t) = \frac{1}{t^2}$. Let N_1, \dots, N_{n-1} be parallel, orthonormal unit normal fields of c that satisfy $\langle dN_i, c' \rangle = 0$ for $i = 1, \dots, n-1$ and let $\bar{c} : (0, b) \rightarrow \mathcal{L}^{n+1}$ be the unique map such that $\langle c, \bar{c} \rangle = -1$ and $\langle c, \bar{c} \rangle = \langle c', N_1, \dots, N_{n-1} \rangle^\perp$. Now define the frame $F : (0, b) \rightarrow O(\mathbb{R}_1^{n+2})$ to map t to the orthogonal transformation $F(t)$ which maps a t -independent pseudo-orthonormal basis $\{o, t, \mathfrak{t}, n_1, \dots, n_{n-1}\}$ of \mathbb{R}_1^{n+2} to the t -dependent basis $\{c(t), \bar{c}(t), c'(t), N_1(t), \dots, N_{n-1}(t)\}$. The Maurer–Cartan form of F is then of the form

$$F^{-1} dF = -t \wedge \mathfrak{t} dt + o \wedge \mathcal{F}^\perp$$

for some $\langle o, t \rangle^\perp$ -valued 1-form \mathcal{F}^\perp .

Next, define the singular factor $R : (0, b) \rightarrow O(\mathbb{R}_1^{n+2})$ by

$$R(t)o = \frac{1}{t} o, \quad R(t)t = t t, \quad R|_{\langle o, t \rangle^\perp} = \text{id}.$$

⁴ This condition means that $\langle N_1, \dots, N_{n-1} \rangle^\perp \cap \mathcal{L}^{n+1}$ is the curvature circle congruence of $\langle c \rangle$.

Then, R has Maurer–Cartan form

$$(R^{-1} dR)_t = -\frac{dt}{t} o \wedge \iota.$$

The product $g := FR$ is a singular frame for the singular lifts $t^{-1}c(t)$ and $t\bar{c}(t)$ of $\langle c(t) \rangle$ and $\langle \bar{c}(t) \rangle$, respectively. It satisfies

$$\lim_{t \rightarrow 0} \langle g(t) \rangle = \lim_{t \rightarrow 0} \langle F(t) R(t) \rangle = \lim_{t \rightarrow 0} \langle F(t) t R(t) \rangle = \langle c(0)\iota^* \rangle \in \mathbb{P}End\left(\mathbb{R}_1^{n+2}\right). \tag{39}$$

Using $\langle g \rangle$ for a gauge transformation (5) of $\lambda\omega$, the gauge-transformed 1-form $g \times \lambda\omega$ reads

$$g \times \lambda\omega = R^{-1}F^{-1}\lambda\omega FR + R^{-1}dR + R^{-1}(F^{-1}dF)R, \\ (g \times \lambda\omega)_t = -\frac{dt}{t} \left(-(\lambda o - \iota) \wedge t + o \wedge \iota \right) + t o \wedge \mathcal{F}_t^\perp + \mathbb{R} \text{ id}. \tag{40}$$

Indeed, $g \times \lambda\omega$ has only a pole of first order and the pure pole form ξ_λ given by

$$\xi_\lambda(t) dt := -\frac{dt}{t} \left(-(\lambda o - \iota) \wedge t + o \wedge \iota \right) + \mathbb{R} \text{ id} \\ = -\frac{dt}{t} (o - \mathfrak{t}) \wedge (\iota - \lambda t) + \mathbb{R} \text{ id}$$

is such that $|g \times \lambda\omega - \xi_\lambda|$ is bounded⁵ on $(0, b)$. Thus, $g \times \lambda\omega$ is a pole form.

The next Lemma, which summarizes the algebraic properties of ξ_λ , can be verified by direct computation.

Lemma 3 *Define*

$$v_\pm := \sqrt{1 - 2\lambda}(\lambda o - \iota) \pm (\lambda o + \iota - 2\lambda t), \tag{41}$$

such that ξ_λ can be written as

$$\xi_\lambda(t) dt = -\frac{dt}{t} \sqrt{1 - 2\lambda} \frac{v_- \wedge v_+}{\langle\langle v_-, v_+ \rangle\rangle} + \mathbb{R} \text{ id} =: -\frac{dt}{t} v \wedge w + \mathbb{R} \text{ id}.$$

If $1 - 2\lambda > 0$, then $\langle v, w \rangle$ is Minkowski and v_\pm are eigenvectors of $v \wedge w$ with real eigenvalues $\pm\sqrt{1 - 2\lambda}$. If $1 - 2\lambda = 0$, then $\langle v, w \rangle$ is degenerate and $v_+ = -v_- \in \langle v, w \rangle$ is null. If $1 - 2\lambda < 0$, then $\langle v, w \rangle$ is spacelike and v_\pm are complex conjugate eigenvectors of $v \wedge w$ with imaginary eigenvalues $\pm\sqrt{1 - 2\lambda} \in i\mathbb{R}$.

In particular, for $\lambda > 0$, the pole form $\lambda\omega$ is of the first kind. For $\lambda \leq 0$, it is of the second kind.

By Proposition 2, Corollary 1 and Proposition 1, for $1 - 2\lambda \geq 0$, the primitives $\Gamma_p^t(g \times \lambda\omega)$ converge as $t \rightarrow 0$ while for $1 - 2\lambda < 0$ they do not have a limit at 0. We treat these cases separately in the next two sections.

5.2 The behaviour at the singularity for $1 - 2\lambda \geq 0$

To determine the behaviour of the Darboux and Calapso transforms of $\langle c \rangle$ at 0, we seek the limits $t \rightarrow 0$ of

$$\langle \hat{c}(t) \rangle = \langle g(t) \rangle \Gamma_t^p(g \times \lambda\omega) \langle g(p) \rangle^{-1} \langle \hat{c}_p \rangle, \tag{42}$$

⁵ In fact, even $\frac{1}{t}|(g \times \lambda\omega - \xi_\lambda)_t|$ is bounded on $(0, b)$, which will be useful in the proof of Theorem 6.

$$\langle c_{\lambda,p}(t) \rangle = \langle g(p) \rangle \Gamma_p^t(g \times \lambda\omega) \langle g(t) \rangle^{-1} \langle c(t) \rangle = \langle g(p) \rangle \Gamma_p^t(g \times \lambda\omega) \langle o \rangle. \tag{43}$$

By (39), $\langle g(t) \rangle$ has a limit at 0. Since $1 - 2\lambda \geq 0$, by Lemma 3 the pole form $g \times \lambda\omega$ is Minkowski or degenerate such that its primitives also have limits at 0. Very little work is required to prove convergence of the Darboux and Calapso transforms at 0 in the following two theorems using the results of Sect. 3.

Theorem 3 *Let $(\langle c \rangle, Q)$ be a polarized curve on $(0, b)$ where $\langle c \rangle$ is regularly extendible to $(-\epsilon, b)$ and Q has a pole of second order at 0. Then, for every $\lambda \in \mathbb{R} \setminus \{0\}$ with $1 - 2\lambda \geq 0$, all λ -Darboux transforms $\langle \hat{c} \rangle$ of $\langle c \rangle$ converge to $\langle c(0) \rangle$ as 0 is approached,*

$$\lim_{t \rightarrow 0} \langle \hat{c}(t) \rangle = \langle c(0) \rangle.$$

Proof Let first $1 - 2\lambda > 0$. To find the behaviour of the Darboux transforms $\langle \hat{c} \rangle$ of $\langle c \rangle$ at 0, we want to take the limit $t \rightarrow 0$ of (42). According to (39) and Proposition 2,

$$\lim_{t \rightarrow 0} \langle g(t) \rangle = \langle c(0) \iota^* \rangle, \quad \lim_{t \rightarrow 0} \Gamma_t^P(g \times \lambda\omega) = \langle v_- k_\lambda(p)^* \rangle$$

for a continuous map $k_\lambda : [0, b) \rightarrow \mathcal{L}^{n+1}$ which satisfies (24) and v_\pm given by (41).

If $\langle \hat{c}_p \rangle \neq \langle g(p) \rangle \langle k_\lambda(p) \rangle$. Then,

$$\begin{aligned} \lim_{t \rightarrow 0} \langle \hat{c}(t) \rangle &= \lim_{t \rightarrow 0} \langle g(t) \rangle \Gamma_t^P(g \times \lambda\omega) \langle g(p) \rangle^{-1} \langle \hat{c}_p \rangle \\ &= \langle c(0) \iota^* \rangle \langle v_- k_\lambda(p)^* \rangle \langle g(p) \rangle^{-1} \langle \hat{c}_p \rangle = \langle c(0) \rangle, \end{aligned}$$

where we used $\iota^*(v_-) \neq 0$.

If on the other hand $\langle \hat{c}_p \rangle = \langle g(p) \rangle \langle k_\lambda(p) \rangle$, then we use (32) to get

$$\lim_{t \rightarrow 0} \langle \hat{c}(t) \rangle = \lim_{t \rightarrow 0} \langle g(t) \rangle \Gamma_t^P(g \times \lambda\omega) \langle k_\lambda(p) \rangle = \lim_{t \rightarrow 0} \langle g(t) \rangle \langle k_\lambda(t) \rangle = \langle c(0) \iota^* \rangle \langle v_+^* \rangle = \langle c(0) \rangle,$$

where in the last equality we used $\iota^*(v_+) \neq 0$.

This proves the theorem for $1 - 2\lambda > 0$. For $1 - 2\lambda = 0$, the proof works completely analogous with Corollary 2 in place of Proposition 2 and (24) instead of (32). \square

For the Calapso transforms, even less work is necessary.

Theorem 4 *Let $(\langle c \rangle, Q)$ be a polarized curve on $(0, b)$ where $\langle c \rangle$ has a regular extension to $(-\epsilon, b)$ and Q has a pole of second order at 0. Then, all λ -Calapso transforms with $1 - 2\lambda \geq 0$ converge as $t \rightarrow 0$.*

Proof In (43), we use Proposition 2 for $1 - 2\lambda > 0$, Corollary 2 for $1 - 2\lambda = 0$ and $v_-(o) \neq 0$ to get

$$\lim_{t \rightarrow 0} \langle c_{\lambda,p}(t) \rangle = \lim_{t \rightarrow 0} \langle g(p) \rangle \Gamma_p^t(g \times \lambda\omega) \langle o \rangle = \langle g(p) \rangle \langle k_\lambda(p) v_-^* \rangle \langle o \rangle = \langle g(p) \rangle \langle k_\lambda(p) \rangle.$$

\square

5.3 The behaviour at the singularity for $1 - 2\lambda < 0$

In this case, ξ_λ takes values in an algebra of infinitesimal Euclidean rotations. Thus, by Proposition 1, the primitives of ξ_λ do not have a limit at 0 and hence by Corollary 2 neither do the primitives of $g \times \lambda\omega$ converge. From this, it follows that the λ -Calapso transforms of $\langle c \rangle$ do not converge.

Theorem 5 *Let $(c), Q$ be a polarized curve on $(0, b)$ such that $\langle c \rangle$ has a regular extension to $(-\epsilon, b)$ and Q has a pole of second order at 0. For $\lambda \in \mathbb{R} \setminus \{0\}$ and $p \in (0, b)$, the λ -Calapso transform normalized at p tends⁶ towards the circular motion*

$$\langle C_{\lambda,p}(t) \rangle := \langle g(p) \rangle \Gamma_p^0(\xi_\lambda \times_p g \times \lambda \omega) \Gamma_p^t(\xi_\lambda) \langle o \rangle$$

whose speed tends to infinity as $t \rightarrow 0$. In particular, it does not have a limit point at 0, but the limit circle

$$\mathfrak{L}[\langle c_{\lambda,p} \rangle] := g(p) \Gamma_p^0(\xi_\lambda \times_p g \times \lambda \omega) \langle o, \iota, \mathfrak{t} \rangle \cap \mathcal{L}^{n+1}. \tag{44}$$

Proof Let Ω and \mathfrak{E}_λ be the orthogonal lifts of ω and ξ_λ , respectively. Since $\Gamma_p(\mathfrak{E}_\lambda \times_p g \times \lambda \Omega)$ is continuous on $[0, b)$, the inner product of the bounded lifts

$$\begin{aligned} c_{\lambda,p}(t) &= g(p) \Gamma_p^t(\mathfrak{E}_\lambda \times_p g \times \lambda \Omega) \Gamma_p^t(\mathfrak{E}_\lambda) o, \\ C_{\lambda,p}(t) &= g(p) \Gamma_p^0(\mathfrak{E}_\lambda \times_p g \times \lambda \Omega) \Gamma_p^t(\mathfrak{E}_\lambda) o \end{aligned}$$

of $\langle c_{\lambda,p} \rangle$ and $\langle C_{\lambda,p} \rangle$, respectively, tends to zero. Since these lifts do not converge to zero at 0, we conclude that $\langle c_{\lambda,p} \rangle$ approaches $\langle C_{\lambda,p} \rangle$.

That $\langle C_{\lambda,p} \rangle$ is a parametrization of the circle $g(p) \Gamma_p^0(\xi_\lambda \times_p g \times \lambda \omega) \langle o, \iota, \mathfrak{t} \rangle$ follows because \mathfrak{E}_λ takes values in the subalgebra $(\langle o - \mathfrak{t} \rangle \wedge \langle \mathfrak{t} - \lambda \mathfrak{t} \rangle)$: The one-parameter group $\Gamma_p(\xi_\lambda)$ of Euclidean rotations moves the point $\langle o \rangle$ on the circle $\langle o, \iota, \mathfrak{t} \rangle \cap \mathcal{L}^{n+1}$ with speed increasing towards infinity as 0 is approached. \square

A generic Darboux transform, in contrast, has a limit at 0. As above, we have

$$\langle \hat{c}(t) \rangle = \Gamma_t^p(\lambda \omega) \langle \hat{c}_p \rangle = \langle g(t) \rangle \Gamma_t^p(g \times \lambda \omega) \langle g(p) \rangle^{-1} \langle \hat{c}_p \rangle. \tag{45}$$

We know that $\langle g(t) \rangle$ converges to $\langle c(0) \iota^* \rangle$, and one might expect that as long as $\Gamma_t^p(g \times \lambda \omega) \langle g(p) \rangle^{-1} \langle \hat{c}_p \rangle$ stays sufficiently far away from the kernel $\langle \mathfrak{t} \rangle$ of $\langle c(0) \iota^* \rangle$, in the limit $\langle c(0) \iota^* \rangle$ forces $\langle \hat{c} \rangle$ towards $\langle c(0) \rangle$. This idea is made precise and affirmed by the following Lemma, formulated in a more general context.

Lemma 4 *For \mathcal{V} an arbitrary subset of \mathbb{R}^m and $q \in \mathbb{R}^m$ a limit point of \mathcal{V} , let $\langle A \rangle : \mathcal{V} \rightarrow \mathbb{P}O(\mathbb{R}_1^{n+2})$ be a map with limit $\langle xy^* \rangle$ at q , where $x, y \in \mathcal{L}^{n+1}$. Let $\langle u \rangle : \mathcal{V} \rightarrow S^n$ be a map so that there is an open neighbourhood U of $\langle y \rangle \in S^n$ such that $\langle u(\mathcal{V}) \rangle \cap U$ is empty. Then the product $\langle A \rangle \langle u \rangle$ has the limit $\langle x \rangle$ at q (Fig. 2).*

Proof Let A be a lift of $\langle A \rangle$ with limit xy^* at q and u the lift of $\langle u \rangle$ that satisfies $\langle\langle u(t), y \rangle\rangle = -1$ for all $t \in \mathcal{V}$. Then, u is bounded with respect to the positive-definite norm $|\cdot|$ by some $R \in \mathbb{R}^+$. Namely, if there was no such bound, then in order to maintain $\langle\langle u(t), y \rangle\rangle = -1$, the point $\langle u(t) \rangle$ would need to get arbitrarily close to $\langle y \rangle$, which contradicts the assumptions.

Now write

$$A(t)u(t) = (A(t) - xy^*)u(t) + xy^*u(t) = (A(t) - xy^*)u(t) - x. \tag{46}$$

Since u is bounded by $R \in \mathbb{R}$, we have

$$|(A(t) - xy^*)u(t)| \leq |A(t) - xy^*| |u(t)| \leq |A(t) - xy^*| R,$$

and so the first term on the right-hand side of (46) has limit 0 at q . Thus, the product Au has the limit $-x$ at q and $\langle A \rangle \langle u \rangle$ has the limit $\langle x \rangle$ at q . \square

Using this Lemma, the following proposition provides a sufficient condition for $\langle \hat{c}(t) \rangle$ to converge to $\langle c(0) \rangle$ as $t \rightarrow 0$.

⁶ $\langle c_{\lambda,p} \rangle$ tends towards $\langle C_{\lambda,p} \rangle$ in the sense that the distance between $\langle c_{\lambda,p}(t) \rangle$ and $\langle C_{\lambda,p}(t) \rangle$ measured with respect to an arbitrary representative metric of the conformal structure of S^n tends to zero as $t \rightarrow 0$.

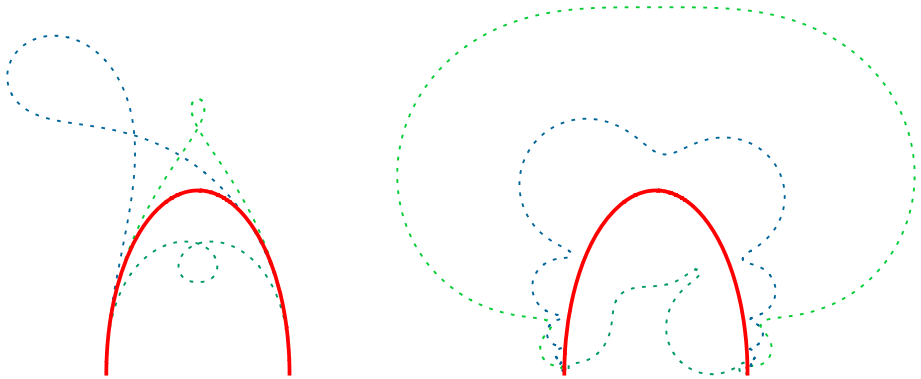


Fig. 2 Darboux transforms (dashed) of a half ellipse (solid) with respect to a polarization with a pole of second order at each end. The spectral parameter λ is greater (smaller) than $\frac{1}{2}$ on the right (left) side

Proposition 3 *Let $(\langle c \rangle, Q)$ be a polarized curve on $(0, b)$ such that $\langle c \rangle$ has a regular extension to $(-\epsilon, b)$ and Q has a pole of second order at 0 . For $\lambda \in \mathbb{R}$ with $1 - 2\lambda < 0$ and $\langle \hat{c}_p \rangle$ not on the limit circle $\mathbb{F}[\langle c_{\lambda, p} \rangle]$ of the λ -Calapso transform normalized at p (see (44)), denote by $\langle \hat{c} \rangle$ the λ -Darboux transform of $\langle c \rangle$ with initial value $\langle \hat{c}(p) \rangle = \langle \hat{c}_p \rangle$. Then, $\langle \hat{c} \rangle$ converges to $\langle c(0) \rangle$ as 0 is approached.*

Proof Fix $p \in (0, b)$ and factorize $\Gamma_p(g \times \lambda\omega)$ as in (21) to write $\langle \hat{c} \rangle$ in the form

$$\begin{aligned} \langle \hat{c} \rangle &= \langle g \rangle \langle u \rangle \text{ with } \langle u \rangle := \Gamma^p(\xi_\lambda) \Gamma^0(\xi_\lambda \times_p g \times \lambda\omega) \langle W \rangle, \\ \langle W \rangle &:= \Gamma_0^p(\xi_\lambda \times_p g \times \lambda\omega) \langle g(p) \rangle^{-1} \langle \hat{c}_p \rangle. \end{aligned}$$

The singular frame $\langle g \rangle$ converges to $\langle c(0) \rangle \iota^*$. We want to apply Lemma 4 to the product $\langle g \rangle \langle u \rangle$, and thus, we need to prove that under the assumption that $\langle \hat{c}_p \rangle$ does not lie on the limit circle $\mathbb{F}[\langle c_{\lambda, p} \rangle]$, there is a $\tilde{b} \in (0, b)$ and neighbourhood $U \subset S^n$ of $\langle \iota \rangle$ such that

$$\forall t \in (0, \tilde{b}) : \langle u(t) \rangle = \Gamma_t^p(\xi_\lambda) \Gamma_t^0(\xi_\lambda \times_p g \times \lambda\omega) \langle W \rangle \notin U. \tag{47}$$

To prove this, we note that, since ξ_λ takes values in $((o - t) \wedge (t - \lambda t) + \mathbb{R} \text{id})$, we have

$$\forall w \in \mathcal{L}^{n+1} \forall t \in (0, b) : \Gamma_t^p(\xi_\lambda) \langle w \rangle \subset \langle w, o - t, t - \lambda t \rangle.$$

From this, we deduce the implication

$$\langle w \rangle \not\subset \langle o, t, t \rangle \Rightarrow \forall t \in (0, b) : \Gamma_t^p(\xi_\lambda) \langle w \rangle \neq \langle t \rangle. \tag{48}$$

By the assumption of the proposition, $\langle W \rangle \not\subset \langle o, t, t \rangle$ and hence, by (48),

$$\forall t \in (0, b) : \Gamma_t^p(\xi_\lambda) \langle W \rangle \neq \langle t \rangle.$$

But then there must also be open neighbourhoods V of $\langle W \rangle$ and U of $\langle t \rangle$ such that

$$\forall t \in (0, b) \forall x \in V : \Gamma_t^p(\xi_\lambda) x \notin U. \tag{49}$$

Due to (18), we can choose $\tilde{b} \in (0, b)$ close enough to 0 such that

$$\forall t \in (0, \tilde{b}) : \Gamma_t^0(\xi_\lambda \times_p g \times \lambda\omega) \langle W \rangle \in V. \tag{50}$$

Using (50) in (49) now yields (47). We can thus apply Lemma 4 to the product $\langle \hat{c} \rangle = \langle g \rangle \langle u \rangle$ restricted to $(0, \tilde{b})$. Since $p \in (0, b)$ was arbitrary, this completes the proof. \square

The characterization of the λ -Darboux transforms excluded in Proposition 3 is independent of the point $p \in (0, b)$, at which the initial condition $\langle \hat{c}(p) \rangle = \langle \hat{c}_p \rangle$ is posed. As soon as $\langle \hat{c}(p) \rangle \in \mathcal{E}[\langle c_{\lambda,p} \rangle]$ holds for one $p \in (0, b)$, it holds for all $p \in (0, b)$.

We finally analyse what happens, when the initial point of the Darboux transform does lie on the limit circle of the Calapso transform. In this case, we cannot apply Lemma 4 and indeed $\langle \hat{c} \rangle$ does not converge to $\langle c(0) \rangle$ as 0 is approached.

Theorem 6 *Let $(\langle c \rangle, Q)$ be a polarized curve on $(0, b)$ where $\langle c \rangle$ has a regular extension to $(-\epsilon, b)$ and Q has a pole of second order at 0. As 0 is approached, any λ -Darboux transform $\langle \hat{c} \rangle$ with $1 - 2\lambda < 0$ converges to $\langle c(0) \rangle$,*

$$\lim_{t \rightarrow 0} \langle \hat{c}(t) \rangle = \langle c(0) \rangle,$$

except when $\langle \hat{c}(p) \rangle$ lies on the limit circle $\mathcal{E}[\langle c_{\lambda,p} \rangle]$ of the Calapso transform $\langle c_{\lambda,p} \rangle$. In that case, $\langle \hat{c} \rangle$ approaches the curvature circle of $\langle c \rangle$ at 0 rotating with frequency increasing towards infinity as 0 is approached.

Proof The first part of the theorem was proved above (see Proposition 3). What remains to be confirmed is the exception. The analysis of this exceptional case is laborious and will occupy us for the rest of this section. We prove the theorem in three steps. First, we show that $\langle \hat{c} \rangle$ approaches the curvature circle of $\langle c \rangle$ at 0. Then, we show that $\langle \hat{c} \rangle$ has at least two distinct limit points on that curvature circle at 0. In the third step, we use that $\langle \hat{c} \rangle$ and $\langle c \rangle$ envelop a circle congruence to argue that $\langle \hat{c} \rangle$ tends towards a rotation on the curvature circle of $\langle c \rangle$ at 0 with frequency increasing towards infinity as 0 is approached.

During this proof, we think of $p \in (0, b)$ and $\lambda \in \mathbb{R}$ with $1 - 2\lambda < 0$ fixed. To simplify notation, we define the integrable 1-form ϕ and its orthogonal lift Φ by

$$\phi := \xi_\lambda \times_p g \times \lambda \omega, \quad \Phi := \Xi_\lambda \times_p g \times \lambda \Omega.$$

In particular, from (20) and (40) we find

$$\Phi_t = \Phi(t) dt = \Gamma_t^t(\Xi_\lambda) t \circ \wedge \mathcal{F}_t^\perp \Gamma_t^p(\Xi_\lambda),$$

such that Φ_t/id is bounded on $(0, b)$.

Let $\langle \hat{c}_p \rangle$ be any point on $\mathcal{E}[\langle c_{\lambda,p} \rangle]$ that is (c.f. Theorem 5),

$$\langle \hat{c}_p \rangle = \langle g(p) \rangle \Gamma_p^0(\phi) \langle \hat{C}_p \rangle \quad \text{with} \quad \hat{C}_p \in \langle o, \iota, \mathfrak{t} \rangle. \tag{51}$$

Then, substitution of (51) in (45) and using $\Gamma^p(g \times \lambda \omega) = \Gamma^p(\xi_\lambda) \Gamma^p(\phi)$ yield

$$\langle \hat{c} \rangle = \langle g \rangle \Gamma^p(g \times \lambda \omega) \Gamma_p^0(\phi) \langle \hat{C}_p \rangle = \langle g \rangle \Gamma^p(\xi_\lambda) \Gamma^0(\phi) \langle \hat{C}_p \rangle. \tag{52}$$

1. Denote by Ξ_λ the orthogonal lift of ξ_λ and let \hat{c} be the lift

$$\hat{c}(t) = \frac{1}{t} g(t) \Gamma_t^p(\Xi_\lambda) \Gamma_t^0(\Phi) \hat{C}_p \tag{53}$$

for some nonzero $\hat{C}_p \in \langle \hat{C}_p \rangle$. In Sect. 5.1, we chose the normal fields $\{N_1, \dots, N_{n-1}\}$ such that $\langle N_1, \dots, N_{n-1} \rangle^\perp$ is the curvature circle congruence of $\langle c \rangle$. Therefore, since \hat{c} is a lift of $\langle \hat{c} \rangle$ that stays away from zero, in order to show that $\langle \hat{c} \rangle$ approaches the curvature circle of $\langle c \rangle$ at 0, it is sufficient to show that the $n - 1$ inner products $\langle \langle \hat{c}, N_i \rangle \rangle$ all tend towards zero. Since $g^{-1}N_i = n_i$ and $\Xi_\lambda(n_i) = 0$ (see Sect. 5.1), the inner product of (53) with N_i reads

$$\langle \langle \hat{c}(t), N_i(t) \rangle \rangle = \frac{1}{t} \langle \langle \Gamma_t^0(\Phi) \hat{C}_p, n_i \rangle \rangle =: \frac{1}{t} \sigma_i(t).$$

Since \hat{C}_p is an element of $\langle o, \iota, \mathfrak{t} \rangle$ and $\mathfrak{n}_i \in \langle o, \iota, \mathfrak{t} \rangle^\perp$, all the functions $\sigma_i(t)$ tend to zero as $t \rightarrow 0$. Since $\Phi(t)/t$ is bounded, $\Phi(t)$ tends to zero as 0 is approached and thus also the first derivatives of all σ_i tend to zero as 0 is approached. An application of the mean value theorem to σ_i then yields that $\langle\langle \hat{c}, N_i \rangle\rangle$ tends to zero and indeed $\langle \hat{c} \rangle$ approaches the curvature circle of $\langle c \rangle$ at 0 as 0 is approached.

2. We now prove that $\langle \hat{c} \rangle$ has at least two limit points at 0, namely $\langle c(0) \rangle$ and $\langle \bar{c}(0) \rangle$, where \bar{c} is as in Sect. 5.1.

To prove that $\langle c(0) \rangle$ is a limit point, let $(t_i)_{i \in \mathbb{N}}$ be a sequence in $(0, b)$ which converges to 0 and is such that

$$\forall i \in \mathbb{N} : \Gamma_{t_i}^p(\xi_\lambda) \langle \hat{C}_p \rangle = \langle o \rangle.$$

In particular,

$$\forall i, j > 0 : \Gamma_{t_i}^{t_j}(\xi_\lambda) = \text{id}.$$

Then, evaluating (52) at t_i and taking the limit $i \rightarrow \infty$ yield

$$\begin{aligned} \lim_{i \rightarrow \infty} \langle \hat{c}(t_i) \rangle &= \lim_{i \rightarrow \infty} \langle g(t_i) \rangle \Gamma_{t_i}^p(\xi_\lambda) \Gamma_{t_i}^0(\phi) \langle \hat{C}_p \rangle \\ &= \langle c(0) \iota^* \rangle \Gamma_{t_i}^0(\xi_\lambda) \langle \hat{C}_p \rangle = \langle c(0) \iota^* \rangle \langle o \rangle = \langle c(0) \rangle. \end{aligned}$$

Thus, $\langle c(0) \rangle$ is a limit point of $\langle \hat{c} \rangle$.

To prove that also $\langle \bar{c}(0) \rangle$ is a limit point, rescale \hat{C}_p if necessary and let $(t_i)_{i \in \mathbb{N}}$ be a sequence in $(0, b)$ which converges to 0 and is such that

$$\forall i \in \mathbb{N} : \Gamma_{t_i}^p(\mathfrak{E}_\lambda) \hat{C}_p = \iota. \tag{54}$$

The lift (53) evaluated at t_i yields the sequence $(\hat{c}(t_i))_{i \in \mathbb{N}}$ with

$$\hat{c}(t_i) = \frac{1}{t_i} g(t_i) \Gamma_{t_i}^p(\mathfrak{E}_\lambda) \Gamma_{t_i}^0(\Phi) \hat{c}_p.$$

We have already proved above that the $n - 1$ quantities $\langle\langle \hat{c}, N_i \rangle\rangle$ converge to zero. We now show that also both $\langle\langle \hat{c}(t_i), \bar{c}(t_i) \rangle\rangle$ and $\langle\langle \hat{c}(t_i), c'(t_i) / \|c'(t_i)\| \rangle\rangle$ converge to zero and that $\langle\langle \hat{c}(t_i), c(t_i) \rangle\rangle$ converges to -1 . It then follows that $\langle \hat{c}(t_i) \rangle$ converges to $\langle \bar{c}(0) \rangle$.

Use $g^{-1}(t)\bar{c}(t) = \frac{1}{t}\iota$ and (54) to write

$$\langle\langle \hat{c}(t_i), \bar{c}(t_i) \rangle\rangle = \frac{1}{t_i^2} \left\langle\left\langle \Gamma_{t_i}^p(\mathfrak{E}_\lambda) \Gamma_{t_i}^0(\Phi) \hat{C}_p, \iota \right\rangle\right\rangle = \frac{1}{t_i^2} \left\langle\left\langle \Gamma_{t_i}^0(\Phi) \hat{C}_p, \hat{C}_p \right\rangle\right\rangle.$$

Now use the integral Eq. (7) for $\Gamma_t^p(\Phi)$ twice and the antisymmetry of elements of $\mathfrak{o}(\mathbb{R}_1^{n+2})$ to get

$$\langle\langle \hat{c}(t_i), \bar{c}(t_i) \rangle\rangle = \frac{1}{t_i^2} \int_{t_i}^0 \int_\tau^0 \left\langle\left\langle \Phi(\tau) \Phi(\tilde{\tau}) \Gamma_{\tilde{\tau}}^0(\Phi) \hat{C}_p, \hat{C}_p \right\rangle\right\rangle d\tilde{\tau} d\tau.$$

Since $\Phi(t)/t$ is bounded, indeed $\langle\langle \hat{c}(t_i), \bar{c}(t_i) \rangle\rangle$ converges to zero as $i \rightarrow \infty$.

To show that the inner product of $\hat{c}(t_i)$ with $c'(t_i) / \|c'(t_i)\|$ converges to zero, we write

$$\langle\langle \hat{c}(t_i), c'(t_i) / \|c'(t_i)\| \rangle\rangle = \frac{1}{t_i} \left\langle\left\langle \Gamma_{t_i}^p(\Xi_\lambda) \Gamma_{t_i}^0(\Phi) \hat{C}_p, \mathfrak{t} \right\rangle\right\rangle.$$

Again use the integral equation for $\Gamma^0(\Phi)$, the boundedness of Φ/t and (54) to conclude that $\langle\langle \hat{c}(t_i), c'(t_i) / \|c'(t_i)\| \rangle\rangle$ converges to zero as $i \rightarrow \infty$.

Finally,

$$\langle\langle \hat{c}(t_i), c(t_i) \rangle\rangle = \left\langle\left\langle \Gamma_{t_i}^p(\Xi_\lambda) \Gamma_{t_i}^0(\Phi) \hat{C}_p, o \right\rangle\right\rangle$$

which converges to $\langle\langle \Gamma_1^p(\Xi_\lambda) \hat{C}_p, o \rangle\rangle = \langle\langle t, o \rangle\rangle = -1$.

Thus, indeed $\lim_{i \rightarrow \infty} \hat{c}(t_i) = \bar{c}(0)$ such that $\langle\bar{c}(0)\rangle$ is also a limit point of $\langle\hat{c}\rangle$.

3. We now show that $\langle\hat{c}\rangle$ tends towards a rotation on the curvature circle of $\langle c \rangle$ at 0. Since both $\langle c(0) \rangle$ and $\langle \bar{c}(0) \rangle$ are limit points of $\langle\hat{c}\rangle$, it then follows that the frequency of this rotation must tend towards infinity.

Choose $\tilde{b} \in (0, b)$ such that $\langle\bar{c}(0)\rangle \notin \langle c((0, \tilde{b})) \rangle$. Then, there is a closed neighbourhood $U \subset S^n$ of $\langle\bar{c}(0)\rangle$ whose intersection with $\langle c((0, \tilde{b})) \rangle$ is empty. Define

$$\mathcal{I} := \{t \in (0, \tilde{b}) \mid \langle\hat{c}(t)\rangle \in U\}.$$

Since $\langle\bar{c}(0)\rangle$ is a limit point of $\langle\hat{c}\rangle$, the set \mathcal{I} is non-empty and has the limit point 0. Now let \bar{c} , c and \hat{c} be spherical lifts of $\langle\bar{c}\rangle$, $\langle c \rangle$ and $\langle\hat{c}\rangle$, respectively. The circle congruence \mathcal{S} enveloped by $\langle\hat{c}\rangle$ and $\langle c \rangle$ is

$$\mathcal{S} = \langle\hat{c}, c, c'\rangle = \left\langle \frac{1}{\langle\langle c, \hat{c} \rangle\rangle} \Pi_{(N_1, \dots, N_{n-1})} \hat{c} - \bar{c}, c, c' \right\rangle.$$

We have showed above that $\Pi_{(N_1, \dots, N_{n-1})} \hat{c}$ converges to zero as 0 is approached. Moreover, we have constructed \mathcal{I} in a way that there exists a constant $C > 0$ such that

$$\forall t \in \mathcal{I} : \langle\langle c(t), \hat{c}(t) \rangle\rangle > C. \tag{55}$$

Therefore, \mathcal{S} restricted to \mathcal{I} converges to the curvature circle $\langle\bar{c}(0), c(0), c'(0)\rangle$ of $\langle c \rangle$ at 0. (55) also implies that inside U the speed of $\langle\hat{c}\rangle$ tends towards infinity because

$$\forall t \in \mathcal{I} : \|\hat{c}'(t)\|^2 = \|\lambda \Omega(c \wedge c')(\hat{c})\|^2 = \frac{\lambda^2}{t^4} \langle\langle c(t), \hat{c}(t) \rangle\rangle^2 \|c'\|^{-2} > \frac{\lambda^2}{t^4} C^2 \|c'\|^{-2}. \tag{56}$$

Now stereographically project S^n to \mathbb{R}^n such that the curvature circle of $\langle c \rangle$ at 0 gets mapped to the unit circle in the $\langle e_1, e_2 \rangle$ -plane with centre the origin. Write the projection \hat{c} of $\langle\hat{c}\rangle$ as

$$\hat{c}(t) = \rho(t) (\sin(\varphi(t))e_1 + \cos(\varphi(t))e_2) + Z(t),$$

where $\varphi : (0, b) \rightarrow \mathbb{R}$ is smooth and $Z(t) \in \langle e_3, \dots, e_n \rangle$ for all $t \in (0, b)$. Since $\langle\hat{c}\rangle$ approaches the curvature circle of $\langle c \rangle$ at 0, the function Z must converge to zero and ρ must converge to 1 as $t \rightarrow 0$. Since moreover the enveloped circle congruence \mathcal{S} restricted to \mathcal{I} converges to the curvature circle of $\langle c \rangle$ at 0, on \mathcal{I} the angle between the tangent to \hat{c} and the tangent to the circle $\sin(\varphi)e_1 + \cos(\varphi)e_2$ must converge to ± 1 , that is,

$$\pm 1 = \lim_{t \rightarrow 0} \left\langle \left\langle \frac{\hat{c}'(t)}{\|\hat{c}'(t)\|}, \cos(\varphi(t))e_1 - \sin(\varphi(t))e_2 \right\rangle \right\rangle = \lim_{t \rightarrow 0} \frac{\rho(t)\varphi'(t)}{\sqrt{\rho'(t)^2 + \rho^2\varphi'(t)^2 + \|Z'(t)\|^2}}.$$

Due to the lower bound (56), the square root in this expression tends to infinity and so also φ' must converge to plus or minus infinity as 0 is approached in \mathcal{I} . Together with

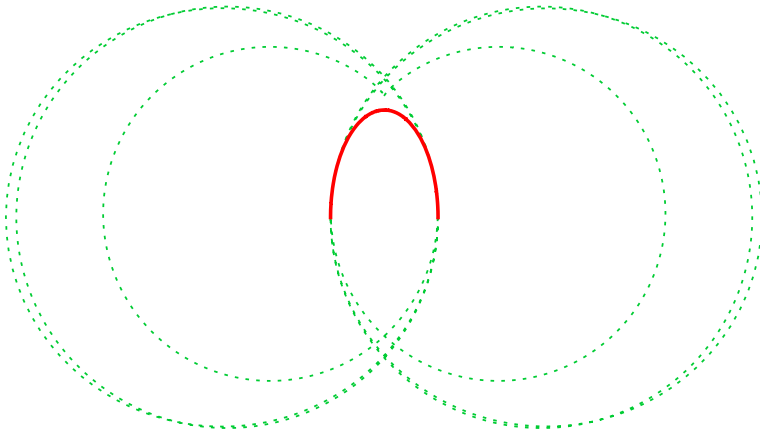


Fig. 3 Darboux transform (dashed) of a half ellipse (solid) with respect to a polarization with a pole of second order at each end. The spectral parameter λ is positive and the initial point $\langle \hat{c}(p) \rangle$ lies on the limit circle of the Calapso transform normalized at p

the result that also $\langle c(0) \rangle$ is a limit point of $\langle \hat{c} \rangle$, this implies that φ converges to plus or minus infinity such that indeed the frequency of the rotation of $\langle \hat{c} \rangle$ on the curvature circle tends towards infinity (Fig. 3).

□

Acknowledgements Open access funding provided by Austrian Science Fund (FWF). The author would like to thank G. Szewieczek for fruitful discussions and S. Fujimori for his hospitality during a visit to Okayama University, where parts of this work were elaborated. Special thanks go to U. Hertrich-Jeromin for his numerous, valuable comments and his support as supervisor of the author’s Ph.D. Thesis. This work has been supported by the Austrian Science Fund (FWF) and the Japan Society for the Promotion of Science (JSPS) through the FWF/JSPS Joint Project Grant I1671-N26 “Transformations and Singularities.”

Open Access This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

1. Bianchi, L.: Vorlesungen über Differentialgeometrie. Teuber, Leipzig (1899). (**Autorisierte deutsche Übersetzung von M. Lukat**)
2. Bobenko, A.I.: Surfaces in terms of 2 by 2 matrices. Old and new integrable cases. In: Harmonic Maps and Integrable Systems, Aspects Math., E23, pp. 83–127. Friedr. Vieweg, Braunschweig (1994)
3. Bobenko, A.I., Suris, Y.B.: On discretization principles for differential geometry. The geometry of spheres. *Uspekhi Mat. Nauk* **62**(1(373)), 3–50 (2007)
4. Bobenko, A.I., Suris, Y.B.: Discrete Differential Geometry, Graduate Studies in Mathematics, vol. 98. American Mathematical Society, Providence (2008)
5. Burstall, F.: Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems. In: Integrable Systems, Geometry, and Topology, AMS/IP Stud. Adv. Math, vol. 36, pp. 1–82. Amer. Math. Soc., Providence (2006)
6. Burstall, F., Hertrich-Jeromin, U., Müller, C., Rossman, W.: Semi-discrete isothermic surfaces. *Geom. Dedic.* **183**, 43–58 (2016)

7. Burstall, F., Hertrich-Jeromin, U., Pedit, F., Pinkall, U.: Curved flats and isothermic surfaces. *Math. Z.* **225**(2), 199–209 (1997)
8. Burstall, F., Santos, S.D.: Special isothermic surfaces of type d . *J. Lond. Math. Soc. (2)* **85**(2), 571–591 (2012)
9. Cieślifski, J., Goldstein, P., Sym, A.: Isothermic surfaces in E^3 as soliton surfaces. *Phys. Lett. A* **205**(1), 37–43 (1995)
10. Dollard, J.D., Friedman, C.N.: *Product Integration with Applications to Differential Equations*, Encyclopedia of Mathematics and Its Applications, vol. 10. Addison-Wesley Publishing Co., Reading (1979)
11. Eisenhart, L.P.: *A Treatise on the Differential Geometry of Curves and Surfaces*. Dover Publications Inc., New York (1960)
12. Ferus, D., Pedit, F.: Curved flats in symmetric spaces. *Manuscr. Math.* **91**(4), 445–454 (1996)
13. Fuchs, A.: *Transformations and singularities of isothermic surfaces*. Ph.D. Thesis, TU Wien (2018) (**in preparation**)
14. Guilfoyle, B., Klingenberg, W.: Proof of the Carathéodory conjecture. ArXiv e-prints (2008)
15. Hertrich-Jeromin, U.: *Introduction to Möbius Differential Geometry*, London Mathematical Society Lecture Note Series, vol. 300. Cambridge University Press, Cambridge (2003)
16. Hopf, H.: *Differential Geometry in the Large*, Lecture Notes in Mathematics, vol. 1000, 2nd edn. Springer, Berlin (1989). (**Notes taken by Peter Lax and John W. Gray, With a preface by S. S. Chern, With a preface by K. Voss**)
17. Levinson, N.: The asymptotic behavior of a system of linear differential equations. *Am. J. Math.* **68**, 1–6 (1946)
18. Müller, C., Wallner, J.: Semi-discrete isothermic surfaces. *Results Math.* **63**(3–4), 1395–1407 (2013)
19. Sharpe, R.W.: *Differential Geometry*, Graduate Texts in Mathematics, vol. 166. Springer, New York (1997)
20. Smyth, B.: Soliton surfaces in the mechanical equilibrium of closed membranes. *Commun. Math. Phys.* **250**(1), 81–94 (2004)
21. Strebel, K.: *Quadratic Differentials*, vol. 5. Springer, Berlin (1984)
22. Terng, C.L., Uhlenbeck, K.: Bäcklund transformations and loop group actions. *Commun. Pure Appl. Math.* **53**(1), 1–75 (2000)