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Erratum

Erratum to: A Finite Dimensional Integrable System Arising in the Study of Shock Clustering

Luen-Chau Li

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA.
E-mail: luenli@math.psu.edu

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It has come to the author's attention recently that the variables $\phi_{r0}(L)$, $r = 1, \ldots, N-1$, as defined on the second line of (4.35) in [L] simply do not work as claimed in Theorem 4.10 (b) and Proposition 4.13 (b) of that work. Regrettably, this is due to overlooking a term in the proof of Theorem 4.10 (b). The purpose of this erratum is to provide the correct definition of the ϕ_{r0}'s to make things work and what is required is a divisor $D_\tau(L)$ different from the one in [L]. Indeed, this new divisor can also be used in constructing the variables that move linearly on the Jacobi variety of the curve. Therefore, while the variables $\phi_{rk}(L)$, $1 \le k \le r-1$, $r = 2, \ldots, N$ in [L] are sound, we will replace them by corresponding quantities constructed by using $D_\tau(L)$ here because it gives us a uniform construction of all the angle variables.

Recall that $L \in \mathfrak{g} = gl(N, \mathbb{R})$ and we have the Lax operator $L_u(h) = hu - L$. Let

$$I(L; h, z) = \det\left(L_u(h) - zI\right) = \sum_{r=0}^{N} \sum_{k=0}^{r} I_{rk}(L) h^k z^{N-r} \tag{1}$$

and denote its homogenization by $\widetilde{I}(L; \zeta, h, z)$, i.e.,

$$\widetilde{I}(L; \zeta, h, z) = \det(hu - \zeta L - zI). \tag{2}$$

Then we have the projective curve

$$C(L) = \{[\zeta : h : z] \in \mathbb{CP}^2 \mid \widetilde{I}(L; \zeta, h, u) = 0\} \tag{3}$$

with affine part $C_a(L) : I(L; h, z) = 0$. In the notation of [L], we will work with $L \in \mathcal{G} \cap \mathcal{U}$ satisfying the additional genericity assumption (GA1). Also, recall that

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$(h) = P_+ - P_-$, $P_\pm = \sum_{i=1}^N P_\pm^i$. In order to define the divisor alluded to above, we have to consider eigenvectors of the transpose $L_u(h)^T$ of the Lax operator. If for $P = [\zeta : h : z] \in C(L)$, we let $\mathcal{M}(L, P) = hu - \zeta L - zI$, then $f_\tau(P) = \ker \mathcal{M}(L, P)^T$ defines the holomorphic eigenvector map $f_\tau : C(L) \longrightarrow \mathbb{CP}^{N-1}$. Let

$$H = \{[z_1 : \cdots : z_N] \in \mathbb{CP}^{N-1} \mid z_1 + \cdots + z_N = 0\}, \tag{4}$$

then the divisor $D_\tau(L) = f_\tau^* H$ is well-defined and is of degree $\frac{N(N-1)}{2} = g + N - 1$, where g is the genus of $C(L)$. Recall that [L]

$$\begin{aligned} \omega_{r+1,-1} &= \frac{z^{N-r-1}}{hI_z(h, z)}\,dh, \;\; r = 1, \dots, N-1, \\ \omega_{r+1,k-1} &= \frac{h^{k-1} z^{N-r-1}}{I_z(L; h, z)}\,dh, \;\; 1 \le k \le r-1, r = 2, \dots, N-1. \end{aligned} \tag{5}$$

As replacement of the variables in [L], we define

$$\begin{aligned} \phi_{r0}(L) &= \int_{D_0(L)}^{D_\tau(L)} \omega_{r+1,-1}, \;\; r = 1, \dots, N-1, \\ \phi_{rk}(L) &= \int_{D_0(L)}^{D_\tau(L)} \omega_{r+,k-1}, \;\; 1 \le k \le r-1, r = 2, \dots, N-1, \end{aligned} \tag{6}$$

where $D_0(L) = (g + N - 1)P_-^1$, and in the definition of $\phi_{r0}(L)$ above, the path of integration going from the point P_-^1 to the points in $D_\tau(L)$ have to avoid the points in P_+^i, $i = 1, \dots, N$.

Theorem 4.10'.

(a) $\{\phi_{rk}, I_{r'k'}\}_R(L) = -\delta_{kk'}\delta_{r',r+1}$, $1 \le k \le r-1, r = 2, \dots, N-1, 0 \le k' \le r-2$, $r' = 2, \dots, N$.
(b) $\{\phi_{r0}, I_{r'k'}\}_R(L) = -\delta_{k'0}\delta_{r',r+1}$, $r = 1, \dots, N-1, 0 \le k' \le r-2, r' = 2, \dots, N$.

Proof. We will give the proof of part (b). For this purpose, recall that [L] we pick $h_0 \in (-1, 1)$, $z_0 \in \mathbb{R}$ such that $[1 : h_0 : z_0] \notin C(L)$, and we set $H_{h_0,z_0}(L) = \log\det(L_u(h_0) - z_0 I)$. Let $P_i = [1 : h_0 : z_i(h_0)] \in C(L)$, $i = 1, \dots, N$. To compute $\{\phi_{r0}, H_{h_0,z_0}\}_R(L)$, it suffices to evaluate it on an open dense subset of L satisfying the following conditions:

(i) the points of $D_\tau(L)$ are distinct, supp $D_\tau(L) \subset C_a(L)$,
(ii) supp $D_\tau(L) \cap \{I_z = 0\} = \emptyset$,
(iii) supp $D_\tau(L) \cap \{P_i\}_{i=1}^N = \emptyset$,
(iv) $\{P_i\}_{i=1}^N \cap \{I_z = 0\} = \emptyset$.

For small values of t, let $L(t)$ be the solution of the Hamiltonian flow generated by H_{h_0,z_0} with initial condition $L(t = 0) = L$, and let $D_\tau(t) = (f_\tau^t)^* H = \sum_{j=1}^{g+N-1} Q_j(t)$, where $f_\tau^t : C(L) \longrightarrow \mathbb{CP}^{N-1}$ is the holomorphic eigenvector map defined by $f_\tau^t(P) =$

Ker $\mathcal{M}(L(t), P)^T$. Let $Q_j(t) = [1 : h_j(t) : z_j(t)]$, $Q_j = Q_j(0)$, $j = 1, \ldots, g+N-1$. Then we have

$$\begin{aligned}\{\phi_{r0}, H_{h_0,z_0}\}_R(L) &= \frac{d}{dt}\bigg|_{t=0} \phi_{r0}(L(t)) \\ &= \sum_{j=1}^{g+N-1} \frac{z_j^{N-r-1}}{h_j I_z(h_j, z_j)} \left.\frac{dh_j(t)}{dt}\right|_{t=0},\end{aligned} \tag{7}$$

where $h_j = h_j(0), z_j = z_j(0)$, $j = 1, \ldots, g + N - 1$. To compute $\left.\frac{dh_j(t)}{dt}\right|_{t=0}$, consider a representative $f_\tau(h, t)$ of the kernel map f_τ^t in a neighborhood of h_j such that $(e, f_\tau(h_j(t), t)) = 0$ for small values of t. (Here e is the vector in $\mathbb{C}^N$ with all components equal to 1.) Differentiating this relation with respect to t at $t = 0$, we have

$$0 = \left(e, \frac{\partial f_\tau}{\partial h}(h_j, 0) \left.\frac{dh_j(t)}{dt}\right|_{t=0} + \frac{\partial f_\tau}{\partial t}(h_j, 0)\right). \tag{8}$$

On the other hand, by differentiating

$$L_u(h, t)^T f_\tau(h, t) = z f_\tau(h, t), \quad L_u(h, t) = hu - L(t), \tag{9}$$

we obtain

$$(L_u(h, t)^T - zI)\left(\frac{\partial f_\tau}{\partial t} + B(h)^T f_\tau(h, t)\right) = 0, \tag{10}$$

where

$$B(h) = \Pi_{\mathfrak{m}}(L_u(h_0) - z_0 I)^{-1} - \frac{h}{h - h_0}(L_u(h_0) - z_0 I)^{-1} \tag{11}$$

from Lemma 4.9 in [L]. Consequently,

$$\begin{aligned}\left(e, \frac{\partial f_\tau}{\partial t}(h_j, 0)\right) &= -\left(e, B(h_j)^T f_\tau(h_j, 0)\right) \\ &= \frac{h_j}{h_j - h_0}(e, (L_u(h_0) - z_0 I)^{-T} f_\tau(h_j, 0))\end{aligned} \tag{12}$$

where we have used the property that $Xe = 0$ for $X \in \mathfrak{m}$. Hence it follows that

$$\begin{aligned}\left.\frac{dh_j(t)}{dt}\right|_{t=0} &= -\frac{\left(e, \frac{\partial f_\tau}{\partial t}(h_j, 0)\right)}{\left(e, \frac{\partial f}{\partial h}(h_j, 0)\right)} \\ &= -\frac{h_j}{h_j - h_0} \frac{\left(e, (L_u(h_0) - z_0 I)^{-T} f_\tau(h_j, 0)\right)}{\left(e, \frac{\partial f_\tau}{\partial h}(h_j, 0)\right)}.\end{aligned} \tag{13}$$

Substitute this in (7) above, we find

$$\begin{aligned}&\{\phi_{r0}, H_{h_0,z_0}\}_R(L) \\ &= -\sum_{j=1}^{g+N-1} \frac{z_j^{N-r-1}}{h_j I_z(h_j, z_j)} \frac{h_j(e, (L_u(h_0) - z_0)^{-T} f_\tau(h_j, 0))}{(h_j - h_0)(e_1, \frac{\partial f_\tau}{\partial h}(h_j, 0))} \\ &= -\sum_{j=1}^{g+N-1} Res_{Q_j} \frac{h(e, (L_u(h_0) - z_0 I)^{-T} f_\tau(h, 0))}{(h - h_0)(e, f_\tau(h, 0))} \omega_{r+1,-1}.\end{aligned} \tag{14}$$

Now the meromorphic 1-form

$$\frac{h(e,(L_u(h_0)-z_0I)^{-T}f_\tau(h,0))}{(h-h_0)(e,f_\tau(h,0))}\omega_{r+1,-1} \tag{15}$$

has poles at the points P_i, $i=1,\ldots,N$, supp $D_\tau(L)$ and possibly at P_-^i, $i=1,\ldots,N$. Thus

$$\begin{aligned}\{\phi_{r0},H_{h_0,z_0}\}_R(L)&\\ =\sum_{i=1}^N Res_{P_i}&\frac{h(e,(L_u(h_0)-z_0I)^{-T}f_\tau(h,0))}{(h-h_0)(e,f_\tau(h,0))}\omega_{r+1,-1}\\ +\sum_{i=1}^N Res_{P_-^i}&\frac{h(e,(L_u(h_0)-z_0I)^{-T}f_\tau(h,0))}{(h-h_0)(e,f_\tau(h,0))}\omega_{r+1,-1}.\end{aligned} \tag{16}$$

To study the behaviour of the meromorphic 1-form in a neighborhood of P_-^i, introduce local coordinate $\xi=h^{-1}$, then $z=\frac{u_i}{\xi}+O(1)$ and so

$$\begin{aligned}&\frac{h(e,(L_u(h_0)-z_0I)^{-T}f_\tau(h,0))}{(h-h_0)(e,f_\tau(h,0))}\omega_{r+1,-1}\\ &\simeq(e,(L_u(h_0)-z_0I)^{-T}e_i)\frac{u_i^{N-r-1}}{\prod_{\mu\neq i}(u_\mu-u_i)}\xi^{r-1}\,d\xi.\end{aligned} \tag{17}$$

Since $r\geq 1$, we conclude that the meromorphic 1-form is holomorphic at P_-^i, $i=1,\ldots,N$. On the other hand,

$$\begin{aligned}&\mathrm{Res}_{P_i}\frac{h(e,(L_u(h_0)-z_0I)^{-T}f_\tau(h,0))}{(h-h_0)(e,f_\tau(h,0))}\omega_{r+1,-1}\\ &=\frac{z_i(h_0)^{N-r-1}}{(z_i(h_0)-z_0)I_z(h_0,z_i(h_0))}.\end{aligned} \tag{18}$$

Therefore, on assembling the calculations, we obtain

$$\begin{aligned}\{\phi_{r0},&H_{h_0,z_0}\}_R(L)\\ &=\sum_{i=1}^N\frac{z_i(h_0)^{N-r-1}}{(z_i(h_0)-z_0)I_z(h_0,z_i(h_0))}\\ &=\lim_{R\to 0}\oint_{|z|=R}\frac{z^{N-r-1}}{(z-z_0)I(h_0,z)}\frac{dz}{2\pi i}-\frac{z_0^{N-r-1}}{I(h_0,z_0)}.\\ &=-\frac{z_0^{N-r-1}}{I(h_0,z_0)}.\end{aligned} \tag{19}$$

But

$$\{\phi_{r0},H_{h_0,z_0}\}_R(L)=\frac{1}{I(h_0,z_0)}\sum_{r',k'}\{\phi_{r0},I_{r'k'}\}_R(L)h_0^{k'}z_0^{N-r'}, \tag{20}$$

hence it follows from the above computation that $\{\phi_{r0},I_{r'k'}\}_R(L)=-\delta_{k'0}\delta_{r',r+1}$. □

From the above Poisson bracket relations and the expression of the Hamiltonian $H = \frac{1}{2}\mathrm{tr}(L(F \circ L))$ in Proposition 4.6 of [L], the following proposition follows.

Proposition 4.13'. *(a)* $\{\phi_{rk}, H\}_R(L) = (-1)^{N+1}\left(\sum_{j=1}^{N} \frac{f(u_j)}{A_j(u_j)} u_j^{N-1-r}\right)\delta_{k,r-1}$, $1 \leq k \leq r-1, r = 2, \ldots, N-1$.

(b) $\{\phi_{r0}, H\}_R(L) = (-1)^{N+1}\left(\sum_{j=1}^{N} \frac{f(u_j)}{A_j(u_j)} u_j^{N-1-r}\right)\delta_{r-1,0}$, $r = 1, \ldots, N-1$.

Note that the Poisson bracket relations in Theorem 4.10' and Proposition 4.13' above only differ from the ones in [L] by a sign. For this reason, the proof of functional independence of the I_{rk}'s for $k = 0, \ldots, r-1$, $r = 1, \ldots, N$ on the open, dense subset of $\mathcal{G} \cap \mathcal{U}$ satisfying (GA1) in Corollary 4.11 of [L] remains the same.

Finally, we correct a typo in equation (4.30) of [L] as follows: the product $\prod_{j=1}^{N} \frac{x-\lambda_j}{\lambda_i-\lambda_j}$ on the left hand side should be replaced by

$$\prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{x-\lambda_j}{\lambda_i-\lambda_j}. \tag{21}$$

Reference

[L] Li, L.-C.: A finite dimensional integrable system arising in the study of shock clustering. Commun. Math. Phys. **340**, 1109–1142 (2015)

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