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Bifurcation locus and branches at infinity of a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$

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Abstract We show that the number of bifurcation values at infinity of a polynomial function $f : \mathbb{C}^2 \to \mathbb{C}$ is at most the number of branches at infinity of a general fiber of f and that this upper bound can be diminished by one in certain cases.

1 Introduction

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function in a fixed coordinate system. It is well known (as being proved originally by Thom [17]), that f is a locally trivial C^∞ fibration outside a finite subset of the target. The smallest such set is called *the bifurcation set of* f and will be denoted here by $B(f)$. The set $B(f)$ might be larger than the set of critical values $f(\mathrm{Sing} f)$, like for instance in the following simple example due to Broughton [1]: $f(x, y) = x + x^2 y$, where $\mathrm{Sing} f = \emptyset$ but $B(f) = \{0\}$, and we say that 0 is a critical value at infinity of f. The set $B_\infty(f)$ of *bifurcation values at infinity*, or *critical values at infinity*, consists of points $a \in \mathbb{C}$ at which the restriction of f to the complement of a large enough ball (centred at $0 \in \mathbb{C}^2$) is not a locally trivial bundle. There are several criteria to detect such a value; one may consult e.g. [2,3,5,16,18,19]. For instance: $a \in B_\infty(f)$ if and only if there exists a sequence of

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points $(p_k)_{k\in\mathbb{N}} \subset \mathbb{C}^2$ such that $\|p_k\| \to \infty$, grad $f(p_k) \to 0$ and $f(p_k) \to a$ as $k \to \infty$.

Upper bounds for $\#B_\infty(f)$ have been found in the 1990's by Lê and Oka [12] in terms of Newton polyhedra at infinity. An estimation in terms of the degree d of f was given by Gwoździewicz and Płoski [8]: if $\dim \mathrm{Sing} f \le 0$ then $\#B_\infty(f) \le \max\{1, d-3\}$. In the general case (dropping the condition $\dim \mathrm{Sing} f \le 0$) we have $\#B_\infty(f) \le d-1$, see e.g. [10,11]. Recently Gwoździewicz [9] proved the following estimation of $\#B_\infty(f)$: if ν_0 denotes the number of branches at infinity of the (reduced) fibre $f^{-1}(0)$, then the number of critical values at infinity other than 0 is at most ν_0. Here we refine and improve this statement by using a different method, in which results by Miyanishi [13,14] and Gurjar [6] play an important role.

For $a \in \mathbb{C}$, let us denote by ν_a the number of branches at infinity of the reduced fiber $f^{-1}(a)$. This number is equal to ν_{gen} for all values $a \in \mathbb{C}$ except finitely many for which one may have either $\nu_a < \nu_{\mathrm{gen}}$ or $\nu_a > \nu_{\mathrm{gen}}$. Let $\nu_{\min} := \inf\{\nu_a \mid a \in \mathbb{C}\}$. Let us denote by b the number of *points at infinity of* f, i.e. $b := \#\overline{f^{-1}(a)} \cap L_\infty$, where L_∞ is the line at infinity $\mathbb{P}^2 \setminus C^2$.

Under these notations, our main result is the following:

Theorem 1.1 *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function of degree d. Then:*

(a) $\#B_\infty(f) \le \min\{\nu_{\mathrm{gen}}, \nu_{\min} + 1\}$.
(b) $\#\{a \in \mathbb{C} \mid \nu_a < \nu_{\mathrm{gen}}\} \le \nu_{\mathrm{gen}} - b$.
(c) $\#\{a \in \mathbb{C} \mid \nu_a > \nu_{\mathrm{gen}}\} \le \nu_{\min}$ *(this remains true even if we count branches with multiplicities).*

In case $\nu_{\mathrm{gen}} > \frac{d}{2}$, we moreover have:

(d) $\#B_\infty(f) \le \min\{\nu_{\mathrm{gen}} - 1, \nu_{\min}\}$.
(e) $\#\{a \in \mathbb{C} \mid \nu_a > \nu_{\mathrm{gen}}\} \le \nu_{\min} - 1$ *(this remains true even if we count branches with multiplicities).*

Remark 1.2 Point (a) of Theorem 1.1 is equivalent to Gwoździewicz's [9, Theorem 2.1]. His result is a by-product of the local study of pencils of curves of Yomdin-Ephraim type. Our method is totally different and allows us to prove moreover several new issues, namely (b)–(e) of Theorem 1.1.

Remark 1.3 As Gwoździewicz remarks, his inequality [9, Theorem 2.1] is "almost" sharp, i.e. not sharp by one. Our new inequality (d) improves by one the inequality (a) under the additional condition $\nu_{\mathrm{gen}} > \frac{d}{2}$, thus yields the sharp upper bound, as shown by the example $f : \mathbb{C}^2 \to \mathbb{C}$, $f(x, y) = x + x^2 y$, where $d = \deg f = 3$, $\nu_{\min} = \nu_{\mathrm{gen}} = 2$, $b = 2$ and $B_\infty(f) = \{0\}$ with $\nu_0 = 3$.

The same example shows that our estimations (b) and (e) are also sharp.

2 Proof of Theorem 1.1

We need here the important concept of *affine surfaces which contain a cylinder-like open subset* which was introduced by Miyanishi [13]. Let us recall it together with some properties which we shall use.

Definition 2.1 [14] Let X be a normal affine surface. We say that X contains a *cylinder-like open subset* U, if there exists a smooth curve C such that $U \cong \mathbb{C} \times C$.

Let X be as in the above definition and let $\pi : U \to C$ be the projection. After [14, p.194], the projection π has a unique extension to a $\mathbb{C}$-fibration $\rho : X \to \bar{C}$, where $\bar{C}$ denotes the smooth completion of the curve C. We have the following important result of Gurjar and Miyanishi:

Theorem 2.2 [6,7,13] *Let X be a normal affine surface with a $\mathbb{C}$-fibration $f : X \to B$, where B is a smooth curve. Then:*

(a) X has at most cyclic quotient singularities.
(b) Every fiber of f is a disjoint union of curves isomorphic to $\mathbb{C}$.
(c) A component of a fiber of f contains at most one singular point of X. If a component of a fiber occurs with multiplicity 1 in the scheme-theoretic fiber, then no singular point of X lies on this component. □

Corollary 2.3 *Let X be a normal affine surface, which contains a cylinder-like open subset U. Then the set $X \setminus U$ is a disjoint union of curves isomorphic to $\mathbb{C}$. Moreover, every connected component l_i of this set contains at most one singular point of X.* □

Let $f : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial function in fixed affine coordinates and denote by $\tilde{f}(x, y, z)$ the homogenization of f by a new variable z, namely $\tilde{f}(x, y, z) = f_d + z f_{d-1} + \cdots + z^d f_0$. Let $X := \{([x : y : z], t) \in \mathbb{P}^2 \times \mathbb{C} \mid \tilde{f}(x, y, z) = tz^d\}$ be the closure in $\mathbb{P}^2 \times \mathbb{C}$ of the graph $\Gamma := \text{graph}(f) \subset \mathbb{C}^2 \times \mathbb{C}$. Then X is a hypersurface and the points at infinity of X (i.e. points outside of Γ) forms precisely the set $\{a_1, \ldots, a_b\} \times \mathbb{C}$, where $\{a_1, \ldots, a_b\}$ are all points at infinity of the curve $f = 0$. In particular if $\rho : \mathbb{P}^2 \times \mathbb{C} \to \mathbb{P}^2$ denotes the first projection, then $\rho(X \setminus \Gamma) = \{a_1, \ldots, a_b\}$.

The second projection $\pi : X \to \mathbb{C}$, $(x, t) \mapsto t$, is a proper extension of f. Let $\nu : X' \to X$ be the normalization of X. Composing ν with π yields $\pi' : X' \to \mathbb{C}$, which is also a proper extension of f. We shall denote it by $\tilde{f}$ in the following.

On the other side composing ν with ρ yields $\rho' : X' \to \mathbb{P}^2$ and $\rho'(X' \setminus \Gamma) = \{a_1, \ldots, a_b\}$, i.e., the points at infinity of X' lie over the points $\{a_1, \ldots, a_b\}$.

Lemma 2.4 *The set $X' \setminus \Gamma$ is a disjoint union of affine curves, $l_1, \ldots, l_r$, each curve l_i is isomorphic to $\mathbb{C}$. On each line l_i there is at most one singular point of X'. Moreover, $b \leq r \leq \nu_{\min}$.*

Proof Let us choose a line $l \subset \mathbb{P}^2$ such that $l \cap \{a_1, \ldots, a_b\} = \emptyset$. Let $X_1 := (\mathbb{P}^2 \setminus l) \times \mathbb{C} \cap X$. The surface X_1 is affine and $X'_1 \setminus \Gamma = \bigcup_{i=1}^r l_i$, where X'_1 denotes the normalization of X_1. The surfaces X' and X'_1 have the same points at infinity since there is no points at infinity of X' which belongs to the line l.

Since the surface X'_1 contains a cylinder-like open subset $U := graph(f_{|\mathbb{C}^2 \setminus l}) \cong \mathbb{C} \times \mathbb{C}^*$ and $X'_1 \setminus U = \bigcup_{i=1}^r l_i$, the first part of our claim follows from Corollary 2.3. Next, the map $\tilde{f}$ restricted to l_i is finite, hence surjective. This implies that every fiber of $\tilde{f}$ has a branch at infinity which intersects l_i. In particular $r \leq \nu_{\min}$. The inequality $r \geq b$ is obvious. □

Denote by $f_i : l_i \cong \mathbb{C} \to \mathbb{C}$ the restriction of $\tilde{f}$ to l_i. It can be identified with a one variable polynomial, the degree of which is equal to the number ν_i of branches of a generic fiber of $\tilde{f}$ which intersect l_i. In particular $\sum_{i=1}^{r} \nu_i = \nu_{\text{gen}}$.

The polynomial f_i of degree ν_i can have at most $\nu_i - 1$ critical points. If a fiber $\tilde{f}^{-1}(a)$ does not contain critical points of any f_i and does not contain singular points of X', then the point $a \notin B_\infty(f)$. This follows from general arguments concerning Whitney stratifications and Thom Isotopy Lemma, like in [3,15,19], but let us outline a short proof here. Firstly, the fiber $\tilde{f}^{-1}(a)$ cannot contain multiple components since otherwise, for some i, the fiber $f_i^{-1}(a)$ will also have a multiple component, thus a singularity, which contradicts our assumption. Therefore the fiber $\tilde{f}^{-1}(a)$ is nonsingular outside some large ball $B(0, R) \subset \mathbb{C}^2$. By the Sard Theorem there is a real value $R' > R$ such that the sphere $\partial B(0, R')$ is transversal to $\tilde{f}^{-1}(a)$. In particular there is a small disc $U(a, \rho)$ such that for every $b \in U(a, \rho)$ the fiber $\tilde{f}^{-1}(b)$ is smooth outside $B(0, R)$ and it is transversal to $\partial B(0, R')$. We can also assume that ρ is so small that $\tilde{f}^{-1}(b)$ does not contain critical points of any of the polynomials f_i, for $i = 1, \dots, r$, and it does not contain any singular point of X'. This means in particular that all these fibers are transversal to all curves $l_i,\ i = 1, \dots, r$. Now take $Y = \tilde{f}^{-1}(U(a, \rho)) \backslash Int(B(0, R'))$. It is a smooth manifold with boundary, where the boundary ∂Y is $\partial B(0, R') \cap \tilde{f}^{-1}(U(a, \rho))$. The set $V := (\bigcup_{i=1}^{r} l_i) \cap Y$ is a smooth submanifold of Y. The mapping $g := \tilde{f}_{|Y} : Y \to U(a, \rho)$ is proper and all fibers of g are transversal to V and to ∂Y. By the Ehresmann Theorem [4] there is a trivialization of g which preserves V and ∂Y. This proves our claim that $a \notin B_\infty(f)$.

Finally we conclude that the bifurcation values at infinity for f can be only images by $\tilde{f}$ of critical points of $f_i,\ i = 1, \dots, r$ and images of singular point of X'. Summing up, we get that f can have at most ν_{gen} critical values at infinity, which shows one of the inequalities of point (a). Moreover, the inequality $\nu_a < \nu_{\text{gen}}$ is possible only if a is a critical value of some polynomial f_i. This means that $\#\{a \in \mathbb{C} \mid \nu_a < \nu_{\text{gen}}\} \le \sum_{i=1}^{r}(\nu_i - 1) \le \nu_{\text{gen}} - r \le \nu_{\text{gen}} - b$, which proves (b).

Let us assume now $\nu_a = \nu_{\min}$. We have $\nu_a \ge \sum_{i=1}^{r} \#\{x \in l_i \mid f_i(x) = a\}$ since in every such point x there is at least one branch at infinity of the fiber $f^{-1}(a)$. Note that if $f_i(x) = a$ then $\text{ord}_x(f_i - a) = \text{ord}_x f_i' + 1$. Thus:

$$\#\{x \in l_i \mid f_i(x) = a\} = \sum_{x \in l_i,\, f_i(x)=a} [\text{ord}_x(f_i - a) - \text{ord}_x f_i'].$$

We have clearly the equality $\sum_{x \in l_i} \text{ord}_x(f_i - a) = \nu_i$. Hence

$$\sum_{x \in l_i,\, f_i(x)=a} [\text{ord}_x(f_i - a) - \text{ord}_x f_i'] = \nu_i - \sum_{x \in l_i,\, f_i(x)=a} \text{ord}_x f_i'.$$

Since $\sum_{x \in l_i} \text{ord}_x f_i' = \nu_i - 1$ we have:

$$\nu_i - \sum_{x \in l_i,\, f_i(x)=a} \text{ord}_x f_i' = 1 + \sum_{x \in l_i,\, f_i(x)\neq a} \text{ord}_x f_i'.$$

Note that:

$$1+\sum_{x\in l_i,\, f_i(x)\neq a} \operatorname{ord}_x f_i' \geq \#\{x\in l_i \mid f(x)\neq a, \text{ and either } f_i'(x)=0 \text{ or } x\in \operatorname{Sing}(X')\}.$$

The number at the right side is greater or equal to the number of critical values at infinity of f different from a. Finally, taking the sum over all $i \in \{1,\dots,r\}$ we get $\#B_\infty(f) \leq \nu_{\min}+1$, which completes the proof of (a).

To prove (c), note that if the fiber $\tilde{f}^{-1}(a)$ does not contain a singular point of X', which lies on some l_i, then the intersection multiplicity $\overline{l_i}\cdot\tilde{f}^{-1}(a)$ is equal to $\nu_i=\deg f_i$, where we consider here $\tilde{f}^{-1}(a)$ as a scheme-theoretic fiber of $\tilde{f}$. Hence the fiber $\tilde{f}^{-1}(a)$ has at most ν_i branches on l_i (even if counted with multiplicity). This implies $\nu_a\leq\nu_{\text{gen}}$. Therefore $\#\{a\in\mathbb{C}\mid \nu_a>\nu_{\text{gen}}\}\leq r\leq\nu_{\min}$.

To prove (d) and (e) it is enough to show that if $\nu_{\text{gen}}>\frac{d}{2}$, then at least one line l_i does not contain singular points of X'. Let d_i be the smallest positive integer such that d_il_i is a Cartier divisor in X' (such a number exists because X' has only cyclic singularities). Since l_i is smooth, we have that $d_i=1$ if and only if the line l_i does not contain any singular point of X', by the following lemma, the proof of which is left to the reader:

Lemma 2.5 *Let X^n be an algebraic variety and let $Z^r\subset X^n$ be a subvariety which is a complete intersection in X^n. If a point $z\in Z^r$ is nonsingular on Z^r, then it is nonsingular on X^n.* □

Now let Z be the closure of Γ in $\mathbb{P}^2\times\mathbb{P}^1$ and let Z' denote its normalization. We have clearly the inclusion $X'\subset Z'$. Let $\Pi : Z'\to\mathbb{P}^2$ the first projection, where the second projection $Z'\to\mathbb{P}^1$ is an extension of $\tilde{f}$ which we will denote by $\tilde{f}'$. Note that for $a\neq\infty$ fibers $\tilde{f}^{-1}(a)$ and $(\tilde{f}')^{-1}(a)$ coincide.

Let $(\tilde{f}')^{-1}(\infty)=S_1\cup\cdots\cup S_k$ (where S_i are irreducible and taken with reduced structure). Recall that $L_\infty=\mathbb{P}^2\backslash\mathbb{C}^2$ is the line at infinity. We have $\Pi^*(L_\infty)=\sum_{i=1}^k m_iS_i+\sum_{i=1}^r e_i\overline{l_i}$. Since $\Pi^*(L_\infty)$ is a Cartier divisor we have $e_i=n_id_i$, where n_i is a positive integer.

Let us assume that every line l_i contains some singular point of X', i.e., that $d_i>1$ for any i. Denoting by $F\subset\mathbb{P}^2$ the closure of a general fiber of f, since Π is a birational morphism, we have:

$$d=F\cdot L_\infty=\Pi^*(F)\cdot\Pi^*(L_\infty)=\left(\tilde{f}'\right)^*(a)\cdot\left(\sum_{i=1}^k m_iS_i+\sum_{i=1}^r e_i\overline{l_i}\right).$$

Note that $\Pi^*(F)\cdot\sum_{i=1}^k m_iS_i=0$ since $|(\tilde{f}')^*(a)|\cap|\sum_{i=1}^k m_iS_i|=|(\tilde{f}')^*(a)|\cap|(\tilde{f}')^*(\infty)|=\emptyset$. Moreover we have $\nu_i=(\tilde{f}')^*(a)\cdot\overline{l_i}$. Thus:

$$d=\sum_{i=1}^r n_id_i\nu_i\geq\sum_{i=1}^r 2\nu_i=2\nu_{\text{gen}}$$

and this ends our proof. □

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References

1. Broughton, S.A.: On the topology of polynomial hypersurface, singularities, Part 1 (Arcata, Calif., 1981). In: Proceedings of Symposia in Pure Mathematics, vol. 40, pp. 167–178. American Mathematical Society, Providence (1983)
2. Chądzyński, J., Krasiński, T.: The gradient of a polynomial at infinity. Kodai Math. J. **26**, 317–339 (2003)
3. Durfee, A.H.: Five definitions of critical point at infinity, singularities (Oberwolfach, 1996), In: Progress in Mathematics, vol. 162, pp. 345–360. Birkhauser, Basel (1998)
4. Ehresmann, C.: Sur les espaces fibrés différetiables. C.R Acad. Sci. Paris **224**, 1611–1612 (1947)
5. Hà, H.V., Lê, D.T.: Sur la topologie des polynômes complexes. Acta Math. Vietnam. **9**(1), 21–32 (1984)
6. Gurjar, R.V.: Remarks on the topology of surface singularities and applications. with an appendix by K. Paranjape. In: Proceedings of the Indo-French Conference on Geometry (Bombay, 1989), pp. 95–102. Hindustan Book Agency, Delhi (1993)
7. Gurjar, R.V., Miyanishi, M.: Automorphisms of Affine Surfaces with $\mathbb{A}^1$-Fibrations. Michigan Math. J. **53**, 33–55 (2005)
8. Gwoździewicz, J., Płoski, A.: Formulae for the singularities at ifinity of plane algebraic curves. Univ. Iagell. Acta Math. **XXXIX**, 109–133 (2001)
9. Gwoździewicz, J.: Ephraim's pencils. Int. Math. Res. Notices **15**, 3371–3385 (2013)
10. Jelonek, Z.: On bifurcation points of a complex polynomial. Proc. AMS. **131**, 1361–1367 (2003)
11. Jelonek, Z., Kurdyka, K.: On asymptotic critical values of a complex polynomial. J. Reine Angew. Math. **565**, 1–11 (2003)
12. Lê, V.T., Oka, M.: Estimation of the Number of the Critical Values at Infinity of a Polynomial Function. Publ. RIMS. Kyoto Univ. **31**, 577–598 (1995)
13. Miyanishi, M.: Singularities of normal affine surfaces containing cylinderlike open sets. J. Algebra **68**, 268–275 (1981)
14. Miyanishi, M.: Open algebraic surfaces. In: proceedings of CRM Monograph Series, vol. 12, pp. viii+259. American Mathematical Society, Providence (2001)
15. Siersma, D., Tibăr, M.: Singularities at infinity and their vanishing cycles. Duke Math. J. **80**(3), 771–783 (1995)
16. Suzuki, M.: Propriétés topologiques des polynômes de deux variables complexes, et automorphismes algébriques de l'espace $\mathbb{C}^2$. J. Math. Soc. Japan **26**, 241–257 (1974)
17. Thom, R.: Ensembles et morphismes stratifiés. Bull. Amer. Math. Soc. **75**, 249–312 (1969)
18. Tibăr, M.: Regularity at infinity of real and complex polynomial functions, singularity theory (Liverpool, 1996) , In: London Mathematical Society Lecture Note Series, vol. 263, pp. 249–264. Cambridge University Press, Cambridge (1999)
19. Tibăr, M.: Polynomials and vanishing cycles. cambridge Tracts in Mathematics, vol. 170, pp. xii+253. Cambridge University Press, Cambridge (2007)