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Results in Mathematics

Sequence Characterization of 3-Dimensional Riordan Arrays and Some Application

Roksana Słowik

Abstract. We propose the characterization of 3-dimensional Riordan arrays with use of three sequences that is analogous to the representation of 2-dimensional Riordan arrays with use of A and Z-sequence. We also suggest an application of this representation for finding totally positive matrices.

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1. Introduction

Let's recall that the Riordan group, introduced in [1], is a group of $\mathbb{N}_0 \times \mathbb{N}_0$ matrices that are identified with pairs of formal power series. Namely, denoting by $\mathcal{F}_0$—the ring of formal power series with nonzero free term, and by $\mathcal{F}_1$—the ring of formal power series with zero free term but nonzero the next term, the Riordan group $\mathcal{R}$ consists of pairs $\mathcal{R}(g, f)$ with $g \in \mathcal{F}_0$, $f \in \mathcal{F}_1$. The multiplication of these pairs is given by

$$\mathcal{R}(g_1(z), f_1(z)) * \mathcal{R}(g_2(z), f_2(z)) = \mathcal{R}(g_1(z) \cdot g_2(f_1(z)), f_2(f_1(z))),$$

and it coincides with multiplication of corresponding matrices.

Recently, one can observe an interest in multi-dimensional matrix algebra [2–4]. Here we would like to focus on three-dimensional matrices. The $(2, 1)$-product $C = [c_{ijk}]$ of $A = [a_{ijk}]$, $B = [b_{ijk}]$, is defined by the formula:

$$c_{ijk} = \sum_{x \geq 0} a_{ixk} b_{xjk}. \tag{1.1}$$

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In this note we are interested in $\mathcal{R}^{\langle 3\rangle}$—the group of 3-dimensional Riordan arrays. It was proved by Cheon and Jin [5] that $\mathcal{R}^{\langle 3\rangle}$ is an extension of $\mathcal{R}$ by $\mathcal{F}_0$. In this group the matrices $R = [r_{nkm}]_{n,k,m\in\mathbb{N}_0}$ are associated with the triples of series (g, f, h) with $g, h \in \mathcal{F}_0,\ f \in \mathcal{F}_1$. The multiplication of such triples is defined as follows:

$$\begin{gathered}\mathcal{R}(g_1(z), f_1(z), h_1(z)) * \mathcal{R}(g_2(z), f_2(z), h_2(z)) \\ = \\ \mathcal{R}(g_1(z)\cdot g_2(f_1(z)), f_2(f_1(z)), h_1(z)\cdot h_2(f_1(z))),\end{gathered} \tag{1.2}$$

Each entry of $R = [r_{nkm}]$ can be found from the relation:

$$r_{nkm} = [z^n]gf^kh^m, \tag{1.3}$$

where $[z^n]f$ denotes the n-th coefficient in the series expansion of f.

Thanks to definition (1.3), multiplication (1.2) corresponds with matrix multiplication given by (1.1).

It is known (see [6,7]) that Riordan arrays can be uniquely determined by two sequences, called A-sequence and Z-sequence. More precisely, starting with $r_{00} = g_0$, all the other entries can be found using the relations:

$$\begin{aligned} r_{n+1,k+1} &= a_0r_{nk} + a_1r_{n,k+1} + a_2r_{n,k+2} + \cdots, \\ r_{n+1,0} &= z_0r_{n0} + z_1r_{n1} + z_2r_{n2} + \cdots. \end{aligned} \tag{1.4}$$

In this paper we wish to give an analogous presentation for 3-dimensional Riordan arrays. Namely, we propose A, Z, and H-sequence, that completely characterize Riordan array:

$$\begin{aligned} r_{n00} &= z_0r_{n-1,0,0} + z_1r_{n-1,1,0} + z_2r_{n-1,2,0} + \cdots \\ r_{nk0} &= a_0r_{n-1,k+1,0} + a_1r_{n-1,k+2,0} + a_2r_{n-1,k+3,0} + \cdots \\ r_{nkm} &= h_0r_{n,k,m-1} + h_1r_{n-1,k,m-1} + h_2r_{n-2,k,m-1} + \cdots. \end{aligned} \tag{1.5}$$

We will show that the following theorem is true.

Theorem 1.1. *Any* 3*-dimensional Riordan array is completely characterized by its A, Z and H sequence given as in* (1.5)*. Moreover*

$$f(z) = z\cdot A(f(z)), \quad g(z) = \frac{g_0}{1 - z\cdot Z(f(z))}, \quad H(z) = h(z). \tag{1.6}$$

After discussing the above presentation, we will propose how it can be used to obtain some totally positive Riordan arrays.

2. The Disscussion

2.1. Sequence Characterization

To prove the main result, it suffices to notice that the below lemma holds.

Lemma 2.1. *The groups*

1. $\mathcal{R}_g^{\langle 3\rangle} := \{(g, f, 1) :\ g \in \mathcal{F}_0,\ f \in \mathcal{F}_1\}$,

2. $\mathcal{R}_h^{\langle 3\rangle} := \{(1, f, h) : \ h \in \mathcal{F}_0, \ f \in \mathcal{F}_1\}$

are isomorphic with $\mathcal{R}$. Moreover

1. *A and Z-sequence of $\mathcal{R}(g, f, 1)$ coincide with A and Z-sequence of $\mathcal{R}(g, f)$,*
2. *H and Z-squence of $\mathcal{R}(1, f, h)$ coincide with A and Z-sequence of $\mathcal{R}(h, f)$.*

Proof. Clearly, the maps $\phi_g : \mathcal{R}_g^{\langle 3\rangle} \to \mathcal{R}$, $\phi_h : \mathcal{R}_h^{\langle 3\rangle} \to \mathcal{R}$ given by

$$\phi_g(\mathcal{R}(g, h, 1)) = (g, h), \qquad \phi_h(\mathcal{R}(1, f, h)) = (h, f)$$

establish the desired isomorphism, and the correspondence of sequences. $\square$

Proof of Theorem 1.1. Comparing (1.4) and (1.5) one can notice that A and Z-sequence of $\mathcal{R}(g, f, h)$ coincide with A and Z-sequence of $\mathcal{R}(g, f)$. Thus, we only need to check the last equality. Using (1.3) we get

$$\begin{aligned} r_{nkm} &= [z^n]gf^k h^m = \sum_{i=0}^{n}([z^i]gf^k \cdot [z^{n-i}]h) = \sum_{i=0} r_{i,k,m-1} \cdot h_{n-i} \\ &= \sum_{j=0}^{n} h_j r_{n-j,k,m-1}. \end{aligned}$$

$\square$

For 2-dimensional Riordan arrays the following result was obtained by He and Sprugnoli.

Theorem 2.2 [8, Thm.3.3,3.4]. *Let $\mathcal{R}(g_1, f_1)$, $\mathcal{R}(g_2, f_2)$ be 2-dimensional Riordan arrays with A, Z-sequences: A_1, Z_1 and A_2, Z_2, respectively. Then A and Z-sequence of the product $\mathcal{R}(g_1, f_1) * \mathcal{R}(g_2, f_2)$ is equal to*

$$A(z) = A_2(z) \cdot A_1\left(\frac{z}{A_2(z)}\right),$$

$$Z(z) = \left(1 - \frac{z}{A_2(z)} Z_2(z)\right) \cdot Z_1(z) + A_1\left(\frac{z}{A_2(z)}\right) \cdot Z_2(z).$$

Based on the above one can prove the following.

Proposition 2.3. *Let $\mathcal{R}(g_1, f_1, h_1)$, $\mathcal{R}(g_2, f_2, h_2)$ be 3-dimensional Riordan arrays with A, Z and H-sequences: A_1, Z_1, H_1 and A_2, Z_2, H_2, respectively. Then A, Z and H-sequence of the product $\mathcal{R}(g_1, f_1, h_1) * \mathcal{R}(g_2, f_2, h_2)$ is equal to*

$$\begin{aligned} A(z) &= A_2(z) \cdot A_1\left(\frac{z}{A_2(z)}\right) \\ Z(z) &= \left[1 - \frac{z}{A_2(z)} Z_2(z)\right] \cdot Z_1\left(\frac{z}{A_2(z)}\right) + A_1\left(\frac{z}{A_2(z)}\right) \cdot Z_2(z) \\ H(z) &= H_2(z) \cdot H_1\left(\frac{z}{H_2(z)}\right). \end{aligned}$$

Proof. By Lemma 2.1, A and Z-sequence of $\mathcal{R}(g,f)$ coincide with A and Z-sequence of $\mathcal{R}(g,f,h)$, so two first equalities hold. By Theorem 1.1, h-sequence of $\mathcal{R}(g,f,h)$ is equal to h, so from the definition and Lemma 2.1, we get that h-sequence of $\mathcal{R}(g_1(z)\cdot g_2(f_1(z)), f_2(f_1(z)), h_1(z)\cdot h_2(f_1(z)))$ is the same as A-sequence of

$$\mathcal{R}(h_1(z)\cdot h_2(f_1(z)), f_2(f_1(z))) = \mathcal{R}(h_1(z), f_1(z)) * \mathcal{R}(h_2(z), f_2(z)).$$

Thus, using Theorem 2.2 again, we get the third equality. □

2.2. Possible Application

In this section we join the representation proposed in the first section with some other issue. Namely, the total positivity of a Riordan matrix. An infinite matrix is said to be totally positive (or shortly TP) if all its minors are nonnegative. In particular, a Toeplitz matrix

$$\begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & & & \ddots \end{bmatrix} \qquad \text{with all } a_n \geq 0,$$

is totally positive if and only if $a(z) = \sum_{n=0}^{\infty} a_n z^n$ has only real (and nonpositive) zeros, and in this case $(a_n)_{n=0}^{\infty}$ is called Pólya frequency sequence.

Let's get back to our matrices. It is obvious that fixing m in (1.3), one obtains a 2-dimensional Riordan array. It is called the m-th layer of $\mathcal{R}(g,f,h)$. According to (1.3), the m-th layer of $\mathcal{R}(g,f,h)$ is equal to $\mathcal{R}(gh^m, f)$. From

$$\mathcal{R}(g(z)h^m(z), f(z)) = \mathcal{R}(h^m(z), z) * \mathcal{R}(g(z), f(z)) = (\mathcal{R}(h(z), z))^m * \mathcal{R}(g(z), f(z)),$$

and the fact that the product of TP matrices is a TP matrix, we get the following conclusion.

Corollary 2.4. *If $\mathcal{R}(g(z), f(z))$ and $\mathcal{R}(h(z), z)$ are totally positive, then every layer of $\mathcal{R}(g(z), f(z), h(z))$ is totally positive.*

Totally positive matrices were considered in the context of A and Z sequences.

It was first proved in [9] (see also [10,11]) that every $2D$ Riordan array $\mathcal{R}(g,f)$ is induced by its production matrix

$$P_{\mathcal{R}(g,f)} = \begin{bmatrix} z_0 & a_0 & & & \\ z_1 & a_1 & a_0 & & \\ z_2 & a_2 & a_1 & a_0 & \\ \vdots & & & & \ddots \end{bmatrix}. \tag{2.1}$$

In particular, if we write U (as in [11]) for the shift matrix:

$$U = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

then $P_{\mathcal{R}(g,f)}$ is the production matrix of the Riordan array $\mathcal{R}(g,f)$ if and only if $U\mathcal{R}(g,f) = \mathcal{R}(g,f)P_{\mathcal{R}(g,f)}$.

From [12] we know that if the production matrix $P_{\mathcal{R}(g,f)}$ is TP, then $\mathcal{R}(g,f)$ is TP as well. Thus, we finish with the following observation.

Corollary 2.5. *If $P_{\mathcal{R}(g,f)}$ given by (2.1) is TP matrix and H is a Pólya frequency sequence, then every layer of $\mathcal{R}(g,f,h)$ is a totally positive matrix.*

Example. It can be checked that for

$$A = (2, 3, 1, 0, 0, 0, 0, \ldots), \qquad Z = (3, 5, 0, 0, 0, 0, \ldots)$$

the production matrix $P_{\mathcal{R}(g,f)}$ is TP (see [12]). Moreover,

$$H = (2, 5, 4, 1, 0, 0, 0, 0, \ldots)$$

is a Pólya frequency sequence. Thus, all the layers of $\mathcal{R}(g,f,h)$

$$L_1 = \begin{bmatrix} 1 & & & & & & \\ 3 & 2 & & & & & \\ 19 & 12 & 4 & & & & \\ 117 & 78 & 36 & 8 & & & \\ 741 & 504 & 272 & 96 & 16 & & \\ 4743 & 3266 & 1830 & 848 & 240 & 32 & \\ \vdots & & & & & & \ddots \end{bmatrix}, \quad L_2 = \begin{bmatrix} 2 & & & & & & \\ 11 & 4 & & & & & \\ 57 & 34 & 8 & & & & \\ 342 & 224 & 92 & 16 & & & \\ 2146 & 1448 & 740 & 232 & 32 & & \\ 13678 & 9376 & 5168 & 2208 & 560 & 64 & \\ \vdots & & & & & & \ddots \end{bmatrix},$$

$$L_3 = \begin{bmatrix} 4 & & & & & & \\ 32 & 8 & & & & & \\ 177 & 88 & 16 & & & & \\ 1015 & 634 & 224 & 32 & & & \\ 6241 & 4156 & 1972 & 544 & 64 & & \\ 39511 & 26922 & 14412 & 5640 & 1280 & 128 & \\ \vdots & & & & & & \ddots \end{bmatrix}, \quad L_4 = \begin{bmatrix} 8 & & & & & & \\ 84 & 16 & & & & & \\ 530 & 216 & 32 & & & & \\ 3047 & 1740 & 528 & 64 & & & \\ 18297 & 11842 & 5128 & 1248 & 128 & & \\ 114464 & 77248 & 39596 & 14128 & 2880 & 256 & \\ \vdots & & & & & & \ddots \end{bmatrix},$$

$$L_5 = \begin{bmatrix} 16 & & & & & & \\ 208 & 32 & & & & & \\ 1512 & 512 & 64 & & & & \\ 9088 & 4624 & 1216 & 128 & & & \\ 54033 & 33264 & 13024 & 2816 & 256 & & \\ 333131 & 220882 & 106976 & 34752 & 6400 & 512 & \\ \vdots & & & & & & \ddots \end{bmatrix}, \quad L_6 = \begin{bmatrix} 32 & & & & & & \\ 496 & 64 & & & & & \\ 4128 & 1184 & 128 & & & & \\ 26584 & 11936 & 2752 & 256 & & & \\ 159762 & 91728 & 32384 & 6272 & 512 & & \\ 974291 & 627092 & 284000 & 84096 & 14080 & 1024 & \\ \vdots & & & & & & \ddots \end{bmatrix}, \cdots$$

are totally positive.

3. Some Closing Comments

Let's finish this short note with some remarks about possible generalizations of the presented notions. In [13] the authors proposed extending the definition of 2-dimensional Riordan array given by

$$r_{nk} = [z^n]gf^k \qquad n, k \in \mathbf{N}_0$$

to all $n, k \in \mathbb{Z}$, and called them *recursive matrices* (in [14] they are also called complementary). Also in 3-dimensional case one can introduce 3-dimensional recursive matrix ${}_{\mathbb{Z}}R = [r_{nkm}]$ whose entries are given by (1.3) for all $n, k, m \in$

$\mathbb{Z}$. It is interesting that, by [15] (see Section 3 of this paper) for such ${}_{\mathbb{Z}}R$ the following identities hold:

$$r_{n+m,k+m,p} = \sum_{j=0}^{n-k} a_j^{(m)} r_{n,k+j,p},$$

$$r_{n+m,k+m,p} = \sum_{j=0}^{n-k} f_{j+m}^{(m)} r_{n-j,k,p},$$

where by $a_j^{(m)}$ we mean $[z^j]A^m$.

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Roksana Słowik
Institute of Mathematics
Silesian University of Technology
Kaszubska 23
44-100 Gliwice
Poland
e-mail: roksana.slowik@gmail.com

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