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A Tale of Bridges: Topology and Architecture

Topology, as its name indicates, is a (mathematical) way of conceiving of TOPOS: the place, the space, all space, and everything included in it. Jean-Michel Kantor evokes a few examples of forms and spaces which should be stimulating for all those interested in the concept of space, architects in particular. In topology, we no longer distinguish between two figures, two spaces, if you can pass from one to the other by means of a continuous deformation – with neither leap nor cut. Knots are a simple way of escaping from the obtuseness of space. Modern techniques of visualization developed for the military or for the Hollywood studios of Lucas Films can integrate the deformations on computer screens: the continued deformations of surfaces are discretized, that is, they are replaced by approximations produced at fixed intervals, then filmed in video. The time of the virtual corresponds to the era of topology, and architects are finding inspiration there.

Introduction

Mathematics is an enormous Ali Baba-type treasure cave in which, down through the centuries and the various world cultures, visitors have come to procure ideas, concepts, forms and results. The many tools available include calculations, curves, theorems, or, at times, new ideas. Architects have always found useful resources there, and some of them have even placed new products there on consignment. Thus it was with perspective – initially a product sold to mathematicians by an architect (Desargues). This Ali Baba's cave is immense – possibly even infinite, so they say. But here we shall limit ourselves to the "Topology" department.

As a first step we shall examine the catalogue. This will enable us to evoke a few examples of forms and spaces which should be stimulating for all those interested in the concept of space, architects in particular. In fact, the vocabulary of architecture has long inspired mathematicians: one of the rare texts in which the Bourbaki group presents its vision of mathematics is entitled "The Architecture of Mathematics" [Le Lionnais 1997]; a term derived from construction, "structure", plays a crucial role in the strategy of this group, which aimed at reconstructing mathematics in the twentieth century while also according an important place to topology.

Topology, as its name indicates, is a (mathematical) way of conceiving of TOPOS: the place, a space, all space, and everything included in it. Let us begin with the space of the Greeks: in the very first reflections on space, and starting with the point that is the atom, we can note a dual point of view which we will encounter again later on. In fact, in ancient Greek there are two words which are used for describing a point: *stigma*, a pinprick puncture[1] and *semion*, a sign. In other words, the point marks a space, or is a sign in that space of something else. With points, the Greeks made lines and volumes, we thus witness the birth of geometry.

The birth of topology on the bridges of Koenisberg.

The first sketches of reflection can be sought, as so often is the case, in Leibnitz: he opposes quantity and form in "Characteristica Geometria", and has the presentiment that we lack an adequate language for speaking of forms. In 1679 he writes to Huygens: "We need another strictly geometrical analysis which can directly express *situm* in the way algebra expresses the Latin *magnitudem*.

And so the term "analysis situs" was coined and would remain in use until the twentieth century.

Topology was born in 1735 (even though the term would only be created as of 1863), when Euler, a native of Basel who had moved to St. Petersburg, reported the following problem:

In Koenisberg (today known as Kaliningrad), there is an island (A) surrounded by a river which is split into two branches (fig. 1).

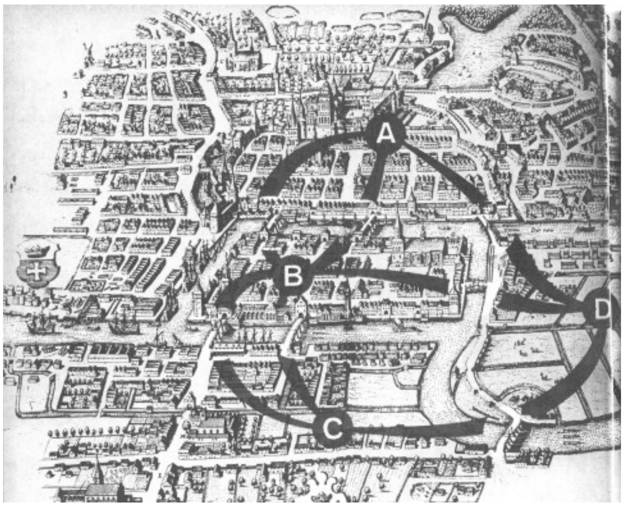

Fig. 1

And Euler was asked the following question: could a person possibly manage to cross one single time over each bridge? Opinion was divided at the time, and Euler gave the general solution – which is valid for any number of bridges and in any distribution of the branches. What is essential here is that he found the solution because he understood that the problem did not depend on the precise map of the city: he stated that it was not a problem of geometry: the distances, the lengths of the bridges for example, and the angles, do not come into play, and Euler establishes the new nature of the problem by using the term "geometry of position", an expression introduced for the first time by Leibnitz for "determining position and for seeking the properties which result from this position, without regards to the sizes themselves" [Euler 1741].

In other words, the quintessence of the problem resides in fig. 2.

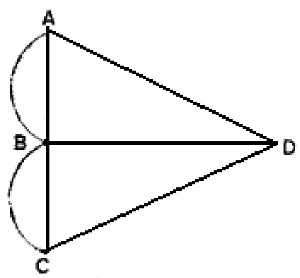

Fig. 2

This is the first diagram and the first manifestation of topology. The problem is reduced to its essence; a geometrical structure is transformed into a more flexible structure, that of topology:

If we replace each bank by a point or a cross (as symbol of the corresponding bank), and each bridge by a line which joins the associated banks, we then replace fig. 1 with fig. 2. The problem is then to draw a path with a pencil from a given summit – let's say A corresponding to the bank A, and ending at a given summit and following once, and only once, each of the lines ; the problem has no solution. In effect, let's imagine that we are following – as we should – the path, and reach a point we'll call B, which corresponds to one of the banks. It is then necessary to leave from there ! But then by coupling the paths of arrival and of departure, we observe that at each intermediary summit, there must be an even number of bridges. As this is not the case in the map of Koenisberg, Euler's excursion is impossible.

To summarize: the Koenisberg topology is given in fig. 2. We can vary the geometric data (the length of the branches, the surface measurement of the islands), deform the map of the city, but the topological structure and the resulting answer does not change.

Through Euler's promenade, mathematics suddenly discovers a brand new sense of freedom, which will then be constantly applied to forms and to spaces: now they can "twist again"!

In this mathematical discipline, we no longer distinguish between two figures or two spaces if it is possible to pass from one to the other by means of a continuous deformation – with neither leap nor cut. Topology is like mathematics made of rubber.

Fig. 3 gives an example which we often summarize by affirming that topology is the field in which you no longer distinguish between the cup and the breakfast bun.

Fig. 3

The use of graphs such as the one in fig. 2 makes it possible to pose – by means of a diagram – the questions in which only the combinational is of importance, in spite of their complexity, for example, all questions that arise from the organization of tasks, or networks (fig. 4). You can well imagine the influence this technique can have on the theory of graphs!

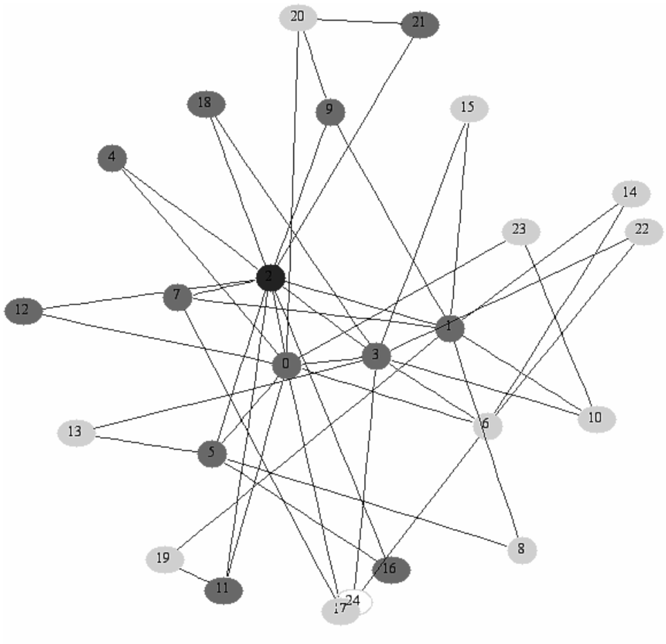

Fig. 4

Classifications – the obsession of topologists

I. Knots. The casual visitor to the "Topology " department of the Ali Baba treasure cave may well have the impression that he can play innocently with the various pieces of string, the knots. But how wrong he is! Nothing is less innocent, as Nabokov says:

> The most arduous of knots is merely a tightly-wound piece of rope that is highly fingernail-resistant. Yet the eye can master it. It was he (Sebastian Knight) this knot, and he was just about to be undone, if only he could find the way to avoid losing the thread. And not simply him, but everything would be worked out, everything he could possibly conceive of in relation to our juvenile notions of space and of time, one and the other of those enigmas invented by man as enigmas, which then come back to strike us: boomerangs of absurdity. [V. Nabokov, *The Real Life of Sebastian Knight*].

The theme of knots has inspired an immense body of literature in many fields – artistic [Coomaraswamy 1944], ethnographic, and scientific – before becoming the object of an advanced mathematical theory [Belpoliti and Kantor 1996; Sossinski 1999].

We may borrow several illustrations from Albert Flocon, an artist and professor from the Bauhaus school, who has both written and drawn on the subject of knots with the imagination of an artist fascinated with topology (fig. 5). We may well wonder what exactly is the cause of this universal interest.

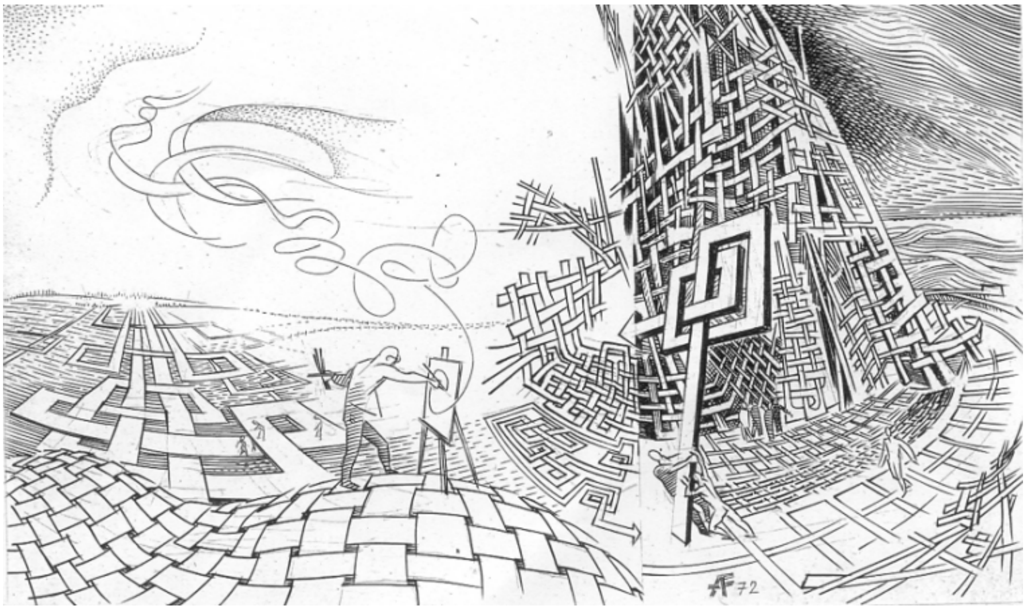

Fig. 5

It is true that knots are a simple way of escaping from the obtuseness of space: the presence of the knot overthrows the milieu itself. From the labyrinths of Leonardo to the medieval illuminations, and from Eskimo games to fishing techniques, the theme of the knot is widespread throughout all of the world's cultures. And this is an elementary example of topology: the deformations of the string are exactly the deformations which are authorized for the topologist.

The mathematician wonders just how to recognize the way that a complex knot is, in fact, undone. Or again, he poses a simple mathematical problem which could easily have been one of Alice's games (the young friend of Lewis Carroll) when she went through the looking glass. Can you deform the clover (hitch) knot of fig. 6a to change it into the clover knot of fig. 6b which happens to be its very image in the looking glass?

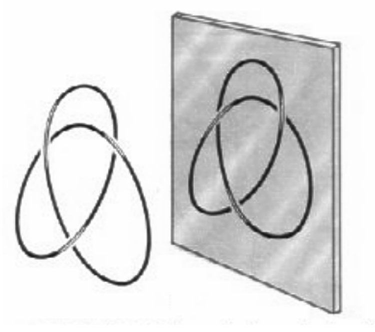

Fig. 6a (left) and 6b (right, mirror image)

In the middle of the nineteenth century, physics attempted to understand electricity, and the environment in which it was propagated: was the space then filled? Did ether exist? A physicist named Thomson, the future Lord Kelvin, imagined any atom as a circular vortex in the midst of the ether. It then becomes crucial to classify these knots, and hopefully to recover the classification of elements of Mendeleieff. Although these reflections weren't successfully concluded in physics (even though you can perceive, in the past few years, an avatar in them in the theory of strings), mathematicians have taken over the question and have obtained, after lengthy efforts, the complete classification of knots.

II. Two Surfaces. Let us consider the sphere, which is the surface of a ball. We cannot continuously deform the sphere in order to transform it into a bun (known as a "torus" in mathematics, fig. 4). Intuition may suggest it, but a demonstration is necessary. Here we shall develop the following argument, which we'll return to later: here is a property which is verified on the sphere and is not on the torus: on the sphere, any loop can be squeezed into a point, as in fig. 7, although this is not the case on the torus. If the two surfaces were equivalent, the property should be true or false for both.

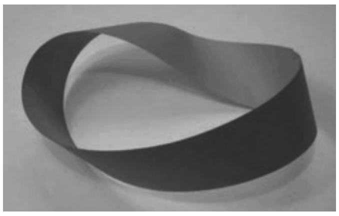

Fig. 7

Mathematicians in the early nineteenth century then asked, "Why not try to apply to surfaces the same classifications used for knots?" They then began by creating a veritable bestiary of surfaces:

A mathematician confided
That a Moebius band is one-sided,
And you'll get quite a laugh
If you cut one in half
For it stays in one piece
When divided.

So goes a student song from Tom Lehrer of Cambridge, Massachusetts.

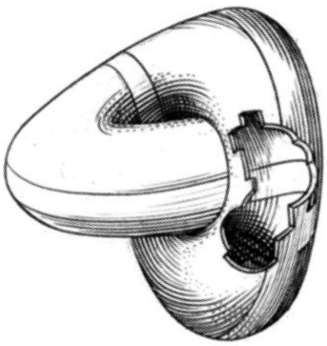

Fig. 8

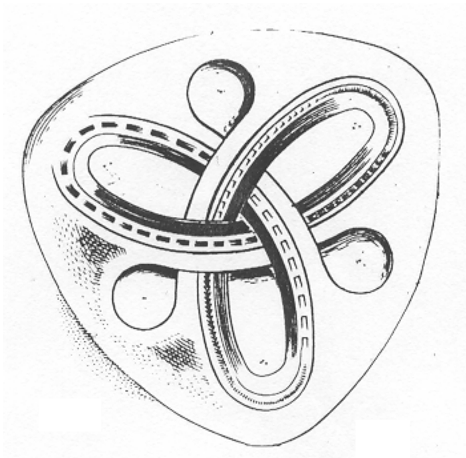

Fig. 9

Flocon even took it upon himself (for fun) to cut the Moebius strip in three (Fig. 9). Klein's bottle (Fig. 8) which fixes neither interior or exterior, evokes another refrain – also from Tom Lehrer:

A mathematician named Klein
Thought the Moebius band was divine,
Said he, "If you'll glue
The edges of two
You'll get a weird bottle like mine

Classification required and was eventually accomplished by Riemann (1826–1866). The result was that surfaces can be classified according the number of holes along them (the type of surface), and that this number can be determined "without leaving the surface." In other words, a little mathematician ant moving across the surface of fig. 8, for example, could determine the type of surface on which it lives, without leaving that surface.

III. In any dimension. The "varieties" introduced by Riemann only locally resemble portions of space (as the sphere locally, but not globally, resembles a piece of the map). In this way, many interesting examples of topology are produced; thus, the ensemble of spatial directions becomes a new topological space (topological space), which then resembles a Moebius strip.

On the other hand, for each of the preceding stages – point, knot, surfaces – we considered figures which had a " degree of liberty": zero for the point, where no movement is possible, one for the knot, and two for the surfaces.

This number of degrees of liberty is called dimension; thus the torus (fig. 7) may then be represented by a pair of numbers, each of which represents an angle: this is a variety of dimension 2.

Inversely, thanks to Riemann we can see spaces of dimension 4 and 5, etc., appearing. As the space-time of Einstein's relativity brought into play a space of dimension 4, at the beginning of the century we were witness to a cultural fashion. Salvador Dali associates the body of Christ with the hypercube (*Corpus Hypercubus*) and Max Weber depicted the "Interior of the Fourth

Dimension". Let us not forget, however, that access to this veritable mental liberation, which opened the path to both modern topology and to a part of contemporary mathematics, has remained limited: the common mental universe is still regulated and limited by the banal coordinates of Descartes.

A few words are in order here to evoke the figure of Henri PoInkaré – the founding father of modern topology – and in particular his famous conjecture, which is possibly on the way to being solved [Perelman]: it consists in characterizing the sphere of dimension 3 (analogous to the sphere of fig. 7) by a property of loops analogous to those described in paragraph III.2.[2]

IV. Forms, Deformations and Animations. Back before geometry was able to make use of the virtual animations of the computer, the various marvels of forms were represented by models made of plaster or steel wire which aided the teaching of geometry. These objects provoked astonishment. The modern techniques of visualization, developed for the military or for the Hollywood studios of Lucas Films, can integrate the deformations on computer screens: the continued deformations of surfaces are discretized, that is, they are replaced by approximations produced at fixed intervals, then filmed in video. The time of the virtual corresponds to the era of topology, and the architects are there taking inspiration (Bernard Cache, himself mentioned by Deleuze with regards to the use of space and territory in [Deleuze 1988]).

Space is our common ground, and it's impossible to break free of it, even in our dreams. The great Master of the universe is an architect, and ever since Galileo we have been imagining Him speaking and communicating in the language of mathematics. And since Riemann and PoInkaré, we can imagine Him dreaming of other spaces, and proceeding to build them as a highly-gifted topologist. This discipline has far from exhausted its resources; Bachelard was already a portender of Lacan and his Borromean knots:

> Flocon has such a collection of paper knots at his place that it makes one want to put them at the disposal of a psychoanalyst. I can imagine that the twisted and knotted strands, with arcs of tension and spirals of release, are the perfect instruments for a study on the connection of conscience [Bachelard 1950].

More recently, the cognitive sciences have been trying to create topological models of brain functioning (an old tradition compared memory to a theater [Yates 2001]).

Topology lends itself to comparisons and metaphors, for its flexibility is inscribed in its very structure: we can deform objects, as long as it's done with gentleness and subtlety!

We have been able to get a feel for an underground network of new ideas, all the way from Bauhaus – represented here by Flocon – to the situationist topology of Asger Jorn [1960]. Perhaps in the future it will be useful for the organization of space, by creating a Floconian bridge between the construction of individual space and the organization of social life.

Notes

1. Indivisible by definition for Euclid; but Leibnitz would add: "the point possesses a place", *situm habens* [Leibniz, *Euclidis Prota*].
2. In more precise terms, PoInkaré's conjecture states that if a smooth variety of dimension 3 is such that any closed loop can be continuously reduced to a point, then it is equivalent to the sphere of dimension three.

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Jean-Michel Kantor is a mathematician working in Paris, doing research in mathematics, its history and philosophy, and active in the diffusion of culture. He is a member of the editorial board of the magazine *La Quinzaine littéraire*. For more, see http://www.math.jussieu.fr/~kantor.