David Reid

# *Teaching Mathematics through Brick Patterns*

There are many elements of architecture that provide teachers and students useful opportunities for mathematical explorations. In this article educator David Reid examines a few aspects of what is possible with only one structure, the brick wall. Mathematics can make us more aware of aspects of the world we might normally ignore. This allows students to develop different view of mathematics, richer than the image of rules and facts that they often have. In the activities Reid describes here, the study of the patterns found in brick walls and pavements makes his students more aware of symmetry as a way of seeing.

## *Introduction*

Mathematics can make us more aware of aspects of the world we might normally ignore. In my work as a teacher educator I use a number of activities that show my students this aspect of mathematics. This allows them to develop different view of mathematics, richer than the image of rules and facts that they often have. In the activities I will describe here, the study of the patterns found in brick walls and pavements makes my students more aware of symmetry as a way of seeing.

## *The Mathematics Of Brick Patterns*

The walls of my classroom are built of concrete blocks arranged in the pattern known to bricklayers as "Running bond", or "Stretcher bond" (see Photograph 1). We begin our investigations of symmetry by looking for transformations that leave this pattern unchanged. As with all periodic patterns, Running bond maps onto itself when translated in two different directions (Fig. 1). This provides the first mathematical distinction that we will use in describing brick patterns: periodic versus non-periodic patterns.

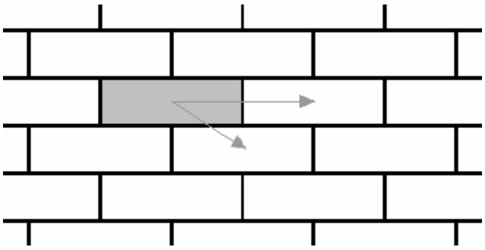
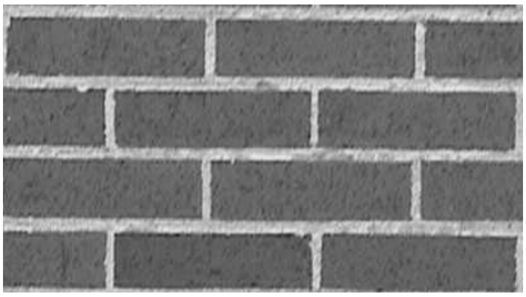


Fig. 1. Translations in Running bond



Photograph 1. Running bond

We then look for other transformations that leave the pattern unchanged. We find three types: vertical reflections through the centre of each brick, horizontal reflections through the centre of each brick, and 180° rotations. At the beginning diagrams (e.g., Fig. 2) are sufficient to describe the locations of mirror lines and centres of rotation . I am always modifying the activities I use, and so when I introduce precise language depends on the students and on the way the activities have developed. When the time comes I distinguish between elements of the bricks, and elements of the pattern. Most importantly, the four segments bordering a brick are its sides, but the sides of a brick might contain several edges of the pattern, an edge being a segment joining two adjacent points (this terminology is used by Grünbaum and Shepard [1987] to describe tilings of the plane). For example, in Running bond each brick is bordered by six edges, as the upper and lower sides of each brick are made up of two edges. The rotations occur at the midpoints of edges and at the centre of each brick (see fig. 2). Those at the midpoints of vertical edges and at the centre of each brick are compositions of the two reflections.

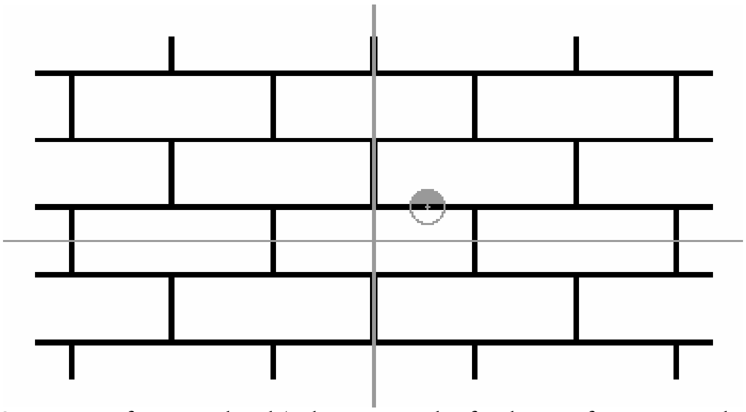
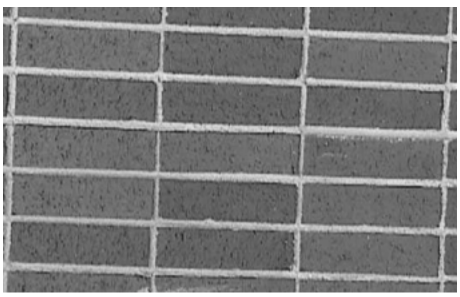


Fig. 2. Symmetries of Running bond (only one example of each type of symmetry is shown)

### *The Brick Pattern Tour*

Having introduced some possible symmetries, I then invite my students to explore the patterns in the walls on the university campus or nearby in the town (see http://plato.acadiau.ca/courses/educ/reid/Geometry/brick/brick-wolf.html for a sample). Sometimes I make this an independent task, but usually I take the class on a tour of some interesting walls. I have them focus first on patterns that are monohedral, that is, the exposed faces of the bricks are congruent.

After Running bond, the next monohedral pattern we encounter on our tour is the one called "Stack bond" (see Photograph 2).



Photograph 2. Stack bond

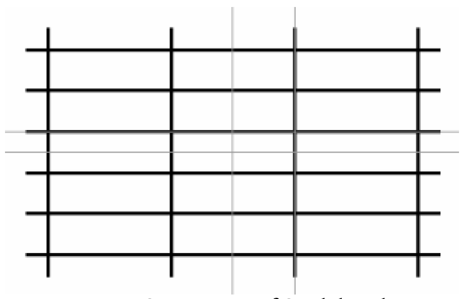


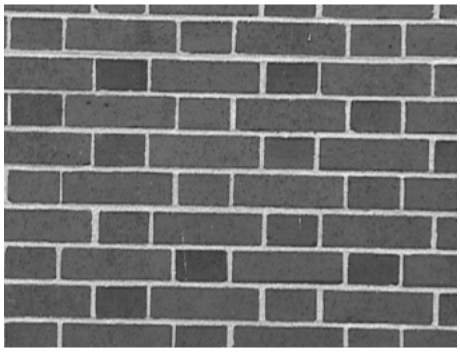
Fig. 3. Symmetries of Stack bond

The symmetries of Stack bond are shown in fig. 3. They include reflections vertically and horizontally though the centre of every brick, and along every edge. There are also 180° rotations that are compositions of two reflections, at the intersection of the reflection lines.

### *Brick Specific Vocabulary*

In addition to mathematical vocabulary to describe brick patterns, it is also useful to introduce my students to some terms used by bricklayers in describing walls and pavements. For example, a brick placed in a wall so that its long face is exposed is called a stretcher and if the short face is exposed it is called a header. Normally a stretcher is a little longer than twice the length of a header, so that two headers separated by a line of mortar are the same length as a stretcher. Photograph 3 shows a non-monohedral pattern called English bond, which is made up of alternating rows (or *courses*) of headers and stretchers.

Photograph 3. English bond, showing courses of headers and stretchers alternating



Photograph 4. Flemish bond, in which stretchers and headers alternate within the same course

| Notation | Name | Explanation | Illustration |
|---|---|---|---|
| S2 | Running bond | Each course is composed of stretchers, and is offset half a stretcher from the course above. | Photograph 1 |
| S0 | Stack bond | Each course is composed of stretchers, and is not offset from the course above. | Photograph 2 |
| S4H | English bond | Courses composed of stretchers alternate with courses composed of headers, and the header courses are offset a fourth of a stretcher from the stretcher course above. | Photograph 3 |
| SH2 | Flemish bond | Every course is composed of stretchers alternating with headers, and is offset by half the length of a stretcher+header from the course above. | Photograph 4 |

Table 1. Notations for four common brick patterns

Bricklayers classify the most common brick patterns into three groups. Patterns made up only of stretchers (e.g., Running bond and Stack bond) are called Stretcher bonds. Patterns in which courses of stretchers alternate with courses of headers are called English bonds. Patterns in which the courses consist of alternating stretchers and headers are called Flemish bonds (see Photograph 4). The arrangements shown in Photographs 1, 3 and 4 are the prototypical patterns of each type. Other patterns in a group are often given names based on the name of the group (e.g., English Cross bond and English Garden Wall bond are types of English bond, see Photographs 9 and 12).

Not all brick patterns have standard names (two different patterns are referred to as "Dutch bond"), and verbal descriptions found in texts on bricklaying are sometimes ambiguous. I have developed a notation to describe patterns more exactly. The letters S and H indicate the pattern of stretchers and headers in each course. Numbers indicate how much each course is shifted over or offset from the previous course, as a unit fraction of the total length of the pattern of headers and stretchers. Table 1 shows the notations for the four patterns illustrated above.

### *Returning to the Tour*

Running bond (S2) has an offset of one half and Stack bond (S0) has offset zero. What happens if the offset is between zero and one half? Another brick pattern I include in my tour has an offset of about one third (see fig. 4 and Photograph 5).

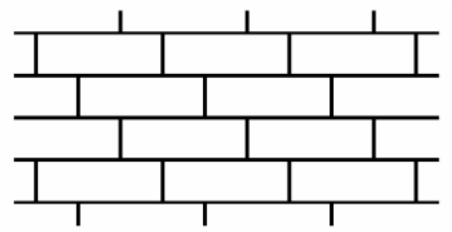


Fig. 4. Raking Stretcher bond[1] shows the pattern S3, with an offset of one third of a stretcher

Photograph 5 shows a pattern with an offset slightly more than one third

Note that Raking Stretcher bond has a "handedness", either it shifts to the right as one moves down one course, or it shifts to the left as one moves down.The most common offsets in Raking Stretcher bond are one fourth and one third of the length of the stretcher (S4 and S3). The symmetries are the same whatever the offset (see fig. 5). They are four rotations of 180 degrees, one at the centre of each brick and three at the midpoints of the edges. As any one of the four is the composition of the other three, only three need to be marked to completely define the set of transformations. This pattern provides an interesting basis for a classroom investigation of the composition of rotations. Composing any two rotations produces one of the translations in the pattern and composing three rotations produces a fourth.

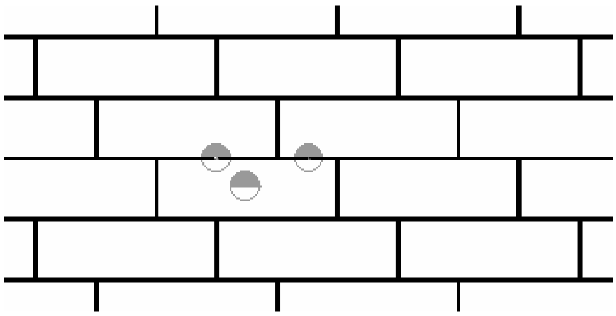
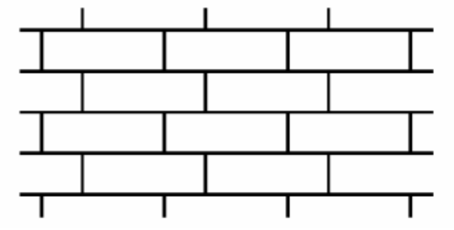


Fig. 5. Symmetries in a Raking Stretcher bond (S3 in this case)

In Raking Stretcher bond the offset is in the same direction in each course. My tour also includes a brick pattern in which the offset alternates left and right (see fig. 6 and Photograph 6). As with Raking Stretcher bond offsets of one third and one fourth are most common. Unlike Raking Stretcher bond these patterns have no handedness.



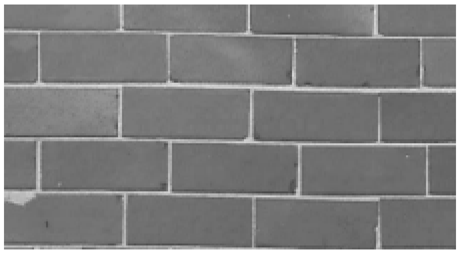


Fig. 6 and Photograph 6. One-Third-Running bond[2] (S3S)

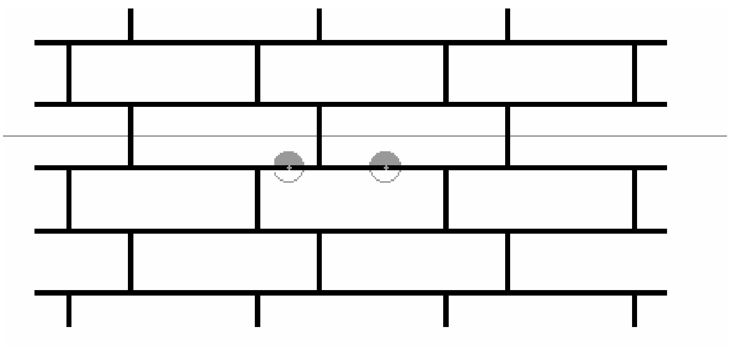
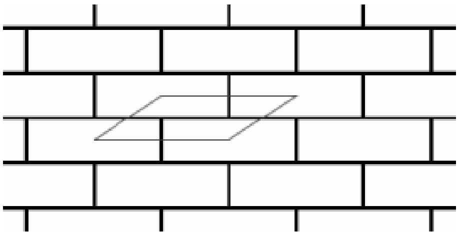


Fig. 7. The symmetries of One-Third-Running bond (S3S) and related patterns

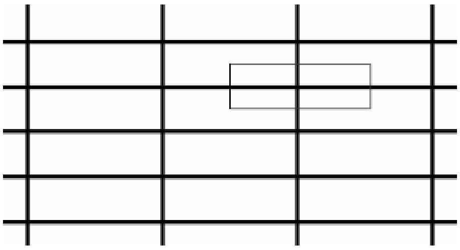
Fig. 7 shows the symmetries of One-Third-Running bond (and related patterns like S4S). They are two 180 degree rotations about the midpoints of the horizontal edges, and a reflection across a horizontal line through the centre of each brick.

S3S can also be the starting point of an interesting classroom investigation. All the patterns on the tour are periodic, but S3S is the first one in which two bricks must be translated to produce it. In the other patterns we have encountered only a single brick has to be translated to produce the entire pattern.

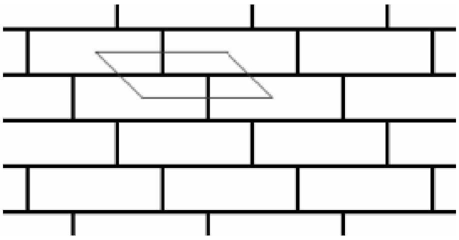
Grünbaum and Shepard [1987] call the parallelogram defined by the two basic translation vectors of a periodic tiling, the period parallelogram of that tiling. In fig. 8 the period parallelograms are shown. For S0, S2 and S3 the period parallelogram has the same area as one brick (see fig. 8, a, b, c). For S3S the period parallelogram has the area of two bricks. The bricks shaded in fig. 8d are the images of the brick shaded under the two translations.



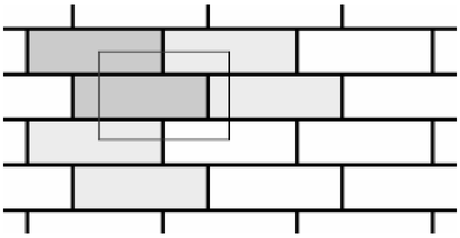
(a) Running bond



(b) Stack bond



(c) Raking Stretcher bond

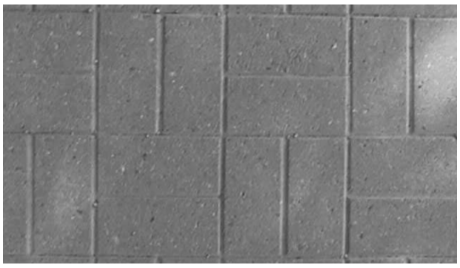


d) One-Third-Running bond

Fig. 8. Period parallelograms

### *Brick Patterns In Pavings*

In addition to looking at bricks in walls on the tour, I also suggest that my students look at bricks used in paving. We encounter two other monohedral patterns in this context. In the first of these, Basketweave, two bricks are placed together to make a square module. These modules are used to form a chequerboard pattern with adjacent modules rotated 90 degrees (see Photograph 7). In the second common paving pattern, Herringbone, the bricks are placed to form zigzags (see Photograph 8) .



Photograph 7. Basketweave

Photograph 8. Herringbone

In Basketweave there are two perpendicular reflection lines through the centres of every module, as well as a 90 degree rotation centred at the corners of the modules (see fig. 9). The figure shows only one reflection line as the composition of the reflection shown and the 90 degree rotation produces the perpendicular reflection.

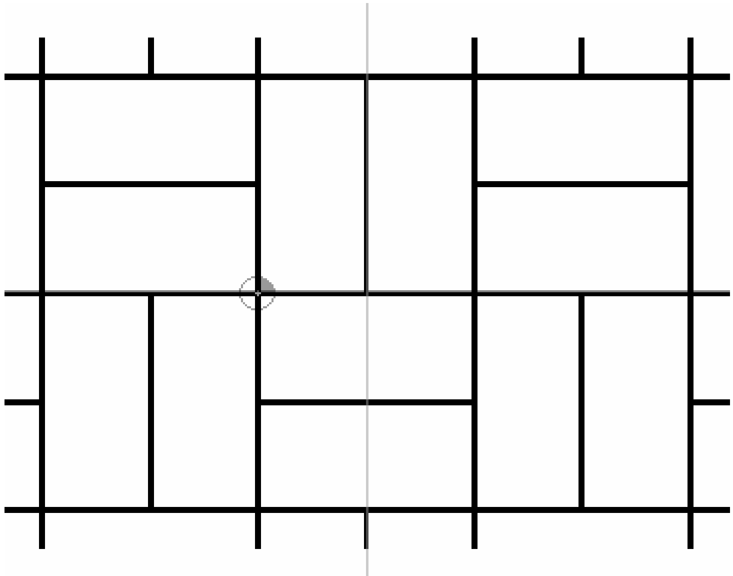
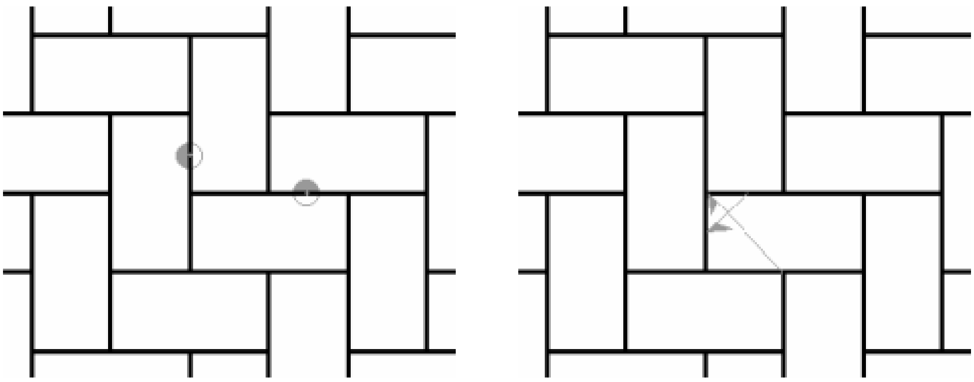


Fig. 9. Symmetries in the Basketweave pattern

In Herringbone the symmetries are harder to identify. My students are usually quick to observe that there are 180° rotations (See fig. 10a) at the centre of some edges. By copying the pattern on tracing paper and flipping it over, they can see that there is a reflectional symmetry of some sort, but investigations with mirrors make it clear there are no reflection lines. Careful work with tracing paper then leads to the discovery of perpendicular glide reflections (see fig. 10b).



(a) (b)

Fig. 10. Symmetries in the Herringbone pattern

A further investigation of the Herringbone pattern can explore how the rotations are produced by the perpendicular glide reflections.

*Back in the Classroom*

As noted above, some deeper investigations into the symmetries of the brick patterns we encounter on our tour are postponed for later work in class. The tour is also the starting point for a number of other investigations into the mathematics of brick patterns, appropriate to various age groups and mathematics content. These include plane symmetry groups, frieze patterns, cultural comparisons, introduction of the concepts of isohedrality and isogonality, and some preliminary analysis of the structural strength of different patterns.

**Plane symmetry groups.** One such investigation is the classification of brick patterns according to plane symmetry groups.

There are many descriptions of the 17 plane symmetries (for example, see Joyce, 1997; Grünbaum and Shepard, 1987; or Washburn and Crowe, 1988). Having studied the symmetries of brick patterns, my students are better able to make sense of the classification of the plane symmetries. They are also ready to investigate whether any of the plane symmetries that they have not observed in a brick pattern could occur.

The six monohedral patterns my students encounter on our tour are those listed by Coxeter (1969) in his brief discussion of symmetries in brick walls. They are examples of the plane symmetry groups cmm, pmm, p2, pmg, pgg, and p4g. Fig. 11 shows another brick pattern made only with stretchers that has symmetry p1. One question for further exploration is:

- Are p1 and the six symmetries Coxeter (1969) lists the only ones possible in brick patterns involving stretchers only? Can you prove this?

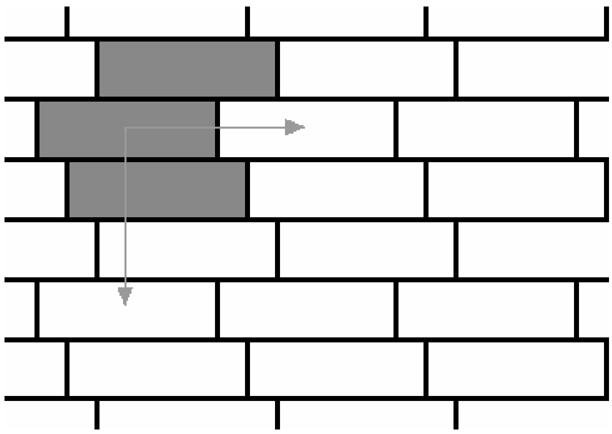


Fig. 11. A brick pattern representing plane symmetry group p1

Many brick patterns are not made up of stretchers only, but instead include headers and stretchers.

This raises a second question:

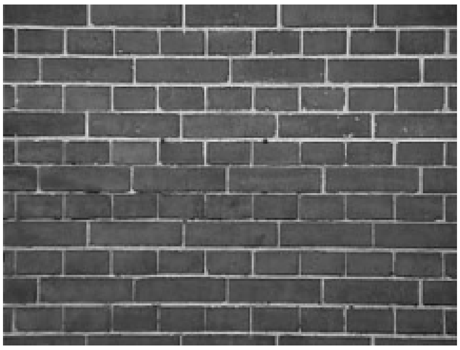
- Are there other symmetries possible in brick patterns involving both stretchers and headers? If not, why not?

These two questions provide the basis for mathematical activity, including proving, in a context that is more directly accessible to students than traditional Euclidean geometry tasks.

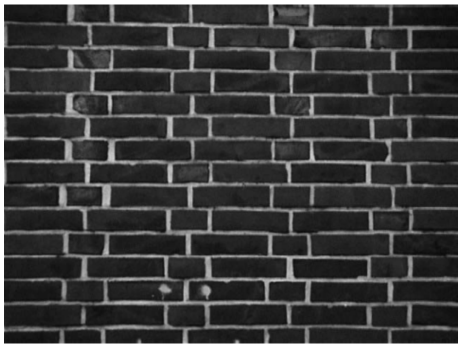
A more extensive exploration of plane symmetry is possible if non-rectangular paving bricks and tiles are considered. I usually give my students a taste of the possibilities during the tour, by examining a paving of octagons and squares, but time does not allow a complete treatment.

**Frieze Patterns.** In many cases walls and paving have borders that offer possibilities for explorations of frieze patterns, and of course other ornaments on buildings are a rich source. In the past I have experimented with basing my tours on friezes instead of on brick patterns, and with mixing the two topics, but I have found that my students find brick patterns both more accessible and interesting. I now leave friezes as a topic for independent study, based on the resources I provide at http://plato.acadiau.ca/courses/educ/reid/Geometry/Symmetry/frieze.html.

**Cultural differences.** During a tour of Wolfville, Nova Scotia, Canada, one sees different brick patterns than if, for example, the tour took place in Hamburg, Germany. In Wolfville the most common patterns are Stretcher bond (S2) and Flemish bond (SH2). In Hamburg the most common patterns are Stretcher bond, English Cross bond (S4H2, see Photograph 9 ) and Monk bond (SHS2, see Photograph 10). A surprising number of brick walls in Hamburg have no pattern to them (see Photograph 11) and hence are nonperiodic.

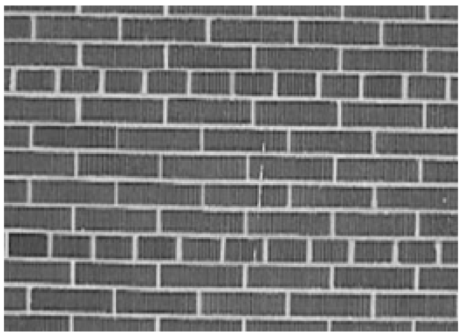


Photograph 9. English Cross bond; S4H2



Photograph 10. Monk bond; SHS2

Photograph 11. "Broken" bond, an arrangement with no pattern



Photograph 12. English Garden Wall bond; S2S2S4H (also known as American bond and Common bond)

A possible classroom activity is to compare the most frequently occurring brick patterns with a class in another community. Email has now made such cross cultural comparisons simple and quick and they are a source of many opportunities for mathematical explorations (see http://www.stemnet.nf.ca/~elmurphy/emurphy/math.html for another example).

**Isohedrality and isogonality.** Advanced students may be interested in exploring the concepts of isohedrality and isogonality in brick patterns. A brick pattern is isohedral if the symmetries of the pattern can map any brick onto any other brick. The six monohedral patterns described above are all isohedral. Patterns with both headers and stretchers cannot be isohedral, as there is no way to map a header onto a stretcher. At best such patterns can be 2-isohedral, that is, there are two disjoint isohedral sets of bricks that make up the entire pattern.

A brick pattern like English Garden Wall bond (S2S2S4H, see Photograph 12) is 4-isohedral. The headers form one isohedral set. The stretchers in courses adjacent to headers form another isohedral set. The stretchers in courses one course away from the headers form a third isohedral set. Finally, the stretchers along the horizontal mirror line between the headers form a fourth isohedral set.

A brick pattern is said to be isogonal if the symmetries of the pattern can map any edge onto any other edge. Because the edges in brick patterns are usually of two or more lengths, most brick patterns are not isogonal. Stack bond, Stretcher bond and Basketweave are 2-isogonal, as all their long edges can be mapped onto any other long edge and all their short edges can be mapped onto any other short edge. In Herringbone all the edges are the same length, but the pattern is also 2-isogonal, as the edges that contain the centres of the rotations (see fig. 10a) cannot be mapped onto the other edges. An interesting exploration is to try to create a brick pattern that is isogonal, or to prove that this cannot be done.

**Structural strength.** The analysis of the structural strength of a brick wall is a very complex task, involving consideration of many factors including the relative strengths of the bricks and the mortar between them. A simple introduction to the kind of mathematical activity involved can be based on the observation that the weakest point in the brick wall is usually the joint between bricks (see Photograph 13) and that brick walls usually fail either because of cracks from the top to the bottom of the wall down the visible face, or through the separation of two layers joined by header bricks.

Photograph 13. English Cross bond showing cracking along joints

These two modes of failure suggest two ways to quantify the strength of a bond. The resistance to vertical cracks can be quantified as the ratio of the length of the shortest possible crack along joints between bricks to the vertical distance. For example, in Photograph 13 the crack travels down 4 courses and has a length of 7 (using the height of each brick as the unit of measure) so

English Cross bond (S4H2) has a ratio of 7:4. Stack bond (S0), on the other hand, has a ratio of 1:1, and because of its weakness it is not used structurally. The resistance to the separation of layers can be quantified by determining what percentage of the pattern is occupied by headers. The more headers, the greater the strength of the bond between the surface layer and the layer behind it. The extreme cases are Header bond (H2) which is 100% headers, and Stretcher bonds (0% headers) which are used only when the bricks form a single layer over some other material (in this case the bricks are connected to the materials behind with metal ties embedded in the mortar between the bricks). These numbers oversimplify the problem of measuring the strength of a brick wall, but they serve as an introduction to two important issues and the use of mathematics in analysing structures.

## *Conclusion*

There are many elements of architecture that provide teachers and students useful opportunities for mathematical explorations. In this article I have only touched on a few aspects of what is possible with only one structure, the brick wall. More could be said about the patterns formed using coloured bricks, or the frieze patterns that are found at the edges of walls, or the patterns possible with non-rectangular bricks. The fact that even something as seemingly simple as a brick wall is rich in mathematics is a lesson I try to teach to future teachers, as an example of a way they might make their future students more aware of the presence of mathematics in the world around them.

## *Notes*

1. I encountered the name "Raking Stretcher bond" for this pattern somewhere, but I can no longer locate the source, and this nomenclature seems not to be general. I would be interested in hearing of other names for this pattern.
2. I encountered this name for this pattern on the web site of the Acme Brick Company (http://www.brick.com). I would be interested in hearing from anyone who knows of other names for it.

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## *About the Author*


David Reid is an associate professor in the School of Education of Acadia University in Nova Scotia, Canada. He teaches courses for future teachers at both the primary and secondary levels, as well as pursuing research into reasoning processes in mathematical activity at all ages. He has a Ph.D. in mathematics education from the University of Alberta. When he applied to study "mathematics and architecture" as an undergraduate he was told he had to pick just one. Since then he has been exploring the connections between mathematics, art and architecture, especially in teaching.