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Linearized field equations of gauge fields from the entanglement first law

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Abstract: In the context of the AdS/CFT correspondence linearized field equations of vector and antisymmetric tensor gauge fields around an AdS background are obtained from the entanglement first law of CFTs. The holographic charged entanglement entropy contains a term depending on the gauge field in addition to the Ryu-Takayanagi formula.

Keywords: AdS-CFT Correspondence, Gauge-gravity correspondence

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Contents

1 Introduction

The idea of entanglement has been discussed in the context of the AdS/CFT correspondence [1–3], which relates a conformal field theory (CFT) in Minkowski spacetime and a gravitational theory in higher dimensional anti de Sitter (AdS) spacetime. In particular, Ryu and Takayanagi proposed a direct connection between the entanglement entropy of a CFT to a dual bulk geometry [4, 5], which was generalized to a covariant form in [6]. (For reviews see [7, 8].)

Recently, the entanglement entropy has been used to understand how the bulk gravitational dynamics is obtained from a CFT [9–24]. In [11, 13] the linearized field equation of the gravitational field around AdS spacetime was derived from a property of the entanglement entropy of the CFT. The entanglement entropy satisfies the entanglement first law [25], which relates a variation of the entanglement entropy and that of the expectation value of the modular Hamiltonian. By rewriting this relation in terms of the bulk gravitational field by the AdS/CFT correspondence one obtains a constraint on the gravitational field, which turns out to be the linearized field equation.

The purpose of this paper is to extend the result of [11, 13] and show that linearized field equations of vector and antisymmetric tensor gauge fields also can be derived from the entanglement first law. Since gravitational theories dual to CFTs, such as supergravity and superstring theories, contain fields other than the gravitational field, it is natural to consider a possibility to derive their field equations from the entanglement first law.

To derive the linearized field equation of a vector field we consider a CFT with a conserved U(1) current J^μ. In the AdS/CFT correspondence the boundary value of the bulk vector field plays a role of a source for this current. Using the charge of this current we can define the charged entanglement entropy [26–28], which satisfies the entanglement first law similar to the first law of thermodynamics for a grand canonical ensemble. By rewriting the first law in terms of the bulk fields we obtain the linearized field equations of the vector field as well as the gravitational field. To rewrite the first law in terms of the bulk fields we follow the approach of [13], which uses the Noether charge of local symmetries of the bulk theory. In [13] the Noether charge for a coordinate transformation by a Killing vector was used. In that case the field equation of only the gravitational field was obtained. Even when matter fields are present in the bulk theory, they contribute to the first law only at higher orders in perturbations and their linearized field equations are not obtained. In our work we consider a U(1) gauge transformation of the vector field which preserves the background configuration in addition to the coordinate transformation. This allows us to obtain the linearized field equation of the vector field from the entanglement first law. In this calculation we find that the entanglement entropy expressed by the bulk fields (3.26) has an extra term depending on the vector field in addition to the Ryu-Takayanagi formula proportional to the area of the extremal surface.

The discussion for a vector field can be generalized to the case of an antisymmetric tensor field. By considering a CFT with conserved antisymmetric tensor current $J^{\mu_1\cdots\mu_n}$ we obtain the linearized field equation of an n-th rank antisymmetric tensor field from the entanglement first law. The charged entanglement entropy (4.16) contains a term depending on the antisymmetric tensor field in addition to the Ryu-Takayanagi formula.

The organization of this paper is as follows. In the next section we discuss the charged entanglement entropy of a CFT and the entanglement first law. In section 3 we consider a bulk theory consisting of a gravitational field and a vector field. We rewrite the entanglement first law in terms of the bulk fields, from which the linearized field equations are derived. In section 4 the linearized field equation of an antisymmetric tensor field is derived from the entanglement first law in a similar way. We conclude in section 5. In appendix A we discuss the holographic renormalization of an antisymmetric tensor field and derive a formula for the one-point function of the CFT current, which we use in the text. In appendix B we discuss another derivation of that formula without using the holographic renormalization calculation.

2 Charged entanglement entropy

We consider a CFT in d-dimensional flat Minkowski spacetime, which has an energy-momentum tensor $T_{\mu\nu}$ and a U(1) current J^μ satisfying

$$\partial_\mu T^{\mu\nu} = 0, \qquad T_\mu{}^\mu = 0, \qquad \partial_\mu J^\mu = 0. \tag{2.1}$$

We assume that this CFT is dual to a $(d+1)$-dimensional classical gravitational theory as discussed in the next section. In order to define the entanglement entropy in the CFT we choose a CFT state $|\psi\rangle$ and a region B on a time slice $t = t_0$. As in [11, 13, 29] we consider the case in which B is a ball of radius R centered at a spatial point x_0^i $(i = 1, 2, \cdots, d-1)$. The state of the region B is described by the reduced density matrix

$$\rho_B = \mathrm{tr}_{\bar{B}}\, \rho_{\text{total}} = \frac{e^{-H_B}}{\mathrm{tr}_B(e^{-H_B})}, \tag{2.2}$$

where $\rho_{\rm total} = |\psi\rangle \langle\psi|$ is the pure density matrix of the full system corresponding to the state $|\psi\rangle$, and ${\rm tr}_{\bar{B}}$ means tracing over states in $\bar{B}$, the complement of B on the time slice $t = t_0$. The density matrix ρ_B can be expressed by the operator H_B called the modular Hamiltonian as in (2.2). The entanglement entropy is defined as the von Neumann entropy of this reduced density matrix $S_B = -\,{\rm tr}_B\,(\rho_B \log \rho_B)$. Using the charge of the U(1) current J^μ we can also define the charged entanglement entropy [26–28]. We first introduce a new density matrix

$$\rho_B(\mu) = \frac{e^{-H_B+\mu Q_B}}{{\rm tr}_B(e^{-H_B+\mu Q_B})}, \tag{2.3}$$

where

$$Q_B = \int_B d^{d-1}x\, J^0 \tag{2.4}$$

is the charge operator in B and μ is a constant. Then, the charged entanglement entropy is defined as

$$S_B(\mu) = -\,{\rm tr}_B\left[\rho_B(\mu) \log \rho_B(\mu)\right]. \tag{2.5}$$

Now, consider an infinitesimal variation of the CFT state $|\psi\rangle \to |\psi\rangle + |\delta\psi\rangle$, which induces a variation of $\rho_B(\mu)$. The first order variation of the charged entanglement entropy (2.5) then gives the first law of the entanglement

$$\delta S_B(\mu) = \delta \langle H_B\rangle - \mu\, \delta \langle Q_B\rangle\,, \tag{2.6}$$

where H_B is the unperturbed modular Hamiltonian, and the expectation value of an operator O in B is defined as

$$\langle O\rangle \equiv {\rm tr}_B\left(\rho_B(\mu)\, O\right) = \langle\psi|\, e^{\mu Q_B} O\, |\psi\rangle\,. \tag{2.7}$$

Here, we have assumed that Q_B commutes with H_B. This is indeed the case in our setup as we will discuss in the next paragraph. The first law (2.6) resembles the first law of thermodynamics for a grand canonical ensemble. The constant μ corresponds to a chemical potential in thermodynamics. In the following we consider the case in which the unperturbed state $|\psi\rangle$ is the CFT vacuum $|0\rangle$. We will show in the next section that the first law (2.6) leads to linearized field equations of bulk gravitational and vector fields.

The modular Hamiltonian H_B is known when the CFT state is the vacuum $|\psi\rangle = |0\rangle$ and B is the ball-shaped region of radius R. It is given by

$$H_B = 2\pi \int_B d^{d-1}x\, \frac{R^2 - (x^i - x_0^i)^2}{2R}\, T_{00}, \tag{2.8}$$

where T_{00} is the 00 component of the energy-momentum tensor $T_{\mu\nu}$. This formula was obtained in [29] as follows. By a conformal transformation the causal development of the region B in d-dimensional Minkowski spacetime is mapped to a hyperbolic cylinder $\mathbb{R} \times H^{d-1}$, where H^{d-1} is a $(d-1)$-dimensional hyperboloid of curvature radius R, and $\mathbb{R}$ represents time. The modular Hamiltonian H_B in (2.8) and the charge operator Q_B in (2.4) are obtained from the Hamiltonian H and the charge operator Q in the hyperbolic cylinder as

$$H_B = 2\pi R\, U^{-1} H U, \qquad Q_B = U^{-1} Q U, \tag{2.9}$$

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where U is the unitary operator which implements the conformal transformation. Since Q is a conserved charge defined on the entire space H^{d-1}, it commutes with the Hamiltonian H. As a consequence H_B and Q_B in (2.9) also commute each other as we mentioned below (2.7). The density matrix (2.3) is then related to the thermal density matrix in the hyperbolic cylinder with temperature $T = (2\pi R)^{-1}$ and chemical potential μ as

$$\rho_B(\mu) = U^{-1} \frac{e^{-H/T+\mu Q}}{\mathrm{tr}(e^{-H/T+\mu Q})} U. \tag{2.10}$$

Therefore, the charged entanglement entropy (2.5) is equal to the thermal entropy of the CFT in the hyperbolic cylinder. By the AdS/CFT correspondence this thermal entropy can be calculated as the entropy of a black hole with a hyperbolic horizon in the bulk. This was done in [13] for the case $\mu = 0$ by using Wald's formula of the horizon entropy [30, 31]. We will generalize it to the case $\mu \neq 0$ in the next section.

3 Linearized field equations

In this section we first rewrite each side of the entanglement first law (2.6) in terms of bulk fields by the AdS/CFT correspondence. Then, the first law will require that perturbations of the bulk fields corresponding to the variation of the CFT state $|\delta\psi\rangle$ should satisfy linearized field equations. This was shown in [11, 13] for the bulk gravitational field in the case $\mu = 0$. We will generalize that result by introducing a bulk vector field corresponding to the CFT current J^μ. The entanglement first law will then require that perturbations of the vector field as well as the gravitational field should satisfy linearized field equations.

The Lagrangian for the gravitational field g_{ab} and a U(1) gauge field A_a in $(d+1)$-dimensional bulk spacetime is[1]

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\left(R + \frac{d(d-1)}{l^2}\right) - \frac{1}{4}\sqrt{-g}F_{ab}F^{ab}, \tag{3.1}$$

where l is a constant characterizing the cosmological constant. We have chosen the gravitational constant as $G = 1$ for simplicity. Under general variations of the fields the Lagrangian changes as

$$\delta\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\left(\delta g^{ab}E_{ab} + 16\pi\delta A_a E^a + \nabla_a v^a\right), \tag{3.2}$$

where

$$\begin{aligned}
E_{ab} &= R_{ab} - \frac{1}{2}g_{ab}R - \frac{d(d-1)}{2l^2}\,g_{ab} - 8\pi T^{\mathrm{bulk}}_{ab}, \\
E^a &= \nabla_b F^{ba}, \\
T^{\mathrm{bulk}}_{ab} &= F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}, \\
v_a &= \nabla^b \delta g_{ab} - g^{cd}\nabla_a \delta g_{cd} - 16\pi F_a{}^b \delta A_b.
\end{aligned} \tag{3.3}$$

[1]We use $a, b, \cdots = 0, 1, \cdots, d$ for $(d+1)$-dimensional coordinate indices.

$E_{ab} = 0$ and $E^a = 0$ are the field equations of g_{ab} and A_a respectively with T^{bulk}_{ab} being the energy-momentum tensor of the vector field. Under general coordinate transformations and U(1) gauge transformations

$$\begin{aligned}\delta_\xi g_{ab} &= \nabla_a \xi_b + \nabla_b \xi_a, \\ \delta_\xi A_a &= \xi^b \partial_b A_a + \partial_a \xi^b A_b + \partial_a \xi,\end{aligned} \tag{3.4}$$

the Lagrangian is invariant up to a total divergence

$$\delta_\xi \mathcal{L} = \partial_a \left(\xi^a \mathcal{L}\right). \tag{3.5}$$

To find a bulk representation of each side of (2.6) we first consider the Noether current corresponding to the local symmetry transformations (3.4) following [30–33]. The Noether current is

$$J^a = \frac{1}{16\pi}\, \bar{v}^a - \xi^a \frac{\mathcal{L}}{\sqrt{-g}}, \tag{3.6}$$

where $\bar{v}^a$ is given by v^a in (3.3) with δg_{ab}, δA_a replaced by $\delta_\xi g_{ab}$, $\delta_\xi A_a$ in (3.4). By (3.2) and (3.5) J^a satisfies

$$\nabla_a J^a = -\frac{1}{16\pi} \delta_\xi g^{ab} E_{ab} - \delta_\xi A_a E^a \tag{3.7}$$

and therefore is divergence free on-shell, i.e., when the field equations $E_{ab} = 0$, $E^a = 0$ are satisfied. As discussed in [34–36] we can construct a new current $\tilde{J}^a$ which coincides with J^a on-shell and is divergence free off-shell. Indeed, the right-hand side of (3.7) can be written as a divergence $\nabla_a S^a$, where

$$S^a = \frac{1}{8\pi} \xi_b E^{ab} - \left(\xi \cdot A + \xi\right) E^a, \tag{3.8}$$

and the new current $\tilde{J}^a = J^a - S^a$ satisfies $\nabla_a \tilde{J}^a = 0$ off-shell. Since $S^a = 0$ on-shell, $\tilde{J}^a$ coincides with J^a on-shell. In terms of differential forms $\nabla_a \tilde{J}^a = 0$ can be written as[2]

$$d\tilde{\boldsymbol{J}} = 0, \qquad \tilde{\boldsymbol{J}} = \tilde{J}^b\, \boldsymbol{\epsilon}_b, \tag{3.9}$$

and we find that $\tilde{\boldsymbol{J}}$ is an exact form $\tilde{\boldsymbol{J}} = d\boldsymbol{Q}$, where

$$\boldsymbol{Q} = \left[-\frac{1}{16\pi} \nabla^b \xi^c - \frac{1}{2} \left(\xi \cdot A + \xi\right) F^{bc}\right] \boldsymbol{\epsilon}_{bc}. \tag{3.10}$$

In (3.9), (3.10) we used the notation

$$\begin{aligned}\boldsymbol{\epsilon}_b &= \frac{1}{d!}\, \epsilon_{b a_1 \cdots a_d}\, dx^{a_1} \wedge \cdots \wedge dx^{a_d}, \\ \boldsymbol{\epsilon}_{bc} &= \frac{1}{(d-1)!}\, \epsilon_{b c a_1 \cdots a_{d-1}}\, dx^{a_1} \wedge \cdots \wedge dx^{a_{d-1}},\end{aligned} \tag{3.11}$$

where $\epsilon_{a_1 \cdots a_{d+1}}$ is the totally antisymmetric tensor with non-vanishing components $\pm\sqrt{-g}$.

We then split the fields as $g_{ab} \to g_{ab} + \delta g_{ab}$, $A_a \to A_a + \delta A_a$, where g_{ab}, A_a are background fields satisfying the field equations $E_{ab} = 0$ and $E^a = 0$, and δg_{ab}, δA_a are small

[2] We use boldface letters to denote differential forms.

perturbations around the background. In the setting of the CFT in the previous section the background corresponds to the vacuum $|\psi\rangle = |0\rangle$ and the perturbations correspond to an infinitesimal variation of the state $|\delta\psi\rangle$. The background corresponding to the CFT vacuum is the AdS metric and a gauge field with vanishing field strength:

$$g_{ab}dx^a dx^b = \frac{l^2}{z^2}\left(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \qquad F_{ab} = 0. \tag{3.12}$$

Here, the coordinate z takes values $z > 0$, and the AdS boundary at infinity $z = 0$ corresponds to the Minkowski spacetime with coordinates x^μ $(\mu = 0, 1, \cdots, d-1)$, in which the CFT is defined. The factor $e^{\mu Q_B}$ in the expectation value in (2.7) means that the vector field A_a has a non-vanishing background proportional to μ for $z \to 0$ [26–28].

The background (3.12) is invariant under the transformations (3.4) when ξ^a is a Killing vector of AdS. As in [13] we use the Killing vector

$$\xi^a\partial_a = -\frac{2\pi}{R}(t-t_0)\left[z\partial_z + (x^i - x_0^i)\partial_i\right] + \frac{\pi}{R}\left[R^2 - z^2 - (t-t_0)^2 - (x^i - x_0^i)^2\right]\partial_t, \tag{3.13}$$

which approaches to the d-dimensional Killing vector corresponding to time translation of the hyperbolic cylinder at the boundary $z = 0$. When we use the gauge field A_a itself instead of the field strength F_{ab}, we also need to consider a compensating U(1) gauge transformation. To find it we note that the transformation of the gauge field in (3.4) can be rewritten as

$$\delta_\xi A_a = \xi^b F_{ba} + \partial_a\left(\xi \cdot A + \xi\right). \tag{3.14}$$

Therefore, the gauge field A_a with $F_{ab} = 0$ is invariant when we choose the U(1) gauge transformation parameter as

$$\xi = \mu' - \xi \cdot A, \tag{3.15}$$

where μ' is an arbitrary constant. Later we will choose this constant as $\mu' = \mu$, where μ is the chemical potential appearing in the entanglement first law (2.6).

We can now construct a $(d-1)$-form $\boldsymbol{\chi}$ from the bulk fields which gives each side of the entanglement first law (2.6) as

$$\int_B \boldsymbol{\chi} = \delta\langle H_B\rangle - \mu\delta\langle Q_B\rangle, \qquad \int_{\tilde{B}} \boldsymbol{\chi} = \delta S_B(\mu). \tag{3.16}$$

Here, B is the ball-shaped region of radius R centered at $x^i = x_0^i$ on a time slice $t = t_0$ in d-dimensional Minkowski spacetime at the boundary $z = 0$:

$$B = \{(z, x^\mu)\,|\, t = t_0,\ z = 0,\ (x^i - x_0^i)^2 \le R^2\}. \tag{3.17}$$

$\tilde{B}$ is the extremal surface in the bulk which has the same boundary as B and is homologous to B. $\tilde{B}$ has the extremal area with respect to the background metric (3.12) and turns out to be a hemisphere

$$\tilde{B} = \{(z, x^\mu)\,|\, t = t_0,\ (x^i - x_0^i)^2 + z^2 = R^2\}. \tag{3.18}$$

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The desired $(d-1)$-form is given by

$$\boldsymbol{\chi} = \delta \boldsymbol{Q} - \frac{1}{16\pi} v^b \xi^c \boldsymbol{\epsilon}_{bc}, \tag{3.19}$$

where $\boldsymbol{Q}$ and v^a are given in (3.10) and (3.3). The transformation parameters ξ^a and ξ are those in (3.13) and (3.15) with $\mu' = \mu$. δ denotes variations of the fields g_{ab}, A_a with the transformation parameters ξ^a, ξ fixed. In the following we will show that this $\boldsymbol{\chi}$ indeed satisfies (3.16). We impose a gauge condition on the perturbations of the fields as

$$\delta g_{zz} = 0, \quad \delta g_{z\mu} = 0, \quad \delta A_z = 0. \tag{3.20}$$

First, let us examine the integral over B of the first equation in (3.16). Using (3.12), (3.13), (3.15) we find

$$\int_B \boldsymbol{\chi} = \int d^{d-1}x \left[\frac{d l^{d-3}}{16R} \left\{ R^2 - (x^i - x_0^i)^2 \right\} \delta g_{00}^{(d-2)} + (d-2)\, l^{d-3} \mu\, \delta A_0^{(d-2)} \right] \Bigg|_{z=0}, \tag{3.21}$$

where we have put

$$\delta g_{\mu\nu}(x,z) = z^{d-2} \delta g_{\mu\nu}^{(d-2)}(x), \qquad \delta A_\mu(x,z) = z^{d-2} \delta A_\mu^{(d-2)}(x) \tag{3.22}$$

for $z \to 0$ so that (3.21) takes a finite value. By the holographic renormalization [37, 38] $\delta g_{\mu\nu}^{(d-2)}$ and $\delta A_\mu^{(d-2)}$ are related to the one-point functions of the energy-momentum tensor $T_{\mu\nu}$ and the current J_μ of the CFT as

$$\delta \langle T_{\mu\nu} \rangle = \frac{d l^{d-3}}{16\pi} \delta g_{\mu\nu}^{(d-2)}, \qquad \delta \langle J_\mu \rangle = (d-2)\, l^{d-3} \delta A_\mu^{(d-2)}. \tag{3.23}$$

The first relation in (3.23) was already used in [11, 13]. The second relation is obtained in appendix A. Substituting (3.23) into (3.21) we obtain

$$\int_B \boldsymbol{\chi} = \int_B d^{d-1}x \left[2\pi \frac{R^2 - (x^i - x_0^i)^2}{2R} \delta \langle T_{00} \rangle - \mu \delta \left\langle J^0 \right\rangle \right]. \tag{3.24}$$

Using (2.8), (2.4) we find that the first equation in (3.16) is indeed satisfied.

Next, let us examine the integral over $\tilde{B}$ of the second equation in (3.16). Using (3.12), (3.13), (3.15) we find

$$\begin{aligned} \int_{\tilde{B}} \boldsymbol{\chi} = \int_B d^{d-1}x &\left[\frac{l^{d-3}}{8R\, z^{d-2}} \left(R^2 \delta^{ij} - (x - x_0)^i (x - x_0)^j \right) \delta g_{ij} \right. \\ &\left. + \frac{l^{d-3}}{z^{d-3}} \mu \left(\delta F_{z0} + \frac{(x - x_0)^i}{z} \delta F_{i0} \right) \right] \Bigg|_{z = \sqrt{R^2 - (x^i - x_0^i)^2}}. \end{aligned} \tag{3.25}$$

As was shown in [11, 13] the first term is the variation of the Ryu-Takayanagi formula proportional to the area A of the extremal surface for the metric $g_{ab} + \delta g_{ab}$. The second term is an additional contribution depending on the gauge field. If we assume that the charged entanglement entropy is given by[3]

$$S_B(\mu) = \frac{1}{4} A - \mu \int_{\tilde{B}} * \boldsymbol{F}, \tag{3.26}$$

where $*\boldsymbol{F}$ is the Hodge dual of $\boldsymbol{F}$, then the second equation in (3.16) is satisfied.

[3]The entanglement entropy of this form was previously used as an order parameter that distinguishes various phases of field theories [39]. We thank Juan F. Pedraza for informing us of this work.

Once we accept (3.26) as the formula for the charged entanglement entropy, we can use it to derive the relations in (3.23) between the expectation values of the CFT operators and the asymptotic values of the fields without using the holographic renormalization calculation. In [13] the first relation in (3.23) for the energy-momentum tensor was indeed derived from the entanglement first law (2.6) with $\mu = 0$ and the Ryu-Takayanagi formula by considering a small size limit of the ball-shaped region B. Similarly, the second relation in (3.23) for the current can be derived from the μ-dependent terms of (2.6) in the same limit. This derivation is discussed in appendix B.

Thus, we have shown that $\boldsymbol{\chi}$ in (3.19) satisfies (3.16) assuming that the charged entanglement entropy is given by (3.26). The entanglement first law (2.6) then requires

$$0 = \int_{\tilde{B}} \boldsymbol{\chi} - \int_B \boldsymbol{\chi} = \int_\Sigma d\boldsymbol{\chi}, \tag{3.27}$$

where Σ is the region enclosed by $\tilde{B}$ and B on the time slice $t = t_0$ satisfying $\partial\Sigma = \tilde{B} - B$. The exterior derivative of $\boldsymbol{\chi}$ in (3.19) is found to be

$$d\boldsymbol{\chi} = \left(-\frac{1}{8\pi} \delta E^{ab} \xi_b + \mu \delta E^a \right) \boldsymbol{\epsilon}_a, \tag{3.28}$$

where δE_{ab} and δE^a are variations of E_{ab} and E^a in (3.3):

$$\begin{aligned} \delta E_{ab} &= \frac{1}{2} \left(\nabla^c \nabla_a \delta g_{bc} + \nabla^c \nabla_b \delta g_{ac} - \nabla^2 \delta g_{ab} - g^{cd} \nabla_a \nabla_b \delta g_{cd} \right) \\ &\quad - \frac{1}{2} g_{ab} \left(\nabla^c \nabla^d \delta g_{cd} - g^{cd} \nabla^2 \delta g_{cd} \right) + \frac{d}{l^2} \delta g_{ab} - \frac{d}{2l^2} g_{ab} g^{cd} \delta g_{cd}, \\ \delta E^a &= g^{ab} \nabla^c \delta F_{cb}. \end{aligned} \tag{3.29}$$

$\delta E_{ab} = 0$ and $\delta E^a = 0$ are the linearized field equations around the background (3.12).[4] Substituting (3.28) into (3.27) we obtain

$$\int_\Sigma dz d^{d-1}x \left(g^{00} \delta E_{00} \xi^0 - 8\pi\mu \delta E^0 \right) = 0, \tag{3.30}$$

where we have used the fact that only the time components of ξ^a and $\boldsymbol{\epsilon}_a$ are non-vanishing on Σ. By requiring that this condition holds for arbitrary R, x_0^i, t_0 and μ we obtain the local conditions $\delta E_{00} = 0$, $\delta E^0 = 0$ (See appendix A of [13]). Moreover, requiring it for any frame of reference we obtain $\delta E_{\mu\nu} = 0$, $\delta E^\mu = 0$ $(\mu, \nu = 0, 1, \cdots, d-1)$.

In [13] it was shown that the remaining gravitational equations $\delta E_{z\mu} = 0$ and $\delta E_{zz} = 0$ are obtained from $\delta E_{\mu\nu} = 0$ and the tracelessness and the conservation of the CFT energy-momentum tensor $T_{\mu\nu}$ in (2.1). Similarly, $\delta E_z = 0$ can be obtained as follows. From the identity $\nabla_a E^a = 0$ and the field equation $\delta E_\mu = 0$ derived above we obtain

$$0 = \nabla_a \left(g^{ab} \delta E_b \right) = \frac{z^{d+1}}{l^2} \partial_z \left(z^{-d+1} \delta E_z \right). \tag{3.31}$$

[4] The on-shell closed form $\boldsymbol{\chi}$ may be understood as a calibration [40]. We thank Eoin Ó Colgáin for pointing it out to us.

Therefore, we find

$$\delta E_z = z^{d-1} C(x), \tag{3.32}$$

where $C(x)$ is an unknown function of x^μ. Using (3.22) and (3.23) we find

$$C(x) = z^{-(d-1)} \delta E_z \Big|_{z=0} = -\frac{z^{3-d}}{l^2} \eta^{\mu\nu} \partial_\mu \partial_z \delta A_\nu \Big|_{z=0} = -l^{-(d-1)} \delta \langle \partial_\mu J^\mu \rangle = 0, \tag{3.33}$$

where we have used the conservation of the CFT current in (2.1). Therefore, we find $\delta E_z = 0$. To summarize, we have obtained all the components of the linearized field equations $\delta E_{ab} = 0$, $\delta E^a = 0$ from the entanglement first law.

4 Antisymmetric tensor field

The discussion in the previous sections for a vector field can be generalized to the case of an antisymmetric tensor field. To derive the linearized field equation of an antisymmetric tensor field we consider a CFT in d-dimensional Minkowski spacetime, which has an energy-momentum tensor $T_{\mu\nu}$ and an n-th rank antisymmetric tensor current $J^{\mu_1 \cdots \mu_n}$ satisfying

$$\partial_\mu T^{\mu\nu} = 0, \qquad T_\mu{}^\mu = 0, \qquad \partial_{\mu_1} J^{\mu_1 \cdots \mu_n} = 0. \tag{4.1}$$

As in section 2 we introduce a density matrix

$$\rho_B(\mu) = \frac{e^{-H_B + \mu_{i_1 \cdots i_{n-1}} Q_B^{i_1 \cdots i_{n-1}}/(n-1)!}}{\mathrm{tr}_B (e^{-H_B + \mu_{i_1 \cdots i_{n-1}} Q_B^{i_1 \cdots i_{n-1}}/(n-1)!})}, \tag{4.2}$$

where $Q_B^{i_1 \cdots i_{n-1}} = \int_B d^{d-1}x J^{0 i_1 \cdots i_{n-1}}$ is the charge operator in B and $\mu_{i_1 \cdots i_{n-1}}$ is a constant. The charged entanglement entropy is defined as in (2.5). It satisfies the entanglement first law

$$\delta S_B(\mu) = \delta \langle H_B \rangle - \frac{1}{(n-1)!} \mu_{i_1 \cdots i_{n-1}} \delta \left\langle Q_B^{i_1 \cdots i_{n-1}} \right\rangle. \tag{4.3}$$

The $(d+1)$-dimensional bulk theory dual to this CFT consists of the gravitational field g_{ab} and an n-th rank antisymmetric tensor field $A_{a_1 \cdots a_n}$. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{16\pi} \sqrt{-g} \left(R + \frac{d(d-1)}{l^2} \right) - \frac{1}{2(n+1)!} \sqrt{-g} F_{a_1 \cdots a_{n+1}} F^{a_1 \cdots a_{n+1}}, \tag{4.4}$$

where the field strength is defined as

$$F_{a_1 \cdots a_{n+1}} = (n+1) \partial_{[a_1} A_{a_2 \cdots a_{n+1}]}. \tag{4.5}$$

Under general variations of the fields the Lagrangian changes as

$$\delta \mathcal{L} = \frac{1}{16\pi} \sqrt{-g} \left(\delta g^{ab} E_{ab} + \frac{16\pi}{n!} \delta A_{a_1 \cdots a_n} E^{a_1 \cdots a_n} + \nabla_a v^a \right), \tag{4.6}$$

where

$$\begin{aligned}
E_{ab} &= R_{ab} - \frac{1}{2} g_{ab} R - \frac{d(d-1)}{2l^2}\, g_{ab} - 8\pi T^{\text{bulk}}_{ab}, \\
E^{a_1\cdots a_n} &= \nabla_b F^{b a_1\cdots a_n}, \\
T^{\text{bulk}}_{ab} &= \frac{1}{n!}\left[F_{a c_1\cdots c_n} F_b{}^{c_1\cdots c_n} - \frac{1}{2(n+1)}\, g_{ab} F^2 \right], \\
v^a &= \nabla^b \delta g_{ab} - g^{cd} \nabla_a \delta g_{cd} - \frac{16\pi}{n!}\, F_a{}^{c_1\cdots c_n} \delta A_{c_1\cdots c_n}.
\end{aligned} \tag{4.7}$$

$E_{ab} = 0$ and $E^{a_1\cdots a_n} = 0$ are the field equations of g_{ab} and $A_{a_1\cdots a_n}$ respectively with T^{bulk}_{ab} being the energy-momentum tensor. Under general coordinate transformations and antisymmetric tensor gauge transformations

$$\begin{aligned}
\delta_\xi g_{ab} &= \nabla_a \xi_b + \nabla_b \xi_a, \\
\delta_\xi A_{a_1\cdots a_n} &= \xi^b \partial_b A_{a_1\cdots a_n} + n \partial_{[a_n} \xi^b A_{a_1\cdots a_{n-1}]b} + n \partial_{[a_1} \xi_{a_2\cdots a_n]} \\
&= \xi^b F_{b a_1\cdots a_n} + n \partial_{[a_1} \left(\xi \cdot A + \xi \right)_{a_2\cdots a_n]},
\end{aligned} \tag{4.8}$$

the Lagrangian is invariant up to a total divergence as in (3.5).

We split the fields into a background and small perturbations around the background: $g_{ab} \to g_{ab} + \delta g_{ab}$, $A_{a_1\cdots a_n} \to A_{a_1\cdots a_n} + \delta A_{a_1\cdots a_n}$. The background is a solution of the field equations $E_{ab} = 0$, $E^{a_1\cdots a_n} = 0$ and is given by the AdS metric g_{ab} in (3.12) and $A_{a_1\cdots a_n}$ satisfying $F_{a_1\cdots a_{n+1}} = 0$. This background is invariant under the local transformations (4.8) when ξ^a is the Killing vector (3.13) and the gauge transformation parameter is

$$\xi_{a_1\cdots a_{n-1}} = \mu_{a_1\cdots a_{n-1}} - \xi^b A_{b a_1\cdots a_{n-1}}, \tag{4.9}$$

where $\mu_{a_1\cdots a_{n-1}}$ is a constant. This constant will be identified with μ in the first law (4.3). Since only the space components $\mu_{i_1\cdots i_{n-1}}$ transverse to the region B appear in (4.3), we set other components to zero.

The $(d-1)$-form $\boldsymbol{\chi}$ which satisfies the analog of (3.16) is given by the same form as (3.19), where v^a is now given in (4.7) and $\boldsymbol{Q}$ is

$$\boldsymbol{Q} = \left[-\frac{1}{16\pi} \nabla^b \xi^c - \frac{1}{2(n-1)!}\, \mu_{a_1\cdots a_{n-1}} F^{a_1\cdots a_{n-1} bc} \right] \boldsymbol{\epsilon}_{bc}. \tag{4.10}$$

We impose a gauge condition on the perturbations of the fields as

$$\delta g_{zz} = 0, \quad \delta g_{z\mu} = 0, \quad \delta A_{z \mu_1\cdots \mu_{n-1}} = 0. \tag{4.11}$$

Integrating $\boldsymbol{\chi}$ over B we obtain

$$\begin{aligned}
\int_B \boldsymbol{\chi} &= \int d^{d-1}x \left[\frac{d l^{d-3}}{16R} \left\{ R^2 - (x^i - x_0^i)^2 \right\} \delta g^{(d-2)}_{00} \right. \\
&\quad \left. + \frac{d-2n}{(n-1)!}\, l^{d-2n-1} \mu^{i_1\cdots i_{n-1}} \delta A^{(d-2n)}_{0\, i_1\cdots i_{n-1}} \right] \\
&= \delta \left\langle H_B \right\rangle - \frac{1}{(n-1)!}\, \mu_{i_1\cdots i_{n-1}} \delta \left\langle Q_B^{i_1\cdots i_{n-1}} \right\rangle,
\end{aligned} \tag{4.12}$$

where we have defined

$$\delta A_{\mu_1\cdots\mu_n} = z^{d-2n}\delta A^{(d-2n)}_{\mu_1\cdots\mu_n} \tag{4.13}$$

for $z \to 0$ and used the result of the holographic renormalization

$$\delta\left\langle J_{\mu_1\cdots\mu_n}\right\rangle = (d-2n)\, l^{d-2n-1}\delta A^{(d-2n)}_{\mu_1\cdots\mu_n} \tag{4.14}$$

discussed in appendix A. On the other hand, integrating $\boldsymbol{\chi}$ over $\tilde{B}$ we obtain

$$\int_{\tilde{B}} \boldsymbol{\chi} = \delta S_B(\mu), \tag{4.15}$$

where

$$S_B(\mu) = \frac{1}{4}A - \int_{\tilde{B}} \boldsymbol{\mu}\wedge *\boldsymbol{F}. \tag{4.16}$$

We assume that this $S_B(\mu)$ corresponds to the charged entanglement entropy in (4.3). It contains a term depending on the antisymmetric tensor field in addition to the Ryu-Takayanagi formula $\frac{1}{4}A$. As in the case of the vector field the relation (4.14) can be derived also from the entanglement first law (4.3) and the formula (4.16) by considering a small size limit of the ball-shaped region B as discussed in appendix B.

The entanglement first law (4.3) requires (3.27) with this $\boldsymbol{\chi}$. The exterior derivative of $\boldsymbol{\chi}$ is found to be

$$d\boldsymbol{\chi} = \left[-\frac{1}{8\pi}\delta E^{ab}\xi_b + \frac{1}{(n-1)!}\,\mu_{b_1\cdots b_{n-1}}\delta E^{ab_1\cdots b_{n-1}}\right]\boldsymbol{\epsilon}_a, \tag{4.17}$$

where δE^{ab} is given in (3.29) and

$$\delta E^{a_1\cdots a_n} = g^{a_1b_1}\cdots g^{a_nb_n}\nabla^c\delta F_{cb_1\cdots b_n}. \tag{4.18}$$

$\delta E^{ab} = 0$ and $\delta E^{a_1\cdots a_n} = 0$ are the linearized field equations. By requiring (3.27) for arbitrary R, x^i_0, t_0 and $\mu_{a_1\cdots a_{n-1}}$ in any frame of reference we obtain d-dimensional components of the linearized field equations $\delta E^{\mu\nu} = 0$, $\delta E^{\mu_1\cdots\mu_n} = 0$. Furthermore, the remaining components $\delta E_{z\mu} = 0$, $\delta E_{zz} = 0$, $\delta E_{z\mu_1\cdots\mu_{n-1}} = 0$ are obtained from the tracelessness and the conservation of the energy-momentum tensor and the current (4.1) as in the case of a vector field. Indeed, from the identity $\nabla_{a_1}E^{a_1\cdots a_n} = 0$ and the field equation $\delta E^{\mu_1\cdots\mu_n} = 0$ we find

$$\delta E_{z\mu_1\cdots\mu_{n-1}} = z^{d-2n+1}C_{\mu_1\cdots\mu_{n-1}}(x), \tag{4.19}$$

where $C_{\mu_1\cdots\mu_{n-1}}(x)$ is an unknown function of x^μ. Using (4.13) and (4.14) we find

$$C_{\mu_1\cdots\mu_{n-1}}(x) = z^{-(d-2n+1)}\delta E_{z\mu_1\cdots\mu_{n-1}}\Big|_{z=0} = -l^{-(d-2n+1)}\delta\left\langle\partial_\nu J^\nu{}_{\mu_1\cdots\mu_{n-1}}\right\rangle = 0 \tag{4.20}$$

and therefore $\delta E_{z\mu_1\cdots\mu_{n-1}} = 0$. Thus, we have obtained all the components of the linearized field equations of g_{ab} and $A_{a_1\cdots a_n}$.

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5 Conclusions

In this paper we have shown that the linearized field equations of vector and antisymmetric tensor gauge fields as well as the gravitational field can be derived from the entanglement first law of a CFT with a conserved current. To rewrite the first law in terms of the bulk fields we followed the approach of [13] and made use of the Noether charges of symmetry transformations. We considered the gauge transformations of the vector and antisymmetric tensor fields as well as the coordinate transformation. This allows us to obtain the linearized field equations of the gauge fields. We found that the bulk representations of the charged entanglement entropy (3.26), (4.16) contain the extra terms depending on the gauge fields in addition to the Ryu-Takayanagi formula.

The derivations of the original Ryu-Takayanagi formula were given in [26, 29]. It would be interesting to study whether our formulae (3.26), (4.16) also can be derived in a similar manner. In [29] the Ryu-Takayanagi formula was derived by using the relation between the entanglement entropy for the ball-shaped region and the thermal entropy in the hyperbolic cylinder $\mathbb{R}\times H^{d-1}$, which we briefly reviewed in section 2. By the AdS/CFT correspondence the thermal entropy of the CFT is then related to the black hole entropy in the bulk, which turns out to equal to the Ryu-Takayanagi formula. In this paper we followed more or less this approach at a linearized order in perturbations. However, we have not discussed a relation of our entropy formulae to black hole entropies. It would be better to clarify this point and to confirm our formulae. Another approach [26] to derive the Ryu-Takayanagi formula uses a bulk generalization of the replica trick. It would also be interesting to check whether this approach gives our entropy formulae.

The approach in this paper to derive linearized field equations from the entanglement first law may be further generalized to other bulk fields related to local symmetries. For instance, the field equation of a Rarita-Schwinger field may be derived from the entanglement first law by considering the local supersymmetry. On the other hand, it is not clear how to derive field equations of bulk fields such as scalar and spinor fields, which are not related to local symmetries. This is an open problem to be studied in future.

A Holographic renormalization

In this appendix we briefly discuss the holographic renormalization [37, 38] of an n-th rank antisymmetric tensor field $A_{a_1\cdots a_n}$ in $d+1$ dimensions. We will obtain the formula (4.14) for the one-point function of the CFT current $J_{\mu_1\cdots\mu_n}$ used in the text. The case of a vector field (3.23) can be obtained by setting $n=1$. The Lagrangian of the antisymmetric tensor field is

$$\mathcal{L} = -\frac{1}{2(n+1)!}\sqrt{-g}\,F_{a_1\cdots a_{n+1}}F^{a_1\cdots a_{n+1}}, \tag{A.1}$$

where $F_{a_1\cdots a_{n+1}}$ is the field strength (4.5) and g_{ab} is the AdS metric in (3.12). We use the gauge condition $A_{z\mu_1\cdots\mu_{n-1}}=0$.

The solution of the field equation derived from this Lagrangian can be expanded for small z as

$$\begin{aligned} A_{\mu_1\cdots\mu_n}(x,z) &= A^{(0)}_{\mu_1\cdots\mu_n}(x) + z^2 A^{(2)}_{\mu_1\cdots\mu_n}(x) + \cdots \\ &\quad + z^{d-2n} A^{(d-2n)}_{\mu_1\cdots\mu_n}(x) + z^{d-2n}\log z^2 B^{(d-2n)}_{\mu_1\cdots\mu_n}(x) + \cdots , \end{aligned} \tag{A.2}$$

where $B^{(d-2n)}_{\mu_1\cdots\mu_n} = 0$ when d is odd. The field equation gives relations among the coefficient functions. The coefficient functions $A^{(m)}_{\mu_1\cdots\mu_n}$ $(m < d-2n)$ and $B^{(d-2n)}_{\mu_1\cdots\mu_n}$ are determined as local functions of $A^{(0)}_{\mu_1\cdots\mu_n}$ by the field equation. In the AdS/CFT correspondence $A^{(0)}_{\mu_1\cdots\mu_n}$ plays a role of the source of the CFT current $J^{\mu_1\cdots\mu_n}$, while $A^{(d-2n)}_{\mu_1\cdots\mu_n}$ is related to the one-point function of the current and represents a CFT state [41, 42].

According to the AdS/CFT correspondence the generating functional of connected correlation functions of the CFT current is given by the classical action evaluated at the solution satisfying the Dirichlet boundary condition $A_{\mu_1\cdots\mu_n}(x, z = 0) = A^{(0)}_{\mu_1\cdots\mu_n}(x)$ [2, 3]. Since the integral over z in the action is divergent near $z = 0$, we need to regularize it and subtract divergences. We regularize the action integral as

$$\begin{aligned} S_{\text{reg}} &= \int_{z>\epsilon} dz d^d x\, \mathcal{L} \\ &= -\frac{1}{2(n+1)!}\int_{z>\epsilon} dz d^d x \left(\frac{l}{z}\right)^{d-2n-1} F_{a_1\cdots a_{n+1}} F^{a_1\cdots a_{n+1}}, \end{aligned} \tag{A.3}$$

where ϵ is a small cut-off parameter. Here and in the following the raising and lowering of indices are done by the flat metric η_{ab}. By integration by parts and using the field equation we can rewrite the regularized action as a d-dimensional integral at $z = \epsilon$

$$S_{\text{reg}} = \frac{1}{2n!}\left(\frac{l}{\epsilon}\right)^{d-2n-1}\int_{z=\epsilon} d^d x\, A^{\mu_1\cdots\mu_n}\partial_z A_{\mu_1\cdots\mu_n}. \tag{A.4}$$

Substituting the expansion (A.2) into this action we find that it contains a finite number of divergent terms, which are local functionals of $A^{(0)}_{\mu_1\cdots\mu_N}(x,\epsilon)$. To remove the divergences we introduce a counterterm

$$S_{\text{ct}} = \int_{z=\epsilon} d^d x\, \mathcal{L}_{\text{ct}}, \tag{A.5}$$

where $\mathcal{L}_{\text{ct}}$ is a local function of $A^{(0)}_{\mu_1\cdots\mu_n}(x,\epsilon)$. This counterterm is chosen such that the renormalized action

$$S_{\text{ren}} = \lim_{\epsilon\to 0}\left(S_{\text{reg}} + S_{\text{ct}}\right) \tag{A.6}$$

is finite. We note that there is an arbitrariness of adding finite terms to the counterterm.

The one-point function of the current is then given by

$$\frac{1}{n!}\langle J_{\mu_1\cdots\mu_n}(x)\rangle = \frac{\delta S_{\text{ren}}}{\delta A^{(0)\mu_1\cdots\mu_n}(x)} = \lim_{\epsilon\to 0}\frac{\delta(S_{\text{sub}} + S_{\text{ct}})}{\delta A^{\mu_1\cdots\mu_n}(x,\epsilon)}. \tag{A.7}$$

Here, we have assumed that the coupling of the gauge field to the current in the CFT Lagrangian has the normalization $\mathcal{L}_{\text{CFT}} = \cdots + \frac{1}{n!}A^{(0)}_{\mu_1\cdots\mu_n}J^{\mu_1\cdots\mu_n}$. Using the regularized action in the form (A.4) we find

$$\frac{\delta(S_{\rm sub}+S_{\rm ct})}{\delta A^{\mu_1\cdots\mu_n}(x,\epsilon)} = \frac{1}{n!}\left(\frac{l}{\epsilon}\right)^{d-2n-1} \partial_z A_{\mu_1\cdots\mu_n}(x,z)|_{z=\epsilon} + \frac{\delta S_{\rm ct}}{\delta A^{\mu_1\cdots\mu_n}(x,\epsilon)}$$
$$\xrightarrow[\epsilon\to 0]{} \frac{1}{n!}(d-2n)\, l^{d-2n-1} A^{(d-2n)}_{\mu_1\cdots\mu_n}(x) + \frac{1}{n!} X_{\mu_1\cdots\mu_n}(A^{(0)}), \tag{A.8}$$

where $X_{\mu_1\cdots\mu_n}(A^{(0)})$ is a function of $A^{(0)}_{\mu_1\cdots\mu_n}$, which depends on a renormalization scheme. Substituting (A.8) into (A.7) we obtain

$$\langle J_{\mu_1\cdots\mu_n}(x)\rangle = (d-2n)\, l^{d-2n-1} A^{(d-2n)}_{\mu_1\cdots\mu_n}(x) + X_{\mu_1\cdots\mu_n}(A^{(0)}). \tag{A.9}$$

Taking a variation of the CFT state corresponds to a variation of $A^{(d-2n)}_{\mu_1\cdots\mu_n}$ keeping $A^{(0)}_{\mu_1\cdots\mu_n}$ fixed [41, 42]. Thus, we obtain (4.14) (and (3.23) for $n=1$) in the text.

B Another derivation of (4.14)

In this appendix we derive the relation (4.14) ((3.23) for the $n=1$ vector field case) for the current without using the holographic renormalization calculation in appendix A. We derive it from the entanglement first law (4.3) and the holographic charged entanglement entropy (4.16) by considering a small size limit of the ball-shaped region B as was done for the energy-momentum tensor in [13].

In the small size limit $R\to 0$ the μ-dependent term of the right-hand side of (4.3) can be calculated as

$$\begin{aligned}(\mathrm{RHS}) &= -\frac{1}{(n-1)!}\,\mu_{i_1\cdots i_{n-1}} \int_B d^{d-1}x\, \delta\left\langle J^{0 i_1\cdots i_{n-1}}(x)\right\rangle \\ &= -\frac{1}{(n-1)!}\,\mu_{i_1\cdots i_{n-1}} \delta\left\langle J^{0 i_1\cdots i_{n-1}}(x_0)\right\rangle \frac{R^{d-1}\Omega_{d-2}}{d-1},\end{aligned} \tag{B.1}$$

where Ω_{d-2} is the volume of a unit sphere S^{d-2}. We have approximated the current by its value at the center x_0^i. The last factor $R^{d-1}\Omega_{d-2}/(d-1)$ is the volume of the region B. Using (4.16) and the gauge condition $A_{z\mu_1\cdots\mu_{n-1}}=0$ the μ-dependent term of the left-hand side of (4.3) is

$$\begin{aligned}(\mathrm{LHS}) &= -\frac{1}{2(n-1)!}\int_{\tilde B} \mu_{i_1\cdots i_{n-1}} \delta F^{ab i_1\cdots i_{n-1}} \boldsymbol{\epsilon}_{ab} \\ &= \frac{1}{(n-1)!}\int_B d^{d-1}x \left(\frac{l}{z}\right)^{d-2n-1} \mu_{i_1\cdots i_{n-1}} \\ &\quad \times \left(\delta F_{z0 i_1\cdots i_{n-1}} + \frac{(x-x_0)^j}{z}\,\delta F_{j0 i_1\cdots i_{n-1}}\right)\Bigg|_{z=\sqrt{R^2-(x^i-x^i_0)^2}}.\end{aligned} \tag{B.2}$$

In the limit $R\to 0$ this must have the same R-dependence as (B.1). It requires that the $z\to 0$ behavior of the field should be

$$\delta A_{\mu_1\cdots\mu_n}(x,z) \sim z^{d-2n}\, \delta A^{(d-2n)}_{\mu_1\cdots\mu_n}(x) \tag{B.3}$$

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as can be seen by rescaling the coordinates z and $(x-x_0)^i$ by R as in [13]. Then, in the limit $R \to 0$ the second term of (B.2) vanishes because of the factor $(x-x_0)^j$ while the first term gives

$$(\text{LHS}) = \frac{d-2n}{(n-1)!} l^{d-2n-1} \mu_{i_1 \cdots i_{n-1}} \delta A^{(d-2n)}_{0 i_1 \cdots i_{n-1}}(x_0) \frac{R^{d-1}\Omega_{d-2}}{d-1}, \tag{B.4}$$

which has the same R-dependence as (B.1). Equating (B.4) to (B.1) we obtain (4.14).

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