ON CLASSIFICATION OF $\mathbb{Z}$–GRADED LIE ALGEBRAS OF CONSTANT GROWTH WHICH HAVE ALGEBRA $\mathbb{C}[h]$ AS CARTAN SUBALGEBRA

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Let us recall the main definitions (see our papers with M. V. Saveliev, Commun. Math. Phys. **126**, no.3 (1989);Phys. Lett.A. **143**, no.3 (1990)).

Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$, and $\mathfrak{g}_0$ is an abelian subalgebra which is called Cartan subalgebra. Contrary to the definition of Kac-Moody algebras we do not suppose that the dimension of $\mathfrak{g}_0$ is finite, but we supply $\mathfrak{g}_0$ with the structure of a commutative associative algebra E over $\mathbb{C}$. By definition the root system is the spectrum of this algebra. Moreover, we suppose that $\mathfrak{g}_{+1}, \mathfrak{g}_{-1}$ are isomorphic to $\mathfrak{g}_0$ as linear spaces. We denote the elements of $\mathfrak{g}_0, \mathfrak{g}_{+1}, \mathfrak{g}_{-1}$ by $x_i(\phi)$ where $\phi \in E,\ i = -1, 0, +1$.

We consider the "local part" $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{+1}$ and suppose that this part generates the Lie algebra $\mathfrak{g}$. Now we shall formulate our axioms: (1) $[x_0(\phi), x_0(\psi)] = 0, \forall \phi, \psi \in E$; (2) $[x_0(\phi), x_{\pm1}(\psi)] = \pm x_{\pm1}(K\phi \cdot \psi)$ where multiplication is in E and K is a linear operator in E (the Cartan operator); (3)$[x_{+1}(\phi), x_{-1}(\psi)] = x_0(\phi \cdot \psi)$; (4) $\forall i \in \mathbb{Z}, \forall x \in \mathfrak{g}_i, x \neq 0, \exists y \in \mathfrak{g}_\varepsilon, \varepsilon = -\operatorname{sgn} i$ and $[x, y] \neq 0$. Let $\mathfrak{g}(E, K)$ be quotient of freely generated by local part Lie algebra factorizing by maximal nilpotent ideal having trivial intersection with $\mathfrak{g}_0$. If $\mathfrak{g}$ has also a property: $\mathfrak{g}_i \cong E, i \in \mathbb{Z}$ (linear isomorphism), we shall say that $\mathfrak{g}(E, K)$ has a constant growth.

Now we return to our problem: which graded Lie algebras have the algebra of polynomials as Cartan subalgebra? Let us suppose that $E = \mathbb{C}[h]$ is the algebra of all complex polynomials of one variable.

Theorem. *If $(Kf)(h) = f(h+1) - 2f(h) + f(h-1)$ then $\mathfrak{g}(\mathbb{C}[h], K)$ is isomorphic as a $\mathbb{Z}$-graded Lie algebra to Lie algebra $<< \partial, L_x >>$ where ∂ is differentiation and L_x is multiplication by x , $[\partial, L_x] = \mathbf{1}$.*

This example is not unique; there are many other Cartan operators which generate algebras of constant growth. The following series was considered by B.Feigin and E.Frenkel. Let $U = U(sl(2))$ be the universal enveloping algebra of $sl(2, \mathbb{R})$ and Δ its Casimir element.Let us consider U as a Lie algebra with gradation generated by $\deg h = 0, \deg f = -1, \deg e = +1$.

Theorem. *If I_c is the ideal generated by $\Delta - c1$ then the $\mathbb{Z}$ graded Lie algebra U/I_c is of the type $(\mathbb{C}[h], K_c)$ for some K_c, $c \in \mathbb{R}$.*

Is this list complete?