A NON REVERSIBLE SEMI-MARTINGALE

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The time-reversal of a semi-martingale may fail to be a semi-martingale. Here is a simple example.

Let B_t be a standard Brownian motion and, inspired by Barlow's example in [1], let ϕ be a measurable function which maps $C[0,1]$ one-to-one into $[0,1]$. Let $T(\omega) = \phi(\{B_t(\omega),\ 0 \le t \le 1\})$, and define

$$X_t = \begin{cases} B_t & \text{if } 0 \le t \le 1 \\ B_1 & \text{if } 1 \le t \le T+1 \\ B_{t-T} & \text{if } t \ge 1+T . \end{cases}$$

Then X is just a Brownian motion with a flat spot of length $T \le 1$ interpolated from $t = 1$ to $t = T+1$. T is $\sigma\{X_s, s \le 1\}$ - measurable, so that it is easy to see that X is a martingale.

Now reverse X from $t = 2$: let $\tilde{X}_t = X_{2-t}$ for $0 \le t \le 2$. Let $(\tilde{F}_t)$ be the natural filtration of $\tilde{X}$. Note that T is $\tilde{F}_1$-measurable, hence so is $\{\tilde{X}_t, t \le 1\}$, since it is just the time-reversal of $\phi^{-1}(T)$. Consequently, $\tilde{F}_t = \tilde{F}_1$ for $t > 1$. Any martingale on these fields will be constant on $(1,2)$ and any semi-martingale will have finite variation there. But $\tilde{X}_t$ has infinite variation on $(1,2)$, so it is not a semi-martingale relative to the $(\tilde{F}_t)$. By Stricker's theorem [2], it can't be a semi-martingale relative to any filtration whatsoever.

References

[1] Barlow, M.T. On Brownian Local Time. Preprint.

[2] Stricker, C. Quasimartingales, martingales locales, semi-martingales, et filtrations naturelles, ZW 39, (1977) p 55-64.