SOME REMARKS ON THE CENTRAL LIMIT THEOREM IN C(S)

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The object of this note is to make two different remarks on central limit theorems (CLTs) of the type proved by Strassen and Dudley (1969), Giné (1974) and Jain and Marcus (to appear). The first of them has to do with speed of convergence: the kind of conditions on the modulus of continuity of a C(S)-valued random variable under which these theorems are proved seem to be also adequate for treating speed of convergence questions by reduction to finite dimension, at least for the sup norm functional and under certain conditions on the covariance of the variable. The second remark consists of an application: one of these CLTs (Giné (1974)) implies Donsker's invariance principle for variables with $2+\delta$ moments, δ positive, or even with less restrictive conditions, but it does not seem to imply the invariance principle for variables with only the second moment finite. The proofs will only be sketched. Complete proofs as well as complementary results may appear elsewhere.

1. Bounds on the speed of convergence. The best result known on the CLT in C(S) is due to Jain and Marcus (to appear) and is as follows: let $\{X_i\}_{i=1}^{\infty}$ be a sequence of centered i.i.d. C(S)-valued random variables, (S,d) compact metric; if

$$(1) \qquad |X_1(\omega,s) - X_1(\omega,t)| \leq M(\omega)e(s,t),$$

where M is a square integrable random variable and e is a continuous pseudo-distance such that $\int_0^1 H^{1/2}(S,e,x)dx < \infty$ (H is the metric entropy of (S,e), i.e. $H(S,e,x) = \log \inf \{n: S=\bigcup_{i=1}^n V_i,\ \mathrm{diam}_e V_i \leq x\}$), then

$$(2) \qquad \text{weak*-}\lim_{n\to\infty} L[n^{-1/2}(X_1+\ldots+X_n)] = L(Z),$$

Z being the centered Gaussian process determined by Cov X_1, which is sample continuous.

Condition (1) with $M \equiv 1$, was introduced by Strassen and Dudley (1969) and, in the present form but with a stronger entropy condition, by Giné (1974).

Here we obtain some weak speed of convergence results for this theorem.

Theorem 1. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of symmetric i.i.d. $C(S)$-valued random variables ((S,d) compact metric) satisfying condition (1) with:

(i) $M \in L_3(\Omega,P)$ and $EM^2=1$, and

(ii) $H(S,e,x) \leq Cx^{2(\alpha-1)}$ for some $\alpha\in(0,1)$, $C>0$ and every x in a neighborhood of zero.

Assume further that

(iii) $E|X_1(s)|^3 < \infty$ for some $s \in S$ and

(iv) $\mathrm{Var}(X_1(s)) \geq \sigma^2 > 0$ for every $s \in S$.

Then,

(3) $|P\{n^{-1/2}||X_1(\omega)+\ldots+X_n(\omega)||_\infty \geq \lambda\}-P\{||Z(\omega)||_\infty \geq \lambda\}|<H(\lambda)(\log n)^{-\alpha/2(1-\alpha)}$,

where $H(\lambda)$ is bounded on bounded sets (and is at most $O(\lambda)$ as $\lambda\to\infty$).

Remarks. The symmetry hypothesis for X_1 may be replaced by: $X_1(s)$ centered for every s and $E(X_1(s)-X_1(t))^{2+\delta}/[E(X_1(s)-X_1(t))^2]^{(2+\delta)/2}>C$ for some $\delta>0$ and $C>0$ and all s,t such that $X_1(s)-X_1(t)$ is non-degenerate. If $M \in L_\infty(\Omega,P)$ or if $H(S,e,x) \leq Cx^{\alpha-1}$, then X_1 can be taken centered instead of symmetric. If instead of (iv), the covariance of X_1 equals the covariance of Brownian motion or the Brownian bridge, then the theorem is still true and in fact $H(\lambda)$ may be taken to be a constant. Under stronger conditions on the metric entropy we can obtain bounds in (3) decreasing as a power of n (if $H(S,e,x)\leq\log x^{-1/\alpha}$ then the bound is of the order of $H(\lambda)n^{-\alpha/6(2-\alpha)}$).

The proof of Theorem 1 is based on the following three lemmas.

Lemma 1. Under the conditions of Theorem 1 there exist constants C' and C'' such that, from some r on and for every n,

(4) $P\{\sup_{e(s,t)\leq 2^{-r}} n^{-1/2}|S_n(s)-S_n(t)|\geq C'2^{-\alpha r}\}<C'2^{-\alpha r-1}+C''EM^3n^{-1/2}$,

where S_n denotes $X_1+\ldots+X_n$.

The proof of this lemma follows the pattern of the proof of probable equicontinuity of $n^{-1/2}S_n$ in the theorem of Jain and Marcus (to appear). The stronger condition (ii) on the metric entropy of (S,e) makes it possible to give explicit values to the positive numbers δ, ε and η in inequality (2.18) there.

Lemma 2. Let (T,d) be a compact metric space and let Z be a centered sample continuous Gaussian process on T such that $EZ^2(t)\geq\sigma^2>0$ for every $t \in T$. Then, the distributions of the random variables $\sup_{t\in T}Z(t)$ and $\sup_{t\in T}|Z(t)|$ are absolutely continuous with respect to Lebesgue measure and their densities have versions which are bounded on bounded sets (and grow at most as $0(\lambda)$ as $\lambda\to\infty$).

Lemma 2 is a direct consequence of some results of Ylvisaker (1965) and (1968).

Given any two points y and z in R^k, let $R(y,z)$ be the k-dimensional parallelogram which vertices are all the points with each coordinate either equal to the corresponding one of y or to the corresponding one of z. With this notation we have:

Lemma 3. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of centered i.i.d. random variables with values in R^k such that $E|x_1|^3<\infty$ and, if $x_1 = (x_{11},\dots,x_{1k})$, $Ex_{1j}^2 > 0$ for $j=1,\dots,k$. Let Δ be the covariance of x_1 and let $\rho_j = E|x_{1j}|^3/(Ex_{1j}^2)^{3/2}$, $j=1,\dots,k$. Then there exists a universal constant C''' such that, for every $y,z\in R^k$,

$$|P\{n^{-1/2}(x_1+\dots+x_n)\in R(y,z)\}-N(0,\Delta)(R(y,z))|<C'''k^{5/3}[n^{-1/2}\Sigma_{i=1}^k\rho_i]^{1/3}.$$

This lemma can be proved similarly to Theorem 3 in Sazonov (1968); the main change consists in the use of Lemma 4 in Paulauskas (1969).

Next we show how these three lemmas are used in the proof of Theorem 1.

Let $\{s_i\}_{i=1}^k$ be a finite subset of S. Then, with $S_n = X_1+\dots+X_n$, we have

$$(5)\qquad \begin{aligned} &|P\{n^{-1/2}||S_n(\omega)||_\infty\geq\lambda\} - P\{||Z(\omega)||_\infty\geq\lambda\}|\leq\\ &[P\{n^{-1/2}||S_n(\omega)||_\infty\geq\lambda\} - P\{\max_i n^{-1/2}|S_n(s_i)|\geq\lambda\}] +\\ &|P\{\max_i n^{-1/2}|S_n(s_i)|\geq\lambda\} - P\{\max_i|Z(s_i)|\geq\lambda\}|+\\ &[P\{||Z(\omega)||_\infty\geq\lambda\} - P\{\max_i|Z(s_i)|\geq\lambda\}]. \end{aligned}$$

Let us call A, B and C respectively the first, second and third summands in the second term of this inequality. The term B can be estimated by means of Lemma 3. If $K = E||X_1(\omega)||_\infty^3$ then, $\rho_i \leq \sigma^{-3}K$ and therefore, for

$$(6)\qquad k \approx n^{1/24},$$

Lemma 3 gives

$$(7)\qquad B \leq 4C'''K^{1/3}\sigma^{-1}n^{-1/12}.$$

Note now that

$$(8)\qquad \begin{aligned} A \leq\ &P\{n^{-1/2}||S_n(\omega)||_\infty\geq\lambda,\ \max_i n^{-1/2}|S_n(s_i)|<\lambda-\varepsilon\} +\\ &[P\{\max_i n^{-1/2}\ |S_n(s_i)|\geq\lambda - \varepsilon\} - P\ \max_i n^{-1/2}\ |S_n(s_i)|\geq\lambda\}]\ . \end{aligned}$$

Denote by A_1 and A_2 respectively the first and second summands in the second term of (8). If $\{s_i\}$ is x-dense in (S,e), then,

$$A_1 \le P\{\sup_{e(s,t)\le x} n^{-1/2}|S_n(s)-S_n(t)|\ge\varepsilon\}.$$

By (6) and hypothesis (ii), we can choose the set $\{s_i\}$ x-dense with x of the order of $(\log n)^{-1/2(1-\alpha)}$. Then, taking $\varepsilon = C'x^\alpha$ in (4), Lemma 1, we obtain the following bound for A_1:

$$(9) \qquad A_1 \le 2^{-1}C'(\log n)^{-\alpha/2(1-\alpha)}.$$

Adding and substracting $P\{\max_i|Z(s_i)|\ge\lambda-\varepsilon\} - P\{\max_i|Z(s_i)|\ge\lambda\}$ to A_2 and applying Lemma 2 and Lemma 3, we have

$$(10) \qquad A_2 \le 8C''K^{1/3}\sigma^{-1}n^{-1/12} + K(\lambda)(\log n)^{-\alpha/2(1-\alpha)}$$

where $K(\lambda)$ is bounded on bounded sets and is at most $O(\lambda)$ as $\lambda\to\infty$. The function $K(\lambda)$ is independent of the set $\{s_i\}$. The quantity C can be bounded as A because the process Z satisfies an inequality similar to (4); the only difference is that Lemma 3 is not needed. Now Theorem 1 follows from (7)-(10).

<u>2</u>. <u>The</u> <u>invariance</u> <u>principle</u>. Theorem 1 in Giné (1974) can be restated for triangular arrays:

<u>Theorem</u> <u>2</u>. Let $\{Y_{n1},\dots,Y_{nk_n}\}_{n=1}^\infty$ be a triangular array $(Y_{n1},\dots, Y_{nk_n}$ independent for each n) of centered, $C(S)$-valued random variables such that $EY_{nk}^2(s) < \infty$ for every s, n and k. Suppose:

(i) there exists a centered Gaussian process Z on S such that the finite dimensional distributions of $S_n = Y_{n1}+\dots+Y_{nk_n}$ converge to the ones of Z, and

(ii) the pseudo-distance

$$e(s,t) = \sup_n||S_n||_{L_2} + \sup_{n,k}||Y_{nk}(s)-Y_{nk}(t)||_\infty$$

is continuous and such that $\int_0^1 H(S,e,x)dx < \infty$.

Then, Z is sample continuous and weak*-$\lim_{n\to\infty} L(S_n) = L(Z)$.

Let now $\{x_i\}_{i=1}^\infty$ be a sequence of i.i.d. random variables such that $Ex_1 = 0$, $Ex_1^2 = 1$ and $E|x_1|^{2+\delta} < \infty$ for some $\delta > 0$ (in fact, $x_1|\log|x_1||^{1/2+\delta}\in L_2(\Omega,P)$ would be enough). Define $s_n = \Sigma_{k=1}^n x_k$ and

$$\xi_n(\omega,t) = n^{-1/2}[s_{[nt]}(\omega) + (nt-[nt])x_{[nt]+1}(\omega)$$

for every natural number n, $\omega\in\Omega$ and $t\in[0,1]$ ($[\cdot]$ denotes the function "greatest integer smaller than or equal to'). $\xi_n(\omega)$ is a $C[0,1]$-valued random variable. Since x_1 is square integrable, the invariance principle asserts that the distribution of $\xi_n(\omega)$ converges weakly (weak*) to the distribution of Brownian motion. We will deduce this fact from Theorem 2.

Set

$$X_{nk}(\omega,t) = \begin{cases} 0 & \text{if} \quad 0<nt\le k-1 \\ n^{-1/2}(t-(k-1)/n)x_k(\omega) & \text{if} \quad k-1<nt\le k \\ n^{-1/2}x_k(\omega) & \text{if} \quad k<nt\le n. \end{cases}$$

Then, $\xi_n = \Sigma_{k=1}^n X_{nk}$. In order to apply Theorem 2 we must discard the trajectories of the processes X_{nk} which are too steep i.e., we must truncate x_k. Define

$$y_k(\omega) = \begin{cases} x_k(\omega) & \text{if} \quad |x_k(\omega)| \le k^{1/(2+\delta)} \\ 0 & \text{otherwise,} \end{cases}$$

Y_{nk} as X_{nk} with x_k replaced by y_k, and $\xi_n' = \Sigma_{k=1}^n Y_{nk}$. It is easy to see that the sequences $\{\mathcal{L}(\xi_n)\}$ and $\{\mathcal{L}(\xi_n'-E\xi_n')\}$ are weak*-convergence equivalent, in fact, that $||\xi_n(\omega) - \xi_n'(\omega) + E\xi_n'||_\infty \to 0$ a.s. In particular, the finite dimensional distributions of $\xi_n'-E\xi_n'$ converge to the ones of Brownian motion and so, in order to obtain the invariance principle for $\{x_i\}$ we only need to check hypothesis (ii) in Theorem 2 for the triangular array $\{Y_{nk}-EY_{nk}\}$. If $[ns] = [nt] = k$, then

$$\xi_n'(s)-\xi_n'(t) = n^{1/2}|s-t|y_k,$$

and if $[ns] = k-1$ and $[nt] = k-r-1$, $r \ne 0$, then

$$\xi_n'(s)-\xi_n'(t) = n^{-1/2}[(k-r-nt)y_{k-r}+y_{k-r-1}+\cdots+(ns-k+1)y_k].$$

Therefore,

$$E(\xi_n'(s)-E\xi_n'(s)-\xi_n'(t)+E\xi_n'(t))^2 \le 2|s-t|^{1/2}. \tag{11}$$

It is equally easy to verify that

$$||Y_{nk}(s)-Y_{nk}(t)||_\infty \le |s-t|^{\delta/2(2+\delta)}. \tag{12}$$

With (11) and (12), the second hypothesis of Theorem 2 is checked and therefore, the law of ξ_n converges weakly to the law of Brownian motion as $n\to\infty$.

<u>Remark</u>. In fact, Theorem 2 holds under the weaker (and best possible) hypothesis of $\int_1^0 H^{1/2}(S,e,x)dx < \infty$ (Giné (to appear)). Then, a simple truncation gives the final result of Jain and Marcus mentioned in Section 1. But this does not seem to imply the invariance principle under the weakest conditions either (only with $x_1|\log|x_1||^{1/4+\delta}\in L_2(\Omega,P)$).

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