Asymptotic Total Cross-Sections and Positivity in Meson-Meson Scattering.

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(ricevuto il 25 Giugno 1979)

Summary. — Starting from the recent experimental analysis of elastic \( \pi \pi \) and \( K \pi \) scatterings, we derive for both processes the (unique) analytic function which best approximates the data, satisfies the requirement imposed by unitarity and generates a constant asymptotic total cross-section. Assuming that the asymptotic regime sets up particularly quickly at imaginary \( v \simeq (s-u)/4 \), we predict the values of these cross-sections. Our prediction is \( \sigma_{\pi \pi}^{\infty} \simeq \sigma_{K \pi}^{\infty} \simeq (9 - 12) \text{ mb} \).

Introduction.

The possibility that a precocious asymptotic regime sets up at imaginary energies in the complex \( v \simeq (s-u)/4 \) plane has been considered in previous papers, where, as a first simple process, nonelastic meson-meson scattering was examined \(^{(1)}\). In that case, stability in the involved extrapolation process was guaranteed by the existence of a finite bound for the asymptotic values of the relevant analytic amplitudes \(^{(2)}\).

In the elastic-scattering case, this property is no longer verified. However, one is still led to a stable extrapolation procedure in the forward direction case exploiting unitarity, i.e. taking into account the positivity of the imaginary part of the scattering amplitude (\(^4\)). In this way, elastic meson-meson scattering was also analysed (\(^5\)). From the overall results, it seemed justified to draw the conclusion that the behaviour of the extrapolated amplitudes along the imaginary \(v\)-axis was, least to say, peculiar and, in all considered cases, consistent with the possibility of a rather precocious asymptotic regime setting along that particular direction.

If we believe that elastic hadron-hadron scattering is asymptotically dominated by the same mechanism, independently of the involved hadron, elastic meson-meson scattering becomes of noticeable interest. In particular, we can investigate in this very simple process the possibility that analytic functions exist which approximate to a given satisfactory extent the available data and generate a constant asymptotic total cross-section. In our scheme, we can then predict such asymptotic values and try to draw more general conclusions.

The aim of this paper is to review in this spirit elastic \(\pi\pi\) and \(K\pi\) scatterings. As we shall see from our combined analysis, the asymptotic properties we would predict are quite similar for the two processes. More precisely, we find that elastic meson-meson data are well reproduced by analytic functions generating constant asymptotic total cross-sections, which, in our interpretation, would be approximately equal for both \(\pi\pi\) and \(K\pi\) scatterings, and of the order of \(\sim (9 \div 12)\) mb.

To illustrate with some instructive examples how positivity works in a general case, we shall devote sect. 1 to a comparison of various extrapolations obtained by using different sets of data. In sect. 2, the main features of our extrapolation along the imaginary \(v\)-axis will be examined and in the short final section some general conclusions will be drawn.

1. – We consider the \(\pi\pi\) and \(K\pi\) elastic amplitudes

\[
\begin{align*}
F^\eta(v) &= 2F^{(\pi\pi \to \pi^+\pi^-)}(v; t = 0) + F^{(\pi\pi \to \pi^0\pi^0)}(v; t = 0), \\
T^\eta(v) &= \frac{\sqrt{6}}{2} [T^{(K^+\pi^- \to K^0\pi^0)}(v; t = 0) + T^{(K^-\pi^+ \to K^0\pi^0)}(v; t = 0)],
\end{align*}
\]

which are even functions of \(v \equiv (s - u)/4m_N^2\) and carry isospin zero in the \(t\)-channel. Owing to the optical theorem, these amplitudes have positive

imaginary parts, $\text{Im } F^0, T^0 (\nu + i\epsilon) > 0$ for $\nu \gg \nu_{\text{th}}$. As a consequence, the stable extrapolation method of ref. (4) applies, allowing one to find the (unique) analytic function $f^0, t^0$ which fulfills the positivity requirement and minimizes the «distance» (we work in the unit circle as in ref. (5)):

\begin{equation}
\chi^2 = \frac{1}{N} \int_{-\pi/2}^{\pi/2} |f, t - h|^2 \frac{g(\theta)}{e^2(\theta)} d\theta, \quad N = \int g d\theta,
\end{equation}

Fig. 1. – a) Real part of the experimental $\pi \pi$ amplitude $F^0$ from ref. (7,8) (●, ○) and its best positivity-constrained analytic approximant on the data region (continuous line). b) The same for the imaginary part.


where \( \varrho(\theta) \) is the density of the data points and \( h(\theta), \epsilon(\theta) \) are the experimental data with related errors.

In ref. (4), to which we shall refer for all the details of our application, the method of ref. (4) was exploited using as an experimental input the analysis of Protopopescu et al. (7) for \( 0.55 < W < 1 \text{ GeV} \) and of Hyams et al. (8) for \( 1 < W < 1.8 \text{ GeV} \). As a result, we found the (unique) analytic function \( f(\nu) \) minimizing eq. (3) (with \( \chi^2 = 1.45 \)) and behaving asymptotically at most like \( \nu \), i.e. generating an asymptotic constant total cross-section. As fig. 1 shows, and we stress this point here, the best analytic approximation \( f(\nu) \) of ref. (5) was not as "perfect" an approximation to the data as one might have in principle expected. In particular, our solution differed somewhat from the available data in the regions around \( W \approx 1 \text{ GeV} \) and \( W \approx 1.5 \text{ GeV} \). Since in the latter region the data we used had troubles with \( P \)-wave unitarity, we considered this departure as an indication of the fact that in our method positivity is indeed a meaningful constraint, i.e. one which actually forbids the solution to approximate too closely data which are not consistent with unitarity.

Meanwhile, a very recent new analysis in the region \( 1 < W < 2.1 \text{ GeV} \) has been performed (9). From this analysis, a set of phases \( \{(\theta)\} \) have emerged, which do not violate unitarity over the whole energy range. Consequently, one expects the new data to be better approximated, e.g., in the region \( W \approx 1.5 \text{ GeV} \) by our optimal solution \( f(\nu) \).

As fig. 2 shows, this is actually the case. The new set of data \( \{(G)\} \) and the new related optimal analytic approximation are everywhere quite close, the only appreciable contribution to \( \chi^2 = 0.62 \) coming from the region around \( W = 1 \text{ GeV} \). This discrepancy, we believe, has nothing to do with unitarity, and is only due to the fact that in this region the two different sets of data ref. (5,9) join in a not smooth way. Both in this new analysis and in that of ref. (5), our solution dislikes the very last experimental values of ref. (5), and fits the first corresponding values of the joining set ref. (5,9) much better. Since this solution behaves asymptotically at most like \( \nu \), we conclude that the available \( \pi\pi \) data, ref. (5,9), are reproduced to the extent shown in fig. 2, consistently with the limitations imposed by unitarity, by an analytic function generating an asymptotic constant total cross-section.

In the K\( \pi \) case, we have performed a similar investigation, using the recent experimental results of Estabrooks et al. (10), which cover the region \( W < 1.9 \text{ GeV} \).

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It turned out that, for these data, the minimization work required by our method was more tedious. This was manifested by the requirement of a greater number of parameters $\theta_i$ (see ref. (7)) and, therefore, by a considerable increase of the computational time. On the other hand, the introduction of each additional parameter $\theta_i$ was accompanied by a continuous increase of the corresponding $\chi^2_0$: for instance, working with six parameters we obtained $\chi^2_0 \approx 3$ (in the $\pi\pi$ case, the optimal amplitude was obtained very rapidly, a number of parameters $\theta_i$ not greater than four being required).

As fig. 3 shows, this trouble was mainly due to the last experimental points, which were very badly approximated, when positivity was imposed. In this region, the $F$-wave was taken in ref. (10) as a reference phase, chosen to cor-

Fig. 2. – The same as in fig. 1, using along $1 \leq W \leq 2.1$ GeV the new $\pi\pi$ data of ref. (9).
respond to the phase of a resonance \(^{(11)}\) of mass 1.78 GeV and width 175 MeV. Since this phase determination seems to us to be somewhat more error affected, we thought it convenient to omit it and to make use only of the experimental information up to \(W = 1.6\) GeV. We must underline, however,

![Figure 3](image)

Fig. 3. - a) Real part of the experimental \(K\pi\) amplitude \(T^0\) from ref. \(^{(10)}\) (•, 4) and its best positivity-constrained approximant on the data region (continuous line). The dashed line represents the best approximant when positivity is not imposed. b) The same for the imaginary part.

that, due to the precision limitation of our computer (16 digits), this does not imply yet any conclusion about the neglected part of the data (*)

As we expected, the omission of the last data points led to a considerable improvement in the extrapolation work, the solution being obtained very quickly with a small number of parameters $\theta_i$, and the minimal $\chi_0^2$ having a reasonable ($\approx 1.26$) value. In fig. 4, we show the corresponding

Fig. 4. – The same as in fig. 3, using the experimental data of ref. (19) only up to $W \leq 1.6$ GeV.

(*) In other words, we can work only with a finite number of terms ($\sim 54$) in the analytic approximation of the amplitude, and the poor description of the data might be due also to the incompleteness of the set used.
solution for the first (A) of the four sets of phases quoted in ref. (10) (as we mentioned in ref. (9), the other sets of phases B, C, D are giving in practice the same extrapolation with no appreciable variation of $\chi_\alpha^2$, whether one uses the additional experimental information up to $W = 1.9$ GeV or not). Since this solution behaves asymptotically at most like $\pi$, we conclude that K$\pi$ data too, at least up to $W = 1.6$ GeV, are satisfactorily reproduced, as fig. 4 shows, by an analytic function compatible with unitarity and generating an asymptotic constant total cross-section. In the next section, we shall try to predict, within our scheme, the values of these cross-sections for both $\pi\pi$ and K$\pi$ scatterings.

2. – The main features of our extrapolation along the imaginary $\nu$-axis have been discussed in ref. (1,2,%,,). Perhaps, they may be well summarized by fig. 5, where the extrapolated values of the forward $\pi\pi$ amplitudes

![Graph](image)

Fig. 5. – a) The modulus of the experimental amplitudes $F_\theta = F(\pi^0\pi^0 \rightarrow \pi^0\pi^0)$ (continuous line) and $F_+ = F(\pi^+\pi^0 \rightarrow \pi^+\pi^0)$ (dashed line) over the data region, ref. (7,9). b) Extrapolated values (with changed sign) of $F_\theta$ (continuous and dashed line) at imaginary $\nu$ values of the same modulus as the corresponding (real) experimental ones.
$F_0 \equiv F(\pi^0\pi^0 \to \pi^0\pi^0)$ and $F_+ \equiv F(\pi^+\pi^0 \to \pi^+\pi^0)$ at imaginary $\nu$ are plotted and compared to the corresponding experimental data at real $\nu$ values of the same modulus. From this figure, one sees that

I) the strong oscillatory behaviour of our amplitudes on the measured real-energy region is completely washed out on the corresponding (of equal modulus in the complex $\nu$-plane) imaginary-energy region, where a quite smooth regime sets up already at the lowest reliable $\nu$ values (where our extrapolation is not affected by the unknown very-low- or very-high-energy details);

II) along the reliable region of imaginary $\nu$-axis explored by us, $1 \leq |W| \leq 2$ GeV, the extrapolated values of $F_+$ and $F_0$ are in practice coincident, although these amplitudes are so different along the real $\nu$-axis; in other words, the contribution due to the $t$-channel $I = 2$ amplitude (the only one by which $F_+, F_0$ differ) is already negligibly small along the whole explored portion of the imaginary $\nu$-axis;

III) the $\nu$-dependence which sets up in the extrapolation region is already of the form $F_{+,0} \sim \text{const} \cdot \nu$; this can be visualized in fig. 6, where we plot the extrapolated values of $F_0/\nu$ and compare them, as usual, with the corresponding experimental values of $F_0/\nu$, eq. (1).

Fig. 6. – Experimental values of $|F_0/\nu|$ at real $\nu$, ref. (7,9), and extrapolated values of $F_0/\nu$ (changed sign) with errors, at the corresponding imaginary $\nu_0$. 
Since our solution behaves like const.\(v\) at infinity, the general conclusion we can draw is that along the whole reliable imaginary-energy extrapolation region, where the \(I_t = 2\) contribution is negligibly small—as one expects at infinity—, a quite peculiar smooth regime, with the same \(v\)-dependence which is assumed at infinity, sets up. These properties are only valid along the imaginary \(v\)-axis: when we move back to the real \(v\)-axis, they are gradually lost.

In the case of \(K\pi\) scattering, quite similar results—with the obvious limitation that the \(I_t = 2\) contributions cannot be investigated—emerge. In fig. 7, which is the analogous of fig. 6 for \(K\pi\) scattering, one sees how well the behaviour \(\sim\) const.\(v\) sets up in this case.

If we believe that a very precocious asymptotic regime is setting up, the next step is to compare the «asymptotic» features of \(\pi\pi\) and \(K\pi\) scatterings. As fig. 8 shows, these features are impressively similar. Again, in spite of the fact that \(T^0\) and \(T^\circ\) are different along the real axis, we find that their extrapolations converge very quickly to an approximately equal value, which corresponds to a total cross-section

\[
9 \text{ mb} \lesssim \sigma^\pi_\infty \sim \sigma^K_\infty \lesssim 1.2 \text{ mb}.
\]
This is quite reasonable a value for $\pi\pi$ scattering, on the basis of all previous evaluations (12). For $K\pi$ scattering, our new prediction would correspond to the naive expectation (for a $SU_3$ singlet pomeron) coming from standard quark-counting conditions.

Fig. 8. – a) The modulus of the experimental amplitudes $(10/3)|F^0/v|$ (continuous line) and $(10/\sqrt{6})|T^0/v|$ (dashed line) at real $v$. b) The corresponding analytic extrapolations (with changed sign) $(10/3)(F^0/v)$ (continuous line) and $(10/\sqrt{6})(T^0/v)$ (dashed line) at imaginary $v$. The normalization is chosen so that the values of the total cross-sections in mb are read on the co-ordinate axis.

3. – Concluding remarks.

We have found that both $\pi\pi$ and $K\pi$ elastic scatterings are reproduced to a satisfactory level, compatibly with unitarity, by analytic functions which generate asymptotic constant total cross-sections. Along the imaginary $v$-axis, these functions (or better, their ratio $F^0/T^0/v$) converge very quickly to a common value for the corresponding total cross-sections, in spite of their completely different features along the measured portion of the real $v$-axis. This result is

independent of the unknown very-low-energy details, as it was fully discussed in ref. (1). It only stems from the assumed analyticity properties and unitarity.

To see whether this apparently precocious asymptotic setting at imaginary energy is a general feature of hadron-hadron scattering, the next step would be to consider processes like hadron-proton collisions. In these cases, complications arise in standard approaches, due to the unknown properties of the very-low-energy region (which may even be, as in the pp case, unphysical). Since we learned from meson-meson scattering that these properties do not affect the region of extrapolation which we are interested in, we expect to be able to overcome this problem without too many troubles. Moreover, we learned from meson-meson scattering that the region where our extrapolation is reliable (i.e. where the extrapolation error is reasonably small) is of the same «length» as the measured one. For proton-proton, this would correspond to $2 < |W| < 60$ GeV. We think that, if a given behaviour set up and remained constant through all this interval, one would be strongly tempted to consider it as a quite serious candidate for being the asymptotic one. Our analysis of proton-proton scattering in this spirit is already at work.

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One of us (CV) wishes to thank the Central Institute of Physics for the hospitality at the Institute for Physics and Nuclear Engineering in Bucharest, where this paper was written.

** RIASSUNTO **

Partendo dai recenti dati sperimentali per l'urto elastico $\pi\pi$ e $K\pi$ si costruisce per entrambi i processi l'unica funzione analitica che meglio approssima questi dati, soddisfa ai requisiti imposti dall'unitarietà e genera una sezione d'urto totale asintotica costante. Nell'ipotesi che il comportamento asintotico a energie immaginarie sia particolarmente precoce, si formula una previsione per i valori delle sezioni d'urto. In tale previsione $\sigma^{\pi\pi}_{\infty} \approx \sigma^{K\pi}_{\infty} \approx (9 \div 12)$ mb.

Асимптотические полные поперечные сечения и положительность в мезон-мезонном рассеянии.

Резюме (*). — Исходя из недавнего экспериментального анализа упругого $\pi\pi$ и $K\pi$ рассеяния, мы выводим для обоих процессов (единую) аналитическую функцию, которая наилучшим образом аппроксимирует экспериментальные данные, удовлетворяет требованиям, вытекающим из унитарности, и приводит к постоянному асимптотическому полному поперечному сечению. Предполагая, что асимптотический режим устанавливается особенно быстро при мнимых $\nu \approx (s - u)/4$, мы предсказываем величины этих поперечных сечений, которые составляют $\sigma^{\pi\pi}_{\infty} \approx \sigma^{K\pi}_{\infty} \approx (9 \div 12)$ мб.

(*) Переведено редакцией.