Summary. – In this paper we develop a theory of prolongations of $G$-structures on a differentiable manifold $M$ to appropriate $\tilde{G}$-structures on the bundle $FM$ of linear frames over $M$. Thus, the different definitions of lifts of tensor fields and connections on $M$ to $FM$ become clear and motivated.

0. – Introduction and notations.

Let $M$ be an $n$-dimensional differentiable manifold, $TM$ its tangent bundle and $\mathcal{F}M$ its frame bundle. Let $Gl(n)$ denote the linear general group.

The theory of lifts to $TM$ of geometric objects, as tensor fields, connections, etc., has been extensively studied for many years, but it was after Morimoto's papers on prolongations of $G$-structures on $M$, $G \subset Gl(n)$, to appropriate $\tilde{G}$-structures on $TM$, $\tilde{G} \subset Gl(2n)$, that the different definitions involved there, as well as many results of that theory, became clear and motivated ([8], [9]).

Recently, K. P. Mok ([6], [7]) has initiated a similar theory of lifts to the frame bundle $\mathcal{F}M$ of geometric objects on $M$; in [1, 2], we have extended and completed Mok's theory, both studies, Mok's and ours, being done in such a way that their results can be closely compared to those in the theory of lifts to $TM$.

At this point, it was natural to ask if it would be possible to develop a theory of prolongations of $G$-structures on $M$ to $\tilde{G}$-structures on $\mathcal{F}M$, with $\tilde{G} \subset Gl(n+n^2)$, that theory being similar to Morimoto's. In the present paper, such a theory of prolongations is constructed, and we show, by studying the prolongations of some classical $G$-structures on $M$, how the different definitions of lifts given in [1, 2] and [6, 7] fit nicely in this general framework. Also, we briefly describe some simple examples which illustrate how this theory may serve as starting point to go further in the study of the differential geometry of $\mathcal{F}M$ for a manifold $M$ with some extra structure.

In the following, $R^n$ will denote the $n$-dimensional Euclidean space, $gl(n)$ the Lie algebra of $Gl(n)$, and $(a_{ij})$ the matrix whose $(i, j)$-entry is $a_{ij}$. For a manifold $M$,
$T_x M$ is the tangent space at $x \in M$; for a map $f: M \to N$, $f_*: T_x M \to T_{f(x)} N$ is the induced map, and if $f$ is a diffeomorphism, $\mathcal{F} f: \mathcal{F} M \to \mathcal{F} N$ will denote the induced diffeomorphism. Finally, for a Lie group $G$, $\mathfrak{g}$ will denote its Lie algebra.

1. The Lie group $J^1_l G(n)$

Through this section, indices $i, j, k, \ldots$ have range in $\{1, 2, \ldots, n\}$ and $\alpha, \beta, \gamma, \ldots$ in $\{1, 2, \ldots, p\}$.

Let $M$ be an $n$-dimensional manifold and $J^1_l M$ the $(n + pn)$-dimensional manifold of 1-jets at $0 \in \mathbb{R}^n$ of differentiable mappings $f: \mathbb{R}^n \to M$ defined on some open neighborhood of $0 \in \mathbb{R}^n$; $j^1(f)$ will denote the 1-jet of $f$ at $0$ and $\pi: J^1_l M \to M$ the target map. If $(U, x^a)$ is a coordinate system in $M$, then the induced coordinate system $(J^1_l U, x^a, x'^a)$ in $J^1_l M$ is defined by setting

$$x^a((j^1(f))) = x^a(0), \quad x'^a((j^1(f))) = \frac{\partial (x^a(f))}{\partial t^a}$$

for any $j^1(f) \in J^1_l U = \pi^{-1}(U)$. Sometimes, it will be useful to write up the coordinate functions in $J^1_l U$ simply as $x^a$, $A = 1, 2, \ldots, n + pn$, with $x^{a+i} = x'^a$.

An equivalent way to describe the points of $J^1_l M$ is the following: there is a canonical one-to-one correspondence $j^1(f) \to [x; X_1, \ldots, X_n]$, where $x = f(0)$ and $X_1$ is the tangent vector at $x$ to the curve $f(0, \ldots, t, \ldots, 0)$, with $t$ at the $a$-th place; in the sequel, we shall write simply $[x; X_1]$ and identify $j^1(f)$ to $[x; X_1]$ if there is no confusion. If $h: M \to M'$ is a differentiable map, then the induced map $h^1: J^1_l M \to J^1_l M'$ is given by

$$h^1([x; X_1]) = [h(x); h_*X_1].$$

Actually, it is well known that if $G$ is a Lie group then there exists a canonical Lie group structure on $J^1_l G$, the target map $\pi: J^1_l G \to G$ being a Lie group homomorphism. Indeed, it is not difficult to check that, in terms of the above identification, the multiplication in $J^1_l G$ is given as follows: for any $[a; X_1], [b; Y_1] \in J^1_l G$,

$$[a; X_1] \cdot [b; Y_1] \equiv [ab; (R_b)_a X_1 + (L_a)_b Y_1].$$

Now, let $[a; X_1] \in J^1_l G$ and denote $B_\alpha \in \mathfrak{g}$ the unique element such that $X_\alpha = (R_\alpha)_a B_\alpha$, for each $\alpha$. Then, the one-to-one correspondence

$$[a; X_1] \in J^1_l G \to [a; B_\alpha] \in \mathfrak{g} \times (\times \mathfrak{g})$$

where $\times \mathfrak{g} \equiv \mathfrak{g} \times \cdots \times \mathfrak{g}$, satisfies

$$[a; X_1] \cdot [a'; X'_1] \to [aa'; B_\alpha + \text{ad } a^{-1} B'_\alpha].$$
Therefore, if $G \times_{\text{ad}} (\times_p G)$ denotes the Lie group semidirect product of $G$ and the abelian Lie group $\times_p G$ via the canonical extension of the adjoint representation of $G$, and $\mathfrak{g} \times_{\text{Ad}} (\times_p \mathfrak{g})$ the Lie algebra semidirect product of $\mathfrak{g}$ and the abelian Lie algebra $\times_p \mathfrak{g}$ via the canonical extension of the adjoint representation of $\mathfrak{g}$, then we deduce

\textbf{Theorem 1.1.} -- (1.2) defines a canonical isomorphism of Lie groups $J^1 G \simeq \simeq G \times_{\text{ad}} (\times_p \mathfrak{g})$ and, consequently, it induces an isomorphism of Lie algebras $J^1 \mathfrak{g} \simeq \simeq G \times_{\text{Ad}} (\times_p \mathfrak{g})$.

\textbf{Theorem 1.2.} -- There exists a canonical embedding of Lie groups $j_p: J^1_p \text{Gl} (n) \rightarrow \text{Gl} (n+p)$.

\textbf{Proof.} -- For any $[a; B] \in J^1_p \text{Gl} (n)$ with $a \in \text{Gl} (n)$ and $B \in \text{gl} (n)$, we define

\begin{equation}
(1.3)
\begin{bmatrix}
a & 0 & \ldots & 0 \\
B_1 a & a & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
B_p a & 0 & \ldots & a
\end{bmatrix}
\end{equation}

and the result follows easily from (1.2) and Theorem 1.1.

Actually, it must be remarked that the definition of $j_p$ by (1.3) above is completely natural in the following sense. Let $G$ be a Lie group of transformations operating on a manifold $M$ through $\phi: G \times M \rightarrow M$, i.e. $\phi (a, x) = a \cdot x$, $a \in G$, $x \in M$, and denote $\phi: G \rightarrow M$, $\tau: \mathbb{R} \rightarrow M$ the induced maps. Then, $J^1_p G$ acts on $J^1_p M$ as a Lie group of transformations through the induced map $\phi^1: J^1_p G \times J^1_p M \rightarrow J^1_p M$ and, in terms of all the previous identifications, this action is given as follows: for any $[x; X] \in J^1_p M$ and $[a; B] \in J^1_p G$ with $B \in \mathfrak{g}$,

\begin{equation}
[a; B] \cdot [x; X] = [a \cdot x; (B_a^o)_{a,x} + (\tau_a x X)]
\end{equation}

where $B_a^o$ stands for the vector field induced on $M$ by $B_a \in \mathfrak{g}$. Therefore, if we identify $J^1_p R^n \simeq R^{n+p}$ through the map $[x; X] \in J^1_p R^n \rightarrow (x^i, X^i_j) \in R^{n+p}$, then it is easy to see that $j_p$ above describes the action of $J^1_p \text{Gl} (n)$ on $R^{n+p}$ which is canonically induced from the standard action of $\text{Gl} (n)$ on $R^n$.

Also, let us remark that, for $p = 1$, $J^1_1 M$ is nothing but the tangent bundle space $TM$ of $M$, and all the previous facts are well known in this case (Morimoto [8]).

And, since the aim of this paper is the study of prolongations of $G$-structures on $M$ to $\mathcal{F} M$, through the rest of the paper we shall restrict ourselves to assume
always \( p = n = \dim M \), denoting simply \( J^1_x M \) as \( J^1 M \) if there is no confusion. Moreover, the following facts must be borne in mind:

(i) \( \mathcal{F}M \) is an open submanifold of \( J^1 M \).

(ii) \( J^1 \text{Gl}(n) \) operates effectively (on the left) as a group of linear transformations of \( R^{n+n^2} \) via the representation \( j_\alpha \) given by (1.3).

(iii) Let \( G \) be a Lie subgroup of \( \text{Gl}(n) \). Then the Lie algebra of \( j_\alpha(J^1_n G) \) consists of all the matrices of the form

\[
\begin{bmatrix}
  A & 0 & \ldots & 0 \\
  B_1 & A & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  B_n & 0 & \ldots & A \\
\end{bmatrix}, \quad A, B_1, \ldots, B_n \in \mathcal{G}.
\]

Finally, let us recall for later use the definition of the complete and vertical lifts of a vector field \( X \) on \( M \) to \( \mathcal{F}M \); if \( \{X^i\} \) denote the local components of \( X \) in \( M \), then

(1) the complete lift \( X^c \) of \( X \) to \( \mathcal{F}M \) (also called natural lift in [5]) is the unique vector field on \( \mathcal{F}M \) locally given by

\[
X^c = X^i \frac{\partial}{\partial x^i} + x^k \frac{\partial X^i}{\partial x^k} \frac{\partial}{\partial x^i}.
\]

(2) the \( \alpha \)-th-vertical lift \( X^{(\alpha)} \) of \( X \) to \( \mathcal{F}M \) (\( \alpha = 1, 2, \ldots, n \)) ([4]) is the unique vector field on \( \mathcal{F}M \) locally given by

\[
X^{(\alpha)} = X^i \frac{\partial}{\partial x^{(\alpha)}_i}.
\]

2. – Imbedding of \( J^1 \mathcal{F}M \) into \( \mathcal{F}J^1 M \).

Let \( P(M, \pi, G) \) be a principal fibre bundle with bundle space \( P \), base space \( M \), projection \( \pi \) and structure group \( G \). Then \( J^1_n P(J^1 M, \pi', J^1 \text{Gl} G) \) is again a principal fibre bundle; in fact, if \( g_{\text{triv}} : U \cap U' \to G \) are the transition functions of \( P \), then \( g_{\mathcal{F}M} : J^1 U \cap J^1 U' \to J^1_n G \) are the transition functions of \( J^1_n P \).

Let \( \mathcal{F}M(M, \pi_M, \text{Gl}(n)) \) be the frame bundle of \( M \), \( J^1_n \mathcal{F}M(J^1 M, \pi'_{\mathcal{F}M}, J^1 \text{Gl}(n)) \) the induced bundle, and \( \mathcal{F}J^1 M(J^1 M, \pi_{J^1 M}, \text{Gl}(n+n^2)) \) the frame bundle of \( J^1 M \).

**Theorem 2.1.** – There exists a canonical injective homomorphism of principal bundles \( j_\alpha : J^1 \mathcal{F}M \to \mathcal{F}J^1 M \) over the identity of \( J^1 M \), with associate Lie group homomorphism \( j_\alpha : J^1 \text{Gl}(n) \to \text{Gl}(n+n^2) \).
PROOF. — Let $U$ be a coordinate neighborhood in $M$, and

$$\varphi_U: \mathcal{F}U \times J^1_s \text{Gl}(n) \to J^s \mathcal{F}U$$

$$\psi_U: J^1 \mathcal{F}U \times \text{Gl}(n + n^2) \to \mathcal{F}J^1 \mathcal{F}U$$

the local trivializations of $J^1_s \mathcal{F}M$ and $\mathcal{F}J^1 \mathcal{F}M$ respectively. Then, we define $(j_M)_\sigma$: $J^1_s \mathcal{F}U \to \mathcal{F}J^1 \mathcal{F}U$ as the composition

$$(j_M)_\sigma = \psi_U \circ (i \times j_\sigma) \circ \varphi_U$$

where $i: \mathcal{F}U \to J^1 \mathcal{F}U$ is the canonical injection. In order to prove that $j_M$ is well defined from these local $(j_M)_\sigma$ it suffices to show that $j_n \cdot J^i \sigma \varphi = J^i \sigma \varphi$, where $J^i \sigma \varphi: U \cap U' \to \text{Gl}(n)$ and $J^i \sigma \varphi: J^1 \mathcal{F}U \cap J^1 \mathcal{F}U' \to \text{Gl}(n + n^2)$ denote the Jacobian matrices of change of coordinates in $M$ and in $J^1 \mathcal{F}M$, respectively, and that can be easily done through a long but straightforward computation which we omit for a sake of brevity.

If we now denote $J^1_s \mathcal{F}M|_{\mathcal{F}M}$ the restriction of $J^1_s \mathcal{F}M$ to the open submanifold $\mathcal{F}M$ of $J^1 \mathcal{F}M$, and remark that the restriction $\mathcal{F}J^1 \mathcal{F}M|_{\mathcal{F}M}$ is canonically isomorphic, as $\text{Gl}(n + n^2)$-bundle, to the frame bundle $\mathcal{F}\mathcal{F}M$ of $\mathcal{F}M$, then from Theorem 2.1 we deduce

THEOREM 2.2. — $j_M$ induces a bundle homomorphism of $J^1_s \mathcal{F}M|_{\mathcal{F}M}$ into $\mathcal{F}\mathcal{F}M$ over the identity of $\mathcal{F}M$ and with associate Lie group homomorphism $j_n$, i.e. for any $X \in J^1_s \mathcal{F}M|_{\mathcal{F}M}$ and $Y \in J^1_s \text{Gl}(n)$,

$$j_M(X \cdot Y) = j_M(X) \cdot j_n(Y).$$

3. — Prolongation of $G$-structures to $\mathcal{F}M$.

Let $G$ be a Lie subgroup of $\text{Gl}(n)$, and denote $\breve{G} = j_n(J^1_s G)$; clearly, $\breve{G}$ is a Lie subgroup of $\text{Gl}(n + n^2)$ isomorphic to $J^1_s G$. Let $P(M; \pi, G)$ be a $G$-structure on $M$; then,

THEOREM 3.1. — If $M$ has a $G$-structure $P$, then $\mathcal{F}M$ has a canonical $\breve{G}$-structure $\breve{P}$.

PROOF. — By virtue of Theorem 2.2, it suffices to set $\breve{P} = j_M(J^1_s P|_{\mathcal{F}M})$. #

DEFINITION 3.2. — $\breve{P}$ will be called the prolongation of the $G$-structure $P$ on $M$ to the frame bundle $\mathcal{F}M$.

Let $M$ and $M'$ be $n$-dimensional manifolds, $f: M \to M'$ a diffeomorphism. Directly from the definitions of the different maps involved there, one checks the commu-
tativity of the diagram

\[
\begin{array}{c}
J^n_0 FM \xrightarrow{J^n_0 f} \mathcal{F}J^1 M' \\
\mathcal{F}f|_M |^\mathcal{F}f|_M
\end{array}
\]

**Theorem 3.3.** Let \( P \) and \( P' \) be \( G \)-structures on \( M \) and \( M' \), respectively, and \( f: M \to M' \) a diffeomorphism. Then, \( f \) is an isomorphism of \( P \) to \( P' \) if and only if \( \mathcal{F}f \) is an isomorphism of \( \mathcal{P} \) to \( \mathcal{P}' \).

**Proof.** Suppose \( f \) is an isomorphism of \( P \) to \( P' \), that is \( \mathcal{F}f(P) = P' \). Then, since \( \mathcal{F}f = \mathcal{F}f|_M \) and \( (\mathcal{F}f)(J^n_0 P|_M) = J^n_0 P'|_M \), we have

\[
\mathcal{F}f(J^n_0 P|_M) = j^M_M((\mathcal{F}f)(J^n_0 P|_M)) = j^M_M(J^n_0 P'|_M) = P'
\]

and therefore \( \mathcal{F}f \) is an isomorphism of \( \mathcal{P} \) to \( \mathcal{P}' \).

Conversely, if \( \mathcal{F}f(P) = P' \), then \( \mathcal{F}f(J^n_0 P|_M) = j^M_M(J^n_0 P'|_M) \); therefore,

\[
j^M_M((\mathcal{F}f)(J^n_0 P|_M)) = j^M_M(J^n_0 P'|_M) \text{ and since } j^M_M \text{ is injective, } (\mathcal{F}f)(J^n_0 P|_M) = J^n_0 P'|_M; \text{ hence } \mathcal{F}f(P) = P'.
\]

**Corollary 3.4.** Let \( P \) be a \( G \)-structure on \( M \) and let \( f \) be a diffeomorphism of \( M \) into itself. Then, \( f \) is an automorphism of \( P \) if and only if \( \mathcal{F}f \) is an automorphism of \( \mathcal{P} \).

Let \( X \) be a vector field on \( M \), \( X^c \) its complete lift to \( \mathcal{F}M \); it is well known that, if \( \varphi \) is the local 1-parameter group generated by \( X \) then \( \mathcal{F}\varphi \), is the local 1-parameter group generated by \( X^c \). Therefore,

**Corollary 3.5.** A vector field \( X \) on \( M \) is an infinitesimal automorphism of a \( G \)-structure \( P \) on \( M \) if and only if \( X^c \) is an infinitesimal automorphism of the prolongation \( \mathcal{P} \) of \( P \).

4. Integrability of the prolongation of \( G \)-structures.

From now on, we shall adopt the following convention of indices: \( i, j, k, \ldots, \alpha, \beta, \gamma, \ldots \) have range in \( \{1, 2, \ldots, n\} \) and \( A, B, C, \ldots \) have range in \( \{1, 2, \ldots, n + n^2\} \).

Let us remind the definition of integrability of a \( G \)-structure.

**Definition 4.1.** Let \( P(M, \pi, G) \) be a \( G \)-structure on \( M \); \( P \) is said integrable if for each point \( x \in U \) there is a coordinate system \( (U, \phi^i) \) with \( x \in U \) such that the frame \( (\frac{\partial}{\partial \phi^i})_x, \ldots, (\frac{\partial}{\partial \phi^i})_x \in P \) for every \( y \in U \).
LEMMA 4.2. – Let \((U, x^i)\) be a coordinate system in \(M\) and \(f: U \to GL(n)\) a differentiable map, \(f^i_j(x)\) being the \((i, j)\)-entry of \(f(x)\) for \(x \in U\). Then, for any \([x; X] \in J^1 U\) with coordinates \((x^i, \dot{x}^a)\), we have

\[
(j_n \cdot f)([x; X]) = \begin{bmatrix}
(f^i_j(x)) & 0 & \cdots & 0 \\
\frac{\partial f^i_j}{\partial x^a} \dot{x}^a_j & (f^i_j(x)) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f^i_j}{\partial x^a} \dot{x}^a_j & 0 & \cdots & (f^i_j(x))
\end{bmatrix}.
\]

PROOF. – Direct from (1.1) and (1.3).

LEMMA 4.3. – Let \((U, x^i)\) be a coordinate system in \(M\), and let \(\varphi: U \to \mathcal{F} M\) be a cross-section given by \(\varphi(x) = (\varphi_j(x)(\partial/\partial x^i)), \) \(x \in U\). Define \(\tilde{\varphi} = j \cdot \varphi_1 : J^1 U \to \mathcal{F} J^1 M\); then \(\tilde{\varphi}\) is also a cross-section which is given, at \(X = [x; X] \in J^1 U\) with \(x = (x^i, \ldots, x^n)\), \(X = \dot{x}^a(\partial/\partial x^a)\), by

\[
\tilde{\varphi}(X) = \left(\varphi_j(x) \left(\frac{\partial}{\partial x^i}\right)_{\dot{x}^a}, \varphi_j(x) \left(\frac{\partial}{\partial x^a}\right)_{\dot{x}^a}\right).
\]

PROOF. – Firstly, from Theorem 2.1, it follows directly that \(\tilde{\varphi}\) is a cross-section. Next, putting \(f(x) = (\varphi_j(x)), \) \(x \in U\), and using Lemma 4.2 above, after some local computations, one obtains \(\tilde{\varphi}(X) = (\varphi_j(x), \dot{x}^a)\) where

\[
(4.1) \quad \ddot{X}_i = \varphi_j(x) \left(\frac{\partial}{\partial x^i}\right)_{\dot{x}^a} + \frac{\partial \varphi_j}{\partial x^a} \dot{x}^a \left(\frac{\partial}{\partial x^a}\right)_{\dot{x}^a}; \quad \ddot{X}_a = \varphi_j(x) \left(\frac{\partial}{\partial x^a}\right)_{\dot{x}^a}
\]

and the Lemma is proved.

Actually, if \(\varphi: U \to \mathcal{F} M\) is a cross-section as above, then the restriction \(\tilde{\varphi} = \varphi|_{\mathcal{F} U}: \mathcal{F} U \to \mathcal{F} J^1 U\) is also a cross-section which, by virtue of (1.4), (1.5) and (4.1), can be expressed by

\[
(4.2) \quad \tilde{\varphi}(X) = ((X_i^0)_x, (X_i^a)_x), \quad X \in \mathcal{F} U
\]

where \(X_j (j = 1, 2, \ldots, n)\) is the local vector field given on \(U\) by

\[
(4.3) \quad X_j(x) = \varphi_j(x) \left(\frac{\partial}{\partial x^j}\right)_x, \quad x \in U.
\]

Therefore, we can state the following

COROLLARY 4.4. – Let \(P\) be an integrable \(G\)-structure on \(M\). Then the prolongation \(P\) of \(P\) to \(\mathcal{F} M\) is also integrable.
Proof. — Immediate from Definition 4.1, (4.2) and (4.3).

Conversely,

Proposition 4.5. — Let \( P \) be a \( G \)-structure on \( M \). If the prolongation \( \tilde{P} \) of \( P \) is integrable, then \( P \) is also integrable.

Proof. — Let \( x_0 \in M \) be an arbitrary point, \((U, x^i)\) a coordinate system with \( x_0 \in U \), and \( \varphi: U \to P \) a local cross-section of \( P \) over \( U \). Then, \( \varphi = \varphi|_\mathcal{F} \) given by (4.2) is a local cross-section of \( \tilde{P} \) over \( \mathcal{F} \).

Now, let \( X_0 \in \mathcal{F} \) be the linear frame at \( x_0 \) given by \( X_0 = ((\partial/\partial x^i)_x) \); then, since \( \tilde{P} \) is integrable, there exists a coordinate system \((\tilde{U}, y^i)\) in \( \mathcal{FM} \) with \( X_0 \in \tilde{U} \), \( \tilde{U} \subset \mathcal{F} \), \( U \), such that \( \varphi_0(x) = ((\partial/\partial y^i)_y) \in \tilde{P} \) for every \( x \in \tilde{U} \). Therefore, \( \varphi_0 \) and \( \varphi_0 \) both are cross-sections of \( \tilde{P} \) over \( \tilde{U} \) and hence there exists \( \tilde{\varphi}: \tilde{U} \to \tilde{G} = J_a(J^*_aG) \) differentiable such that, for every \( x \in \tilde{U} \),

\[
\varphi_0(X) = \varphi(X) \cdot \tilde{\varphi}(X).
\]

In fact, by virtue of (1.3), there exist differentiable maps \( g: \tilde{U} \to G \) and \( B_a: \tilde{U} \to G \), \( a = 1, 2, \ldots, n \), such that

\[
g(X) = \begin{bmatrix} g(X) & 0 & \cdots & 0 \\ B_1(X)g(X) & g(X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_n(X)g(X) & 0 & \cdots & g(X) \end{bmatrix}, \quad X \in \tilde{U}.
\]

Now, if we put \( \varphi(x) = (\varphi^i(x)(\partial/\partial x^i)_x) \) for \( x \in U \), and \( g(X) = (g^i(X)), B_a(X) = (B^i_a(X)) \) and \( g^a(x) = B^a_b(X)g^b(X) \) for \( X \in \tilde{U} \), then, using (4.2) and (4.3), we can write (4.4) as follows: for each \( x \in \tilde{U} \)

\[
\left( \frac{\partial}{\partial y^i} \right)_x = (g^i_t X^c_t + g^i_s X^{(s)}_t)(X), \quad \left( \frac{\partial}{\partial y^{2^a+1}} \right)_x = (g^i_t X^{(s)}_t)(X)
\]

from where we get

\[
0 = \left[ \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^{2^a+1}} \right] = -g^i_t X^{(s)}_t(g^i_t) X^c_t + \text{terms in } X^{(s)}_t
\]

and, since \( X^c_t, X^{(s)}_t \) are linearly independent in \( \tilde{U} \),

\[
g^i_t X^{(s)}_t(g^i_t) = 0
\]

which implies

\[
(4.5) \quad X^{(s)}_a(g^i_t) = 0.
\]
But the matrix \((g'_{i})\) is non-singular at any point \(x \in U\), so from the definition of \(X^{(a)}\) and (4.5) we deduce that \(g = (g_{i}^{b})\) does not depend on the coordinates \(x_{\alpha}\) on \(\tilde{U}\); hence, there exists a family of differentiable functions \(a_{i}^{b}\) on \(\pi_{M}(\tilde{U}) \subset U\) such that

\[
g_{i}^{b} = (a_{i}^{b} : \pi_{M})|_{\tilde{U}}.
\]

We now define \(n\) vector fields on \(\pi_{M}(\tilde{U})\) by setting

\[
W_{j} = a_{j}^{i}X_{i}.
\]

Then,

\[
[W_{j}, W_{i}] = \left( g_{j}^{k} \frac{\partial g_{i}^{a}}{\partial x_{k}} - g_{i}^{a} \frac{\partial g_{j}^{a}}{\partial x_{k}} \right) X_{k}.
\]

On the other hand,

\[
0 = \left[ \frac{\partial}{\partial y^{i}}, \frac{\partial}{\partial y^{j}} \right] = \left( g_{j}^{i} \frac{\partial g_{i}^{a}}{\partial x_{k}} - g_{i}^{a} \frac{\partial g_{j}^{a}}{\partial x_{k}} \right) X_{k} + \text{terms in } X^{(a)}
\]

and, thus,

\[
g_{j}^{i} \frac{\partial g_{i}^{a}}{\partial x_{k}} - g_{i}^{a} \frac{\partial g_{j}^{a}}{\partial x_{k}} = 0.
\]

Hence, \([W_{j}, W_{i}] = 0\) and therefore \(\{W_{1}, ..., W_{n}\}\) is a natural frame on \(\pi_{M}(\tilde{U})\); that is, there exists a coordinate system \((\tilde{x}^{1}, ..., \tilde{x}^{n})\) on \(\pi_{M}(U)\) such that \(W_{i} = \partial / \partial \tilde{x}^{i}\).

Now, from (4.7) and (4.3), we get \((\partial / \partial \tilde{x}^{i}) = a_{i}^{j}(\partial / \partial \tilde{x}^{j})\) on \(\pi_{M}(\tilde{U})\) and therefore the cross-section \(\phi: \pi_{M}(\tilde{U}) \rightarrow \mathcal{F}M\) given by

\[
\phi(x) = \left( \left( \frac{\partial}{\partial \tilde{x}^{1}} \right)_{x}, ..., \left( \frac{\partial}{\partial \tilde{x}^{n}} \right)_{x} \right), \quad x \in \pi_{M}(\tilde{U})
\]

satisfies \(\phi(x) = g(x)\phi(x)\), where \(g(x) = g(X)\) for any \(X \in \tilde{U}\) with \(\pi_{M}(X) = x\). Since \(g(X) \in G\) for any \(X \in \tilde{U}\), then \(\phi\) is in fact a cross-section of \(P\) and the proposition is proved.

Combining Corollary 4.4 and Proposition 4.5, we have

**Theorem 4.6.** Let \(P\) be a \(G\)-structure on a manifold \(M\). Then \(P\) is integrable if and only if its prolongation \(\tilde{P}\) to \(\mathcal{F}M\) is integrable.

5. **Prolongation of some classical \(G\)-structures.**

Let \(P\) be a \(G\)-structure on \(M_{1}(U, x^{a})\) a local coordinate system in \(M_{1}\), and \(\varphi: U \rightarrow P\) a cross-section. Then, \(\varphi\) defines a local field of frames \(\{X_{1}, ..., X_{n}\}\) adapted to \(P\) and given by (4.3); hence, the local field of coframes \(\{\theta^{1}, ..., \theta^{n}\}\) dual to \(\{X_{a}\}\) is
given by

\begin{equation}
\theta^i = \psi^i_j \, dx^j
\end{equation}

where \((\psi^i_j)\) denotes the inverse of the matrix \((\psi^i_j)\). On the other hand, \(\varphi\) induces a cross-section \(\hat{\varphi}: \mathcal{F}U \to \hat{P}\) given by (4.2); therefore, the local field of frames adapted to \(\hat{P}\), which is defined from \(\hat{\varphi}\), is \(\{X^a, X^a_{(n)}\}\) and, consequently, its dual local field of coframes \(\{\hat{\theta}^a\}\) is given by

\begin{equation}
\hat{\theta}^a = \psi^a_i \, dx^i, \quad \hat{\theta}^a_{(n)} = \frac{\partial \psi^a_i}{\partial x^i} \, dx^i + \psi^a_i \, d\hat{x}^i.
\end{equation}

(1) \textbf{G-structures defined by tensor fields of type (1, 1).}

Let \(\varphi: \text{Gl}(n) \to \text{Gl}(R^n)\) be the canonical representation of \(\text{Gl}(n)\) into \(R^n\), \(u \in \text{End}(R^n)\) an arbitrary element and \(G_u\) the isotropy group of \(u\) with respect to \(\varphi\). Let \(\tilde{u} = J^1 u \in \text{End}(R^{n+1})\), \(R^{n+1}\) identified to \(R \times R^n\), the induced linear map; if \(u = (u^i_j)\) is the matrix expression of \(u\), then \(\tilde{u}\) is given by

\begin{equation}
\tilde{u} = \begin{bmatrix}
(u^i_j) & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & (u^i_j)
\end{bmatrix}
\end{equation}

and therefore the following lemma is obvious:

\textbf{Lemma 5.1.} \textit{Let} \(\tilde{u} = J^1 u \in \text{End}(R^{n+1})\) \textit{be the linear map induced by} \(u \in \text{End}(R^n)\). \textit{Then, if} \(\text{rank } u = r\), \(\text{rank } \tilde{u} = (n + 1)r\). \textit{Moreover, if} \(u\) \textit{satisfies a polynomial equation} \(Q(u) = 0\), \(\text{then} \tilde{u}\) \textit{satisfies the same equation, that is} \(Q(\tilde{u}) = 0\).

\textbf{Proposition 5.2.} \textit{Let} \(G_u\) \textit{be the isotropy group of} \(u\) \textit{with respect to the canonical representation of} \(\text{Gl}(n)\) \textit{into} \(R^{n+1}\), \(R^{n+1}\) \textit{identified to} \(R \times R^n\). \textit{Then} \(G_u \subset G_u^*\).

\textbf{Proof.} \textit{Direct from (5.3)}.

\textbf{Theorem 5.3.} \textit{If} \(M\) \textit{admits a} \(G_u\text{-structure}, \text{then} \mathcal{F}M\) \textit{admits a} \(G_u^*\text{-structure}. \text{Moreover, if the} G_u\text{-structure on} M \text{is integrable, then the induced} G_u^*\text{-structure on} \mathcal{F}M \text{is also integrable}.

\textbf{Proof.} \textit{From Theorem 3.1 we deduce that} \(\mathcal{F}M\) \textit{admits a} \(G_u^*\text{-structure which, by virtue of Proposition 5.2, can be extended to a} \(G_u^*\text{-structure. The assertion on the integrability follows from Corollary 4.4}.

\#}
Let $P$ be a $G$-structure on $M$, and let $F$ be the tensor field of type $(1,1)$ on $M$ associated to $P$. If $(U, x^i)$ is a coordinate system in $M$, and if $\{X^i\}, \{\theta^j\}$ are the local field of frames induced by a cross-section $\varphi: U \to P$, then $F$ is locally given in $U$ by

$$F = F_i^j \frac{\partial}{\partial x^i} \otimes dx^j = u^i_j X_i \otimes \theta^j.$$

Similarly, let $\bar{F}$ denote the tensor field of type $(1,1)$ on $\bar{F}M$ associated to $\bar{P}$, extension of the prolongation $\bar{P}$ of $P$ (Theorem 5.3); then, with respect to the local fields of frames induced on $\bar{F}U$ by the prolongated section $\varphi$, from (4.2) and (5.2), we have

$$\bar{F} = u^i_j X_i^\alpha \otimes \delta^\beta_j \delta^\alpha_i + \delta^\beta_j X^\alpha_i \otimes \delta^\alpha_i \delta^\beta_j,$$

from where, taking into account the identity $F_i^j = \varphi^k_i u^i_k \varphi^j_k$, a straightforward computation leads to the following local expression of $\bar{F}$ on $\bar{F}U$ with respect to the canonical coordinates $\{x^i, \dot{x}^i\}$:

$$\bar{F} = F_i^j \frac{\partial}{\partial x^i} \otimes dx^j + \frac{\partial F_i^j}{\partial \dot{x}^k} \frac{\partial}{\partial \dot{x}^k} \otimes \frac{\partial}{\partial x^i}.$$

Comparing (5.4) with Mok's definition of the complete lift $F^c$ of $F$ to $\bar{F}M$ ([7], p. 78), we have $\bar{F} = F^c$.

Summing up, we can state

**Theorem 5.4.** Let be $u \in \text{End}(R^n)$, $P$ a $G$-structure on $M$, and $F$ the tensor field of type $(1,1)$ induced by $P$ on $M$. Then the complete lift $F^c$ of $F$ to $\bar{F}M$ defines the $G_2$-structure on $\bar{F}M$ given in Theorem 5.3. Moreover, if $F$ defines on $M$ a polynomial structure of rank $r$, then $F^c$ defines on $\bar{F}M$ a polynomial structure of rank $(n + 1)r$ and same structural polynomial.

**(II)** $G$-structures defined by tensor fields of type $(0,2)$.

Let be $u \in \otimes_2 (R^n)^n$, $J^1 u: J^1 R^\times \times J^1 R^\times \to J^1 R$ the induced map, and let $\pi: J^1 R \simeq R^{n+1} \to R$ be the map defined by setting

$$\pi(s, c_1, \ldots, c_n) = \sum_{s=1}^n c_s.$$

Then, we define $\bar{u}: R^{n+1} \times R^{n+1} \to R$ as the composition $\bar{u} = \pi \cdot J^1 u$.

**Lemma 5.5.** $\bar{u} \in \otimes_2 (R^{n+1})^n$, i.e. $\bar{u}$ is bilinear. Moreover, if $u$ is symmetric (resp. skew-symmetric), then $\bar{u}$ is also symmetric (resp. skew-symmetric), and if rank $u = r$, then rank $\bar{u} = 2r$. 
PROOF. - It suffices to check from the definition that if \( u = (u_{ij}) \) is the matrix expression of \( u \), then the matrix expression of \( \tilde{u} \) is

\[
\tilde{u} = \begin{bmatrix}
0 & (u_{11}) & \cdots & (u_{nn}) \\
(u_{11}) & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
(u_{nn}) & \cdots & 0 & 0
\end{bmatrix}.
\]

Now, let \( G_u \) (resp. \( G_{\tilde{u}} \)) be the isotropy group of \( u \in \otimes_2 (\mathbb{R}^n)^* \) (resp. \( \tilde{u} = \pi \cdot J^1 u \)) with respect to the canonical representation of \( \text{Gl}(n) \) into \( \mathbb{R}^n \) (resp. of \( \text{Gl}(n + n^2) \) into \( \mathbb{R}^{n+n^2} \)), and denote \( \tilde{G}_u = j_s(J^2 G_u) \). Then, direct computations show that \( \tilde{G}_u \subset G_{\tilde{u}} \) and, therefore, Theorem 5.3 is still valid here with the obvious changes of spaces under consideration. Hence, we deduce in particular

**Corollary 5.6.** - If \( M \) has an almost-symplectic structure (resp. a symplectic structure) then \( \mathcal{F}M \) has an induced almost-presymplectic structure (resp. presymplectic structure) of rank \( 2n \).

Let be \( u \in \otimes_2 (\mathbb{R}^n)^* \), \( P \) a \( G_u \)-structure on \( M \), \( \tilde{P} \) the induced \( G_{\tilde{u}} \)-structure on \( \mathcal{F}M \), and \( g \) (resp. \( \tilde{g} \)) the tensor field of type \((0, 2)\) on \( M \) (resp. on \( \mathcal{F}M \)) associated to \( P \) (resp. to \( \tilde{P} \)). Then, with the obvious local notations, we have in \( U \subset M \),

\[
g = g_{ij} \, dx^i \otimes dx^j = u_{ij} \tilde{g}^i \otimes \tilde{g}^j
\]

and, in \( \mathcal{F}U \subset \mathcal{F}M \), with respect to the adapted frames,

\[
\tilde{g} = \sum_{\alpha=1}^{2n} (u_{ij} \tilde{g}^i \otimes \tilde{g}^j + u_{ij} \tilde{g}^i \otimes \tilde{g}^j)
\]

from where, taking into account the identity \( g_{\alpha\beta} = u_{ij} \psi^i_\alpha \psi^j_\beta \), we get

\[
(5.5) \quad \tilde{g} = \sum_{\alpha=1}^{2n} \left( \tilde{x}^\alpha \frac{\partial g_{\alpha\beta}}{\partial x^\alpha} dx^\beta \otimes dx^\beta + g_{ij} dx^i \otimes dx^j + \tilde{g}_{ij} dx^i \otimes dx^j \right)
\]

local expression of \( \tilde{g} \) on \( \mathcal{F}U \) with respect to the canonical coordinates \( \{x^i, \tilde{x}^\alpha\} \). Comparing (5.5) with our definition of complete lift \( g^c \) of \( g \) to \( \mathcal{F}M \) ([1], p. 247), we have \( \tilde{g} = g^c \).

Summing up, we can state

**Theorem 5.7.** - Let be \( u \in \otimes_2 (\mathbb{R}^n)^* \), \( \tilde{u} = \pi \cdot J^1 u \in \otimes_2 (\mathbb{R}^{n+n^2})^* \), \( P \) a \( G_u \)-structure on \( M \), and \( g \) the tensor field of type \((0, 2)\) induced by \( P \) on \( M \). Then the complete lift \( g^c \) of \( g \) to \( \mathcal{F}M \) defines the \( G_{\tilde{u}} \)-structure induced by \( P \) on \( \mathcal{F}M \).
Let be \( V = \mathbb{R}^n, \{e_i, 1 \leq i \leq n\} \) the canonical basis of \( \mathbb{R}^n \), \( W \) the \( k \)-dimensional subspace of \( V \) generated by \( \{e_1, \ldots, e_k\} \) and \( \text{Gl}(V, W) \) the Lie group of all \( a \in \text{Gl}(V) \) such that \( a(W) = W \).

Then, through the identification \( J^1 V \cong \mathbb{R}^{n+1} \), \( J^1 W \) is identified to the \((n + 1)k\)-dimensional subspace of all points of the form

\[
\begin{bmatrix}
x^i \\
x^l
\end{bmatrix}
\quad \text{with } x^i = X^i = 0 \quad \text{for } k + 1 \leq i \leq n \quad \text{and} \quad 1 \leq x \leq n.
\]

Hence, one easily proves

**Lemma 5.8.** Let \( G = \text{Gl}(V, W) \) and \( \mathcal{G} = j_s(J^1 G) \). Then \( \mathcal{G} \subset \text{Gl}(J^1 V, J^1 W) \).

Thus, since an \( n \)-dimensional manifold \( M \) admits a \( k \)-dimensional differentiable distribution if and only if it admits a \( \text{Gl}(V, W) \)-structure, from the previous results one follows

**Proposition 5.9.** Let \( D \) be a \( k \)-dimensional differentiable distribution on \( M \); then, there exists on \( \mathcal{F}M \) a \((n + 1)k\)-dimensional differentiable distribution \( \mathcal{D} \) canonically induced by \( D \). Moreover, \( D \) is completely integrable if and only if \( \mathcal{D} \) is so also.

**(IV) \( G = \text{Sl}(n, \mathbb{R}) \).**

**Lemma 5.10.** Let \( G = \text{Sl}(n, \mathbb{R}) \) and \( \mathcal{G} = j_s(J^1 G) \); then, \( \mathcal{G} \subset \text{Sl}(n + n^2, \mathbb{R}) \).

**Proof.** Direct from (1.3). 

Therefore, we deduce

**Theorem 5.11.** If \( M \) has a \( \text{Sl}(n, \mathbb{R}) \)-structure, then \( \mathcal{F}M \) has a canonical \( \text{Sl}(n + n^2, \mathbb{R}) \)-structure.

6. **Diagonal prolongation of \( G \)-structures to \( \mathcal{F}M \).**

Let \( G \) be a Lie subgroup of \( \text{Gl}(n) \), and let us denote \( G_0 = G \times \cdots \times G \) the diagonal product of \( G \) by itself \((n + 1)\) times.

There is a non-canonical way to prolongate \( G \)-structures on \( M \) to \( \mathcal{F}M \) for any closed subgroup \( G \) of \( \text{Gl}(n) \). Actually, we have

**Proposition 6.1.** Let \( P(M, \pi, G) \) be a \( G \)-structure on \( M \), \( G \) being a closed subgroup of \( \text{Gl}(n) \). Then there exists a (non canonical) \( G_0 \)-structure \( \tilde{P}_0 \) on \( \mathcal{F}M \) called the diagonal prolongation of \( P \) to \( \mathcal{F}M \).

Proof. – Firstly, we notice that $G_0$ is a closed subgroup of $\tilde{G} = j_n(J^*_n G)$ and, by virtue of Theorem 1.1, $\tilde{G}/G_0$ is diffeomorphic to the product $\times_n \mathbb{R}$; hence, $\tilde{G}/G_0$ is diffeomorphic to an Euclidean space and, therefore, contractible. So, if $\tilde{P}$ denotes the prolongation of $P$ to $\tilde{M}$ (Definition 3.2), then the bundle $\tilde{P}/\tilde{G}_0 (\tilde{M}, \pi, \tilde{G}/G_0)$, associated to $\tilde{P}$ with fibre $\tilde{G}/G_0$ and group $\tilde{G}$, admits a (non-canonical) global cross-section $\sigma: \tilde{M} \to \tilde{P}/\tilde{G}_0$; and, associated to $\sigma$, there is a $G_0$-subbundle $\tilde{P}_0$ of $\tilde{P}$ which defines the announced $G_0$-structure on $\tilde{M}$.

Nevertheless, there is a canonical procedure to construct a diagonal prolongation $\tilde{P}_0$ of $P$ to $\tilde{M}$ if we simply assume the existence on $M$ of a linear connection. This procedure runs as follows.

Let $\Gamma$ be a linear connection on $M$, $\Gamma_\pi^\pi$ its local components with respect to the coordinate system $(U, x^i)$. Recall that, if $X$ is a vector field on $M$ with local components $\{X_i\}$ in $U$, then the horizontal lift $X^h$ of $X$ to $\tilde{M}$ is the unique vector field on $\tilde{M}$ locally expressed with respect to the induced coordinates $\tilde{x}_i$ on $U$ by

$$X^h = X^i \left( \frac{\partial}{\partial x^i} - \Gamma^i_{jk} \frac{\partial}{\partial \tilde{x}_j} \right).$$

Now, let $G$ be a Lie subgroup of $Gl(n)$, and define the injective homomorphism of Lie groups $i: G \to J^*_n G$ by setting $i(a) = [a; 0]$, $a \in G$. Then, $G_\pi = (i \circ i)(G)$. Let $P(M, \pi, G)$ be a $G$-structure on $M$, $\varphi: U \to P$ a local cross-section and $\{X_i\}$, given by (4.3), the local field of frames associated to $\varphi$, and hence adapted to $P$. Then, using connection $\Gamma$, we associate to $\varphi$ a local cross-section $\varphi_\Gamma: \tilde{U} \to \tilde{P}$ by setting

$$\varphi_\Gamma(x) = ((X_i^\pi)_x, (X_i^\pi)_x), \quad X \in \tilde{P}.$$
Proof. - The direct implication follows from the fact that, if $\varphi$ preserves $\tilde{P}_0$ then $(\varphi f)_{\ast} X^\mu$ is also horizontal with respect to $\tilde{\Gamma}$. Conversely, if $f$ preserves $\tilde{\Gamma}$ then $(\varphi f)_{\ast} X^\mu = (f_{\ast} X)^\mu$ for any vector field $X$ on $M$, and the result follows. 

The infinitesimal version of the later theorem can be stated as follows:

Corollary 6.5. - Let $X$ be a vector field on $M$ which is an infinitesimal automorphism of $P$. Then, $X^\ast$ is an infinitesimal automorphism of $\tilde{P}_0$ if and only if $X$ is an infinitesimal $\tilde{\Gamma}$-transformation.

In order to describe two examples of diagonal prolongations with respect to a connection $\tilde{\Gamma}$ of $G$-structures on $M$, let us remark that, if $\varphi: U \to P$ is a local cross-section of $P$ over a coordinate neighborhood $U$, then the induced field of frames is $\{X^\mu_i, X^{(u)}_i\}$ and its dual $\{\tilde{\theta}^i, \tilde{\phi}^i\}$ is given by

$$\tilde{\theta}^i = \psi^i_j \, dx^j, \quad \tilde{\phi}^i = \tilde{\theta}^{u+i} = \psi^i_k \Gamma^k_{ij} x^j \, dx^i + \psi^i_k \, dx^k.$$

(I) $G$-structures defined by tensor fields of type $(1,1)$.

Take $u \in \text{End} (R^n)$, $\tilde{\varphi} = J^4 u$ and $\tilde{G}_u = j_s J_s^4 G_u$ as in (I), Section 5. Then, $(G_u)_0 \subset \subset \tilde{G}_u \subset G_u$ and therefore the diagonal prolongation $\tilde{P}_0$ with respect to a connection $\tilde{\Gamma}$ of a $G_u$-structure $P$ on $M$ can be extended to a $G_u$-structure $\tilde{P}$ on $\tilde{\mathcal{M}}$.

Then, if $F$ (resp. $\tilde{F}$) denotes the tensor field of type $(1,1)$ on $M$ (resp. on $\tilde{\mathcal{M}}$) associated to $P$ (resp. to $\tilde{P}$),

$$(6.1) \quad \tilde{F} = F^i_j \frac{\partial}{\partial x^j} \otimes dx^i + (F^i_j \Gamma^k_{ij} - F^j_k \Gamma^i_{jk}) X^k \frac{\partial}{\partial x^i} \otimes dx^j + \partial^2 \frac{\partial F^i_j}{\partial x^a} \otimes dx^j$$

is the local expression of $\tilde{F}$ in $\tilde{\mathcal{F}} U, x', x''_a, \{F^i_j\}$ being the local components of $F$ in $(U, x')$.

At this point, we would like to refer the reader again to our paper [1], where the horizontal lift $F^H$ and the diagonal lift $F^D$ of a tensor field $F$ on $M$ of type $(1, s)$, $s \geq 1$, to $\tilde{\mathcal{M}}$ with respect to $\tilde{\Gamma}$ have been defined. There, we proved that both lifts $F^H$ and $F^D$ coincide for $s = 1$, and this fact, apparently surprising because it does not hold for $s \geq 2$, becomes clear since $\tilde{F} = F^H$.

The integrability conditions of the diagonal prolongation with respect to $\tilde{\Gamma}$ of this sort of $G$-structures on $M$ have been studied also in [1], in terms of the Nijenhuis torsion of $F^H$.

(II) $G$-structures defined by tensor fields of type $(0,2)$.

As in (II), Section 5, let $u \in \otimes_2 (\mathbb{R}^n)^s$, and define $\hat{u}: J^1 R^n \times J^1 R^n \to R$ by setting

$$(6.2) \quad \hat{u}([x; X_a], [y; Y_a]) = u(x, y) - \sum_{a=1}^s u(X_a, Y_a)$$
for any \([x; X_a], [y; Y_a] \in J^1 R^c \cong R^{n+s'}\). From (6.2), one easily follows that the matrix expression for \(\tilde{u}\) is

\[
\tilde{u} = \begin{bmatrix}
u & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \nu
\end{bmatrix}
\]

and, therefore, Lemma 5.5 still holds here. Moreover, a direct computation shows that \((G_a) \subset G_c\) and, hence, the diagonal prolongation \(\tilde{P}_a\) of a \(G_s\)-structure \(P\) on \(M\) with respect to connection \(\Gamma\) can be extended to a \(G_c\)-structure \(\tilde{P}_s\) on \(\mathcal{F}M\).

Hence, if \(g\) (resp. \(\tilde{g}\)) is the tensor field of type \((0, 2)\) on \(M\) (resp. on \(\mathcal{F}M\)) associated to \(P\) (resp. to \(\tilde{P}\)), the standard procedure leads to the following local expression for \(\tilde{g}\):

\[
\tilde{g} = g_{ij} \eta^i \otimes \eta^j + \delta^k \eta_i \eta^j \otimes \eta^k
\]

where \(\{g_{ij}\}\) are the local components of \(g\) and

\[
\eta^i = dx^i, \quad \eta^j = l^i_{\alpha} x^\alpha dx^i + dx^i.
\]

Comparing (6.3) with the so called diagonal lift \(g^0\) of \(g\) to \(\mathcal{F}M\) with respect to \(\Gamma\) as defined in [1], we see that \(\tilde{g} = g^0\). Also, let us remark that, if \(G_c = O(n)\), the orthogonal group, then \(\tilde{P}\) defines on \(\mathcal{F}M\) the Riemannian metric introduced by MoK ([6], p. 19); some properties of this metric on \(\mathcal{F}M\) have been studied in [1] and [6]. In a forthcoming paper [3], the curvature of the Riemannian manifold \((\mathcal{F}M, g^0)\) will be studied.

7. - Lifts of G-connections to \(\mathcal{F}M\).

Let \(G\) be a Lie subgroup of \(GL(n)\), \(P(M, \pi, G)\) a \(G\)-structure on \(M\) and \(\Gamma\) a linear connection on \(M\) with covariant differentiation \(\nabla\). Let \(U\) be an arbitrary coordinate neighborhood on \(M\) and \(\{X_i\}\) a local field of frames on \(U\) adapted to \(P\). Then, for any vector field \(Y\) on \(M\) assume that

\[
\nabla_X Y = Y^a A^i_a X_a
\]

holds, the matrix \((Y^a A^i_a)\) belonging to the Lie algebra \(\mathfrak{g}\) of \(G\), where \(Y = Y^a X_a\); under these assumptions, \(\nabla\) is said to be a \(G\)-connection relative to the \(G\)-structure \(P\) and the coefficients \(A^i_a\) in (7.1) are called the components of \(\nabla\) with respect to the adapted frame \(\{X_i\}\).

Now, let us recall that MoK in [7] has introduced the so called complete lift \(\nabla^c\)
to $\mathcal{F}M$ of a linear connection $\nabla$ on $M$ as the unique linear connection on $\mathcal{F}M$ verifying

$$\nabla^\sigma \cdot Y^\sigma = (\nabla_x Y)^\sigma$$

for any vector fields $X, Y$ on $M$. Actually, $\nabla^\sigma$ also satisfies the identities

$$\nabla^\sigma x_\tau, Y^\rho = \nabla^\sigma x_\tau Y^{(a)} = (\nabla_x Y)^{(a)} , \quad \nabla^\sigma x_\tau Y^{(b)} = 0 , \quad 1 \leq x, \beta \leq n .$$

On the other hand, from (4.2) we know that, if $\{X_i\}$ is a local field of frames on $U$ adapted to a $G$-structure $P$ on $M$, then $\{X_i^c, X_i^{ca}\}$ is a local field of frames on $\mathcal{F}U$ adapted to the prolongation $\tilde{P}$ of $P$ to $\mathcal{F}M$ (Definition 3.2); hence, a simple computation shows:

$$\nabla^\sigma x_\tau X_i^c = A_i^a X_a^{(a)} + \sum_{a=1}^n \frac{\partial A_i^a}{\partial x^a} X_a^{(a)} , \quad \nabla^\sigma x_\tau X_i^{ca} = A_i^a X_a^{(a)}$$

from where we deduce

**Theorem 7.1.** Let $\nabla$ be a $G$-connection relative to a $G$-structure $P$ on $M$. Then, the complete lift $\nabla^\sigma$ of $\nabla$ to $\mathcal{F}M$ is a $G$-connection relative to the prolongation $\tilde{P}$ of $P$ to $\mathcal{F}M$.

Similarly, let us define the horizontal lift $\nabla^H$ of $\nabla$ to $\mathcal{F}M$ of a linear connection $\nabla$ on $M$ as the unique linear connection on $\mathcal{F}M$ verifying the relations

$$\nabla^H \cdot Y^H = (\nabla_x Y)^H , \quad \nabla^H \cdot Y^{(a)} = (\nabla_x Y)^{(a)}$$

$$\nabla^H x_\tau Y^H = \nabla^H x_\tau Y^{(b)} = 0$$

for any vector fields $X, Y$ on $M$, and where horizontal lifts of vector fields are considered with respect to $\nabla$.

Now, if $\{X_i\}$ denotes as before a local field of frames adapted to a $G$-structure $P$ on $M$, then from Section 6 we know that $\{X_i^H, X_i^{H(1)}\}$ is adapted to the diagonal prolongation $\tilde{P}$ of $P$ (Definition 6.3). Therefore,

$$\nabla^H x_\tau X_i^H = A_i^a X_a^H , \quad \nabla^H x_\tau X_i^{H(1)} = A_i^a X_a^{H(1)}$$

$$\nabla^H x_\tau X_i^{H(1)} = 0 , \quad \nabla^H x_\tau X_i^{H(2)} = 0$$

and from these identities one easily follows

**Theorem 7.2.** Let $\nabla$ be a $G$-connection relative to a $G$-structure $P$ on $M$. Then the horizontal lift $\nabla^H$ of $\nabla$ to $\mathcal{F}M$ is a $G_0$-connection relative to the diagonal prolongation $\tilde{P}$ of $P$. 

We must emphasize that, following the usual terminology for the tangent bundle theory of prolongations and lifts, \( \nabla^u \) is called the horizontal lift of \( \nabla \) and not the diagonal lift as it might seem to be more appropriate according to Theorem 7.2. Also, it must be remarked that a different approach to the definition of the horizontal lift of \( \nabla \) to \( \mathcal{F}M \) has been developed in [2]; in fact, the lifted connection as defined in [2], say \( \tilde{\nabla}^u \), coincides with \( \nabla^u \) given by (7.2) if and only if \( \nabla \) is symmetric; nevertheless, both \( \tilde{\nabla}^u \) and \( \nabla^u \) have the same geodesics.

8. – Final remarks.

Finally, let us show by means of some simple examples, how the framework we have built up along the present paper gives new outlooks to go further on the study of the differential geometry of \( \mathcal{F}M \) when \( M \) has some extra structure.

(1) Let \( J \) be an almost complex structure on \( M \), i.e. \( J^2 = -I \), and let \( \nabla \) be a linear connection on \( M \). Then, on \( \mathcal{F}M \) there exist two different almost complex structures, namely \( J^c \) and \( J^u \); \( J^c \) depends only on \( M \) and \( J \) while \( J^u \) depends also on \( \nabla \). If, moreover, \( \nabla \) is assumed almost complex, i.e. \( \nabla J = 0 \), then \( \nabla^c \) (resp. \( \nabla^u \)) is also almost complex with respect to \( J^c \) (resp. to \( J^u \)).

(2) Let \( G \) be a Riemannian metric on \( M \) and let \( \nabla \) be the Levi-Civita connection of \( G \). Then the diagonal lift \( G^0 \) of \( G \) to \( \mathcal{F}M \) with respect to \( \nabla \) is also a Riemannian metric; moreover, \( \nabla^u \) is a metric connection, i.e. \( \nabla^u G^0 = 0 \), but it is not the Levi-Civita connection of \( G^0 \) unless the metric \( G \) is flat (see [2]). An study of some properties of the Riemannian manifold \( (\mathcal{F}M, G^0) \) can be found in [6] and [3].

(3) Combining (1) and (2), it can be proved that, if \( G \) is hermitian with respect to \( J \), then \( G^0 \) is so also with respect to \( J^u \) (see [1]). There, in [1], we have also proved that the frame bundle of a flat Kähler manifold is a Kähler manifold.

(4) Recall that, if \( P \) is a \( G \)-structure on \( M \) and \( \nabla \) a \( G \)-connection relative to \( P \), then the linear holonomy group \( \Psi \) of \( \nabla \) is a subgroup of \( G \). And conversely, if \( \Psi \) is the linear holonomy group of a linear connection \( \nabla \) on \( M \), then \( M \) admits a \( \Psi \)-structure \( P \) and \( \nabla \) is a \( \Psi \)-connection relative to \( P \). Therefore, bearing in mind Theorems 7.1 and 7.2, we deduce

**Theorem 8.1.** – Let \( \Psi \) be the linear holonomy group of a linear connection \( \nabla \) on \( M \). Then the linear holonomy group of \( \nabla^c \) (resp. of \( \nabla^u \)) is \( \tilde{\Psi} = j_c(\Psi) \) (resp. \( \Psi^u = \Psi \times \cdots \times \Psi \)).

Also, recalling that a linear connection \( \nabla \) is flat if and only if its linear holonomy group is discrete, we obtain

**Corollary 8.2.**

\[ \nabla \text{ is flat} \iff \nabla^c \text{ is flat} \iff \nabla^u \text{ is flat}. \]
REFERENCES