On the Genus of a Hyperplane Section of a Geometrically Ruled Surface (*).

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Summary. - In this paper we estimate the minimal genus of hyperplane sections of a geometrically ruled surface.

Introduction.

Let $D$ be a divisor on a geometrically ruled surface $\pi: X \to C$. If $C_0$ is a minimal section and $f$ is fiber on $X$ we can write $D = aC_0 + bf$. For a fixed number $a$ we have studied two related problems:

I) What is the minimal $b$ (call it $b_a$) such that $D = aC_0 + bf$ is very ample?

II) What is the minimal genus $\lambda_a$ of a very ample divisor $D$?

For $g = g(C) = 0$ (see [Ha, Corollary, V.2.18]) we have $b_a = ae + 1$ and $\lambda_a = (1/2)a(a - 1)e$, where $e = -C_0 \cdot C_0$ is an invariant of $X$.

In this paper we obtain some answers for $g > 1$. In particular if $g = 1$ our answer (§ 6) is sharp i.e.

$$b_a = \begin{cases} ae + 3 & \text{if } e > 0 \text{ and any } a \text{ or } e = -1 \text{ and } a < 3 \\ 1 - (a/2) + e(a) & \text{if } e = -1 \text{ and } a > 4 \end{cases}$$

where

$$e(a) = \begin{cases} 1 & \text{if } a \text{ even} \\ (1/2) & \text{if } a \text{ odd} \end{cases}$$

and

$$\lambda_a = \begin{cases} (1/2)a(a - 1)e + 3a - 2 & \text{if } e > 0 \text{ and any } a \text{ or } e = -1 \text{ and } a < 3 \\ (a - 1)e(a) + a & \text{if } e = -1 \text{ and } a > 4 \end{cases}.$$
For \( g \geq 2 \) we found (§ 5) that if \( e > 0 \)

\[
 ae + 1 \leq b_e \leq ae + 2g + 1 \\
(a(a-1)/2)e + ag \leq \lambda_e \leq (a(a-1)/2)e + (3a-2)g
\]

and if \( e < 0 \)

\[
(1/2)ae + e(ae) \leq b_e \leq (1/2)ae + 2g + e(ae) \\
ag + (e(ae) - 1)(a - 1) \leq \lambda_e \leq (3a - 2)g + (e(ae) - 1)(a - 1).
\]

For the case \( g = 2 \) we can improve the above bounds (see § 7). In particular for \( e > 0 \), \( b_e = ae + 5 \) and \( \lambda_e = (1/2)e(a-1)e + 6a - 4 \).

Our results are very useful in the study of smooth, connected, projective, ruled surfaces with the genus of a hyperplane section less than or equal to seven. See [Li1], [Li2], [Bi-Li].

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0. Background material.

The notation, throughout this paper, is essentially that used in [Ha].

(0.1) Let \( X \) be an analytic space. We let \( \mathcal{O}_X \) denote its structure sheaf and let \( h^0(X) = \dim H^0(X, \mathcal{O}_X) \). If \( X \) is a complex manifold, we let \( \mathcal{K}_X \) denote its canonical bundle.

(0.2) Let \( X \) be a smooth connected projective surface. Let \( D \) be an effective Cartier divisor on \( X \). We denote by \( L(D) \), the holomorphic line bundle associated to \( D \). If \( L \) is a holomorphic line bundle on \( X \), \( L \) denotes the linear system of Cartier divisors associated to \( L \). Of course if \( L \) is non-empty then \( L(D) = L \) for \( D \in |L| \). Let \( E \) be a second holomorphic line bundle on \( X \), then \( E \) denotes the evaluation of the cup product, \( C_1(L) \wedge C_1(E) \) on \( X \), where \( C_1(L) \) and \( C_1(E) \) are the Chern classes of \( L \) and \( E \) respectively. If \( D \in |L| \) and \( C \in |E| \), it is convenient to let \( D \cdot C = D \cdot E = L \cdot C = L \cdot E \). We often let \( g = g(L) = (1/2)(L \cdot L + K_X \cdot L + 2) \), which is called the adjunction formula. If there is a smooth \( D \in |L| \), then

\[
 g = g(L) = h^{1,0}(D).
\]

(0.3) Let \( L \) be a line bundle on a projective variety. We say \( L \) is spanned if \( \Gamma(L) \) is generated by its global sections. By [Ha, lemma 7.8] this is equivalent to saying that \( \Gamma(L) \) is base-point-free. We say \( L \) is very ample if \( L \) is spanned and
the map \( \varphi: X \to \mathbb{P}_C^d \) associated to \( \iota(L) \) is an embedding. We say that \( L \) is *ample* if some power of \( L \) is very ample.

(0.4) Let \( D \) be an effective divisor on a smooth connected, projective surface \( X \), \( D \) is *\( k \)-connected* if \( D \cdot D > 0 \) and for every decomposition \( D = D_1 + D_2 \) into effective divisors \( D_1, D_2 > k \).

(0.5) *(Ruled surfaces).* Let \( C \) be a smooth curve of genus \( g \) and \( \pi: X \to C \) a (geometrically) ruled surface. A section of \( X \) is a map \( \sigma: C \to X \) such that \( \pi \circ \sigma = \text{id}_C \). The image of \( \sigma \) is a divisor \( D_\sigma \) which we will also call a section. Let \( C_\sigma \subseteq X \) be a section, and let \( f \) be a fiber, then \( \text{Pic } X \cong \mathbb{Z} \oplus \pi^* \text{Pic } C \), where \( \mathbb{Z} \) is generated by \( C_\sigma \).

Also \( \text{Num } X \cong \mathbb{Z} \oplus \mathbb{Z}_d \) is generated by \( C_\sigma \) and \( f \) with \( C_\sigma \cdot f = 1 \) and \( f \cdot f = 0 \). For any ruled surface there exist a rank 2 vector bundle on \( C \), \( p: E \to C \) such that \( \text{P}(E) \cong X \) and viceversa. We have \( \text{P}(E) \cong \text{P}(E') \) if and only if there is a line bundle \( L \) such that \( E' = E \oplus L \). Moreover it is always possible to write \( X = \text{P}(E) \) with \( H^0(C, E) \neq 0 \) and \( H^0(C, E \oplus L) = 0 \), for every line bundle with \( \deg L < 0 \).

Such an \( E \) is said to be *normalized.* It is not necessarily unique but \( \deg E \) is uniquely determined and is an invariant of \( X \). Let \( e \) be the divisor on \( C \) corresponding to \( \mathcal{L} E \). Set \( e = -\deg e = -\deg \mathcal{L} E \). We fix a section \( C_\sigma \) of \( X \) with \( \mathcal{L}(C_\sigma) = \mathcal{O}_D(1) \). We have \( C_\sigma^2 = \deg e = -e \) and \( C_\sigma \cdot f = 1 \). If \( b \) is any divisor on \( C \), then we denote the divisor \( \pi^* b \) by \( b_f \).

Thus any element of \( \text{Pic } X \) can be written \( aC_\sigma + bf \) with \( a, b \in \mathbb{Z} \) and \( b \in \text{Pic } C \). Any element of \( \text{Num } X \) can be written \( aC_\sigma + bf \) with \( a, b \in \mathbb{Z} \).

If \( D_\sigma = aC_\sigma + b_f \), \( h = 1, 2 \) we get

\[
\begin{align*}
D_1 \cdot D_2 &= a_1 b_2 + a_2 b_1 - a_1 a_2 e \\
D_1^2 &= 2a_1 b_1 - a_2^2 e.
\end{align*}
\]

Moreover since

\[(0.6)\]
\[K_x = -2C_\sigma + (2g - 2 - e)f\]

we get

\[(0.7)\]
\[K_x^2 = 8(1 - g) .\]

If \( D = aC_\sigma + bf \) and setting \( h(D) = \dim H^i(X, L(D)) \), \( i > 0 \) then by the Riemann-Roch Theorem we have

\[(0.8)\]
\[h^i(D) - h^{i+1}(D) = (a + 1)(b - (ae/2) - g + 1) .\]

Let \( D = aC_\sigma + bf \) be a divisor on \( X \). Then \( D \) is ample if and only if

\[(0.9)\]
\[a > 0 \quad \text{and} \quad b > \begin{cases} ae & \text{if } e > 0 \\ (1/2)ae & \text{if } e < 0 . \end{cases} \]
1. Vanishing theorems.

Let \( \pi: X \to C \) be a ruled surface. Let \( X \simeq \mathbb{P}(E) \), where \( E \) is a normalized rank 2 vector bundle. Set \( L(e) = A^2 E \). If \( \deg e = -\epsilon \) then we write (numerically) \( L(e) = L(-\epsilon) \). We have

\[
E = E^* \otimes A^2 E = E^* \otimes L(e)
\]

where \( E^* \) is dual to \( E \). Let \( S^a E \) be the \( a \)-symmetric product of \( E \). Then we have

\[
H^i(X, aC_0 + bI) \simeq H^i(C, S^a E^* \otimes L(b)), \quad i > 0
\]

\[
S^a E \simeq S^a E^* \otimes (A^2 E)^{\otimes a} = S^a E^* \otimes L(\epsilon)
\]

Since \( E \) is normalized we have

\[
H^i(C, E \otimes L(b)) \simeq H^i(X, C_0 + bI) = 0
\]

for any \( b < 0 \). As before we set \( h^i(D) = \dim H^i(X, L(D)) \). Let \( D = aC_0 + bI \) be a divisor on \( X \). Then by the Kodaira Vanishing Theorem we have \( h^i(D) = 0 \) if \( D - K_X \) is ample. Therefore using (0.1) and (0.9) we get \( h^i(D) = 0 \) if

\[
a > 1 \quad \text{and} \quad b > \begin{cases} (a + 1)e + 2g - 2 & \text{if } \epsilon > 0 \\ (1/2)ae + 2g - 2 & \text{if } \epsilon < 0 \end{cases}
\]

By Serre duality, (1.1) and (1.2) we get

\[
H^i(X, aC_0 + bI) \simeq H^i(X, aC_0 + (ae - b + 2g - 2)I)
\]

\[
H^i(X, aC_0 + bI) \simeq H^i(X, aC_0 + (ae - b + 2g - 2)I).
\]

**Theorem 1.1.** Let \( D = aC_0 + bI \) be a divisor on \( X \), with \( a > 1 \). We have \( h^i(D) = 0 \) if

\[
b > \begin{cases} ae + 2g - 2 & \text{if } a = 1 \text{ and any } \epsilon \text{ or } a > 2 \text{ and } \epsilon > 0 \\ (1/2)ae + 2g - 2 & \text{if } a > 2 \text{ and } \epsilon < 0 \end{cases}
\]

and \( h^4(D) > 0 \) if

\[
b < (1/2)ae + g - 1.
\]

**Proof.** (1.7) follows from (0.8). Consider now (1.6). The case \( \epsilon < 0 \) was already done in (1.4). We prove the case \( \epsilon > 0 \) by induction. By (1.3) and (1.5) we get \( h^4(D) = 0 \) if \( a = 1 \), \( b > \epsilon + 2g - 2 \) and any \( \epsilon \).
Suppose (1.6) true for \( a - 1 \). We have the short exact sequence

\[ 0 \rightarrow L(D - C_0) \rightarrow L(D) \rightarrow L(-ae + b) \rightarrow 0 \]

since \( L(D)|_{C_0} \cong L(-ae + b) \). Then

\[ H^1(X, L(D - C_0)) \rightarrow H^1(X, L(D)) \rightarrow H^1(C, L(-ae + b)) \rightarrow 0. \]

We have \( b > ae + 2g - 2 > (a - 1)e + 2g - 2 \) since \( e > 0 \). Thus by induction \( H^1(X, L(D - C_0)) = 0 \). Moreover \( -ae + b > 2g - 2 \) implies \( H^1(C, L(-ae + b)) = 0 \). Hence \( h^1(D) = 0 \).

**Theorem 1.2.** Let \( D = aC_a + bf \) be a divisor on \( X \), with \( a > 1 \). Then \( h^0(D) = 0 \) if

\[ b < \begin{cases} 0 & \text{if } a = 1 \text{ and any } e \text{ or } a > 2 \text{ and } e > 0 \\ (1/2)ae & \text{if } a > 2 \text{ and } e < 0 \end{cases} \]

and \( h^0(D) > 0 \) if

\[ b > (1/2)ae + g - 1. \]

**Proof.** (1.9) follows from (0.8). Consider now (1.8). The case \( a = 1 \) is just (1.3). By (1.5) and (1.6) we get (1.8) in the case \( a > 2 \).

**2. Very ample line bundles on ruled surfaces.**

Let \( D = aC_a + bf \) be a divisor on a ruled surface \( \pi: X \rightarrow C \), with \( a > 1 \). We set \( f_x = \pi^{-1}(x) \) for \( x \in C \).

**Lemma 2.1.** If \( h^1(D - f_x) = 0 \) then \( L(D)|_{f_x} = \mathcal{O}_D(a) \).

**Proof.** Since \( h^1(D - f_x) = 0 \) we have

\[ 0 \rightarrow H^0(X, L(D - f_x)) \rightarrow H^0(X, L(D)) \rightarrow H^0(X, L(D)|_{f_x}) \rightarrow 0. \]

If \( D'|_{f_x} \neq 0 \) for some \( D' \in |L(D)| \) we would have \( L(D)|_{f_x} \cong \mathcal{O}_D(a) \). But \( D'|_{f_x} = 0 \) for every \( D' \in |L(D)| \) implies \( \beta = 0 \), hence \( h^0(D - f_x) = h^0(D) \). On the other hand from (0.8) we get \( h^0(D) = h^0(D - f_x) + (a + 1) + h^0(D) \) which implies \( h^0(D) > h^0(D - f_x) \) since \( a > 1 \) and \( h^0(D) > 0 \). Therefore \( L(D)|_{f_x} = \mathcal{O}_D(a) \).

**Proposition 2.2.** If \( h^1(D - f_x) = 0 \) then \( L(D) \) is spanned.
Proof. Since $h^i(D - f_x) = 0$ for every $x$, by lemma 2.1 we have

$$0 \rightarrow L(D - f_x) \rightarrow L(D) \rightarrow \mathcal{O}_{P_x}(a) \rightarrow 0$$

since $\mathcal{O}_{P_x}(a)$ is very ample for $a > 1$ we get that $L(D)$ is spanned. □

**Proposition 2.3.** If $h^i(D - f_x) = 0$ and $h^i(D - 2f_x) = 0$ then $L(D)$ is very ample.

**Proof.** We have to prove that $|L(D)|$ separates points and tangent vectors.

**Case 1.** $P$ and $Q$ (or $P$ and $t$) not in the same fiber. Let $f_P$ and $f_Q$ be the fibers which $P$ and $Q$ are on respectively. Since $h^i(D - f_P - f_Q) = 0$ and $L(D - f_P)|_{f_Q} = \mathcal{O}_{f_Q}(a)$ we have

$$0 \rightarrow H^i(X, L(D - f_P - f_Q)) \rightarrow H^i(X, L(D - f_P)) \rightarrow H^o(P, \mathcal{O}_{P}(a)) \rightarrow 0.$$

So we can find $D' \simeq D - f_P$ such that $Q \notin D'|_{f_Q}$ i.e. $Q \notin D'$. Hence $Q \notin D' + f_P \simeq D$ but $P \in D' + f_P$. In the case ($P$ and $t$) we do the same considering $P = Q$. Then we get $P \in D' + f_P$, but $2P \notin D' + f_P$ so $t$ is not a tangent vector to $D' + f_P$ at $P$.

**Case 2.** $P$ and $Q$ (or $P$ and $t$) are both in the same fiber $f_x$ for some $x \in C$. From (2.1) we can find $D' \simeq D$ such that $P \in D'|_{f_x}$ but $Q \notin D'|_{f_x}$ (if $P \in D'$ but $2P \notin D'|_{f_x}$). Hence $P \in D'$ and $Q \notin D'$ (or $P \in D'$ but $t$ is not tangent to $D'$ at $P$).

**Corollary 2.4.** — $D$ is spanned if

$$b > \begin{cases} 
    \frac{ae + 2g - 1}{2} & \text{if } a = 1 \text{ and any } e \text{ or } a > 2 \text{ and } e > 0 \\
    (1/2)a + 2g - 1 & \text{if } a > 2 \text{ and } e < 0
\end{cases}$$

and $D$ is very ample if

$$b > \begin{cases} 
    ae + 2g & \text{if } a = 1 \text{ and any } e \text{ or } a > 2 \text{ and } e > 0 \\
    (1/2)a + 2g & \text{if } a > 2 \text{ and } e < 0
\end{cases}.$$

3. — On the 3-connectedness of a divisor on a ruled surface.

Let $D = aC_0 + bf$ be a divisor on a ruled surface. If $D = D_1 + D_2$ we have $D_1 = aC_0 + (b - \tilde{y})f$ and $D_2 = (a - x)C_0 + (b - \bar{y})f = \tilde{x}C_0 + \bar{y}f$. Assume $D^2 = (2b - ae) > 0$, i.e. $a > 0$ and $b > (1/2)ae$. In order to prove that $D$ is 3-connected we have to prove that for any decomposition $D = D_1 + D_2$ with $D_1 \simeq 0$ and $h^0(D_1) > 1$ we get that $D_1 \cdot D_2 > 3$. 

Lemma 3.1. - Assume that $h^0(xC_0 + yf) > 1$, then

\[ b) \quad y > \begin{cases} 
(1/2)xe & \text{if } x \geq 2 \text{ and } e < 0 \\
0 & \text{if } x = 0, 1 \text{ and } e < 0 \text{ or } e > 0 \text{ and any } x.
\end{cases} \]

Proof. - a) If $x < 0$ then $h^0(D_1) = 0$. It is enough to prove it for $x = -1$, since $h^0(D_1 + C_b) = 0$ implies $h^0(D_1) = 0$ for $x < -2$. We have

\[ 0 \rightarrow L(D_1) \rightarrow L(yf) \rightarrow L(yf)|_{C_b} \simeq L(y) \rightarrow 0. \]

Since $h^0(yf) = h^0(y)$ and $H^0(X, L(yf)) \rightarrow H^0(C, L(y))$ is surjective, we have $h^0(D_1) = 0$.

b) If $x = 0$, then $h^0(D_1) = 0$ if $y < 0$. Therefore if $x_1 = 0$ and $h^0(D_1) > 0$ it follows that $y > 0$. If $x > 1$ then by (1.8) we get part b). \( \square \)

Proposition 3.2. - Assume that $e < 0$ and $a > 3$. Then $D$ is 3-connected if

\[ b > \begin{cases} 
0 & \text{if } e = -1 \text{ and } a = 3 \\
(1/2)ae + 1 & \text{otherwise}.
\end{cases} \]

Proof. - By lemma 3.1 $x$ and $\bar{x}$ are non-negative.

Case 1. - Assume $x = 0$ (or $\bar{x} = 0$) from

\[ D_1 \cdot D_2 = \begin{cases} 
y(a - 2x) + x(-(a - x)e + b) \\
g(a - 2\bar{x}) + \bar{x}(-(a - \bar{x})e + b)
\end{cases} \]

we get $D_1 \cdot D_2 = ya$ (or $= \bar{g}a$). Since $a > 2$ and $y > 0$ ($\bar{g} > 0$) we obtain $D_1 \cdot D_2 > 2$.

Case 2. - Assume $x = 1$ (or $\bar{x} = 1$). From $a - 2x > 0$ and $y > 0$ and (3.1) we have $D_1 \cdot D_2 > b - (a - 1)e$. If $(a, e) = (3, -1)$ then $b > 0$ so $-(a - 1)e + b > 2$, hence $D_1 \cdot D_2 > 2$. If $(a, e) \neq (3, -1)$ then $b > (1/2)ae + 1$ so

\[ D_1 \cdot D_2 > b - (a - 1)e > 1 + (1/2)ae - (a - 1)e = 1 - (a - 2)(ae/2) > 2, \]

then $D_1 \cdot D_2 > 2$.

Case 3. - $2 < x < a - 2$ and $a - 2x > 0$ (or $2 < \bar{x} < a - 2$ and $-(a - 2x) = a - 2\bar{x} > 0$) we treat only the part $a - 2x > 0$. The other part is similar. In this case $y = xe/2$ so $D_1 \cdot D_2 > (xe/2)(a - 2x) + x(-(a - x)e + b) = x(b - (1/2)ae)$. If $(a, e) = (3, -1)$ then $b > 0$ and we have $D_1 \cdot D_2 > 3$. If $(a, e) \neq (3, -1)$ then $D_1 \cdot D_2 > x(b - (ae/2)) > 2(b - (ae/2)) > 2$. So $D_1 \cdot D_2 > 2$. \( \square \)
4. – Very ampleness by Bombieri’s method.

We would like to find new conditions for $L$ to be very ample. In order to do this we shall use the following theorem.

**Theorem 4.1.** – Let $L$ be a line bundle over a surface $X$. We put $L_0 = L \otimes K_X^{-1}$. If i) $h^0(L_0) \geq 7$; ii) $L_0 \cdot L_0 > 10$; iii) $L_0$ is 3-connected, then $L$ is very ample.

**Proof.** – See [VdV].

Theorem 4.1 has been proved using a method of Bombieri. See also [Be], [Bo], [So1] and [So2].

We will apply Theorem 4.1 for $L = L(D)$ where $D = a C_0 + b f$ is a divisor over a ruled surface $X$. Then $L_0 = L(D_0)$ where

$$D_0 = D - K_x = a C_0 + b f = (a + 2) C_0 + (b - 2g + 2 + \varepsilon) f.$$

We are interested in the case $\varepsilon < 0$ and $a > 2$. By (0.8) we have

$$h^0(D_0) = (a_0 + 1)(b_0 + 1 - g - (a_0 \varepsilon/2)).$$

Since $h^0(D_0) > 0$ we have $h^0(D_0) > 7$ if

\begin{equation}
(4.1) \quad b > 7/(a + 3) + (a \varepsilon/2) + 3g - 3.
\end{equation}

By (0.6) we have $L_0 \cdot L_0 = 2a_0(b_0 - (1/2)a_0\varepsilon) = 2(a + 2)(b - 2g + 2 - (a\varepsilon/2))$. Therefore $L_0 \cdot L_0 > 10$ when

\begin{equation}
(4.2) \quad b > 5/(a + 2) + (a \varepsilon/2) + 2g - 2.
\end{equation}

Moreover by Proposition 3.2 we have $L_0$ is 3-connected if

\begin{equation}
(4.3) \quad b > (a \varepsilon/2) + 2g - 1.
\end{equation}

We set

$$K_1 = 7/(a + 3) + (a \varepsilon/2) + 3g - 3, \quad K_2 = 5/(a + 2) + (a \varepsilon/2) + 2g - 2, \quad K_3 = (a \varepsilon/2) + 2g - 1 + \varepsilon(a), \quad K_4 = K_3 + 1,$$

where

$$\varepsilon(n) = \begin{cases} 1 & \text{if } n \text{ even} \\ 1/2 & \text{if } n \text{ odd}. \end{cases}$$
Using Theorem 4.1 we have $D$ is very ample if $b > \sum K_0 = \max \{K_1, K_2, K_3\}$. We have $K_1 > K_2$. For $g = 1$ we have $K_0 = K_3$. For $g = 2$

$$K_0 = \begin{cases} 
K_1 & \text{if } \begin{cases} 
a = 2, 3, 5, 7, 9 \text{ and } e = -1 \\
a = 2, 3 \text{ and } e = -2 
\end{cases} \\
K_3 & \text{otherwise}.
\end{cases}$$

For $g > 3$ then $K_0 = K_3$. Therefore by Theorem 4.1 and Corollary 2.4, in the case $e < 0$ and $a \geq 2$, $D$ is very ample when

$$b > \min \{K_0, K_4\} = K$$

and

$$K = K_0 = K_3 \text{ if } g = 1$$

$$K = \begin{cases} 
K_1 = K_4 & \text{if } \begin{cases} 
a = 2, 3, 5, 7, 9 \text{ and } e = -1 \\
a = 2, 3 \text{ and } e = -2 
\end{cases} \\
K_3 & \text{otherwise}.
\end{cases}$$

For $g > 3$ we have $K = K_3$.

5. - The genus of a very ample divisor on a ruled surface.

Let $D = aC_0 + bf$ be a very ample divisor on a ruled surface $X$. Let $b = \deg b$ and $\gamma = g(D)$. Then by the Adjunction formula we have $2\gamma - 2 = (D - K_0)$ where $K_0 = -2C_0 + (K_0 + e)f$. Therefore

$$(D + K_0)D = 2(a - 1)(b - 1 - (1/2)ae) + 2ag - 2$$

and hence

$$\gamma = (a - 1)(b - 1 - (1/2)ae) + ag.$$ 

We set $\lambda_a = \lambda_a(C, X)$ and $b_a = b_a(C, X)$ which are respectively the minimum genus and the minimum $b$ of a very ample divisor $D = aC_0 + bf$ on a ruled surface $X$ over the curve $C$. We have

$$\lambda_a = (a - 1)(b_a - 1 - (1/2)ae) + ag.$$ 

So finding $\lambda_a$ is equivalent to finding $b_a$. The next step is finding an estimate for $b_a$ (or $\lambda_a$). We are interested in the case $a \geq 2$. Since if $yD$ is very ample it is
ample. Hence

\[ b_\varepsilon > \begin{cases} 
  ae & \text{if } \varepsilon > 0 \\
  (1/2)ae & \text{if } \varepsilon < 0 
\end{cases} \]

and by corollary 2.4 we have

\[ b_\varepsilon < \begin{cases} 
  ae + 2g + 1 & \text{if } \varepsilon > 0 \\
  (1/2)ae + 2g + \varepsilon(ae) & \text{if } \varepsilon < 0 
\end{cases} \]

Therefore if \( \varepsilon > 0 \)

\[ \tag{5.2} ae + 1 < b_\varepsilon < ae + 2g + 1 \]

and

\[ \tag{5.3} (a(a - 1)/2)\varepsilon + a\lambda = \lambda < a(a - 1)/2 + (3a - 2)g \]

if \( \varepsilon < 0 \)

\[ \tag{5.4} (1/2)ae + \varepsilon(ae) < b_\varepsilon < (1/2)ae + 2g + \varepsilon(ae) \]

and

\[ \tag{5.5} a\lambda + (\varepsilon(ae) - 1)(a - 1) < \lambda < (3a - 2)g + (\varepsilon(ae) - 1)(a - 1). \]

If \( g = 0 \) then \( \varepsilon > 0 \) and \( b_\varepsilon = ae + 1 \) hence

\[ \tag{5.6} \lambda = (1/2)a(a - 1)e \]

In the case \( g = 1 \) or \( 2 \) we can improve the lower bound. By the short exact sequence

\[ 0 \to L(D - C_0) \to L(D) \to L(D)|_{C_1} \simeq L(ae + b) \to 0 \]

we get that \( L(D) \) very ample implies \( L(ae + b) \) very ample.

In the case \( g = 1 \) or \( 2 \), \( L(ae + b) \) is very ample if and only if \( b > ae + 2g \). If \( \varepsilon > 0 \) we have \( ae + 2g + 1 > ae + 1 \) and

\[ \tag{5.7} b_\varepsilon = ae + 2g + 1, \]

\[ \tag{5.8} \lambda_\varepsilon = (1/2)a(a - 1)e + (3a - 2)g. \]

If \( \varepsilon < 0 \) we have

\[ (1/2)ae \begin{cases} > ae + 2g & \text{if } a > -4g/e \\
< ae + 2g & \text{if } a < -4g/e 
\end{cases} \]
So

\[(5.9) \quad b_a > \begin{cases} \frac{a e + 2g + 1}{2} & \text{if } a < -4g/e \\
(1/2)ae + e(a) & \text{if } a > -4g/e . \end{cases}\]

6. - The case \( g = 1 \).

If \( e > 0 \) we have (5.7) and (5.8). It only remains to study the case \( e = -1 \).

By (5.4), (5.9) and (4.4) we have for \( a > 2 \)

\[(6.1) \quad 1 - (a/2) + e(a) > b > \begin{cases} a + 3 & \text{if } a < 3 \\
-(a/2) + e(a) & \text{if } a > 4 . \end{cases}\]

We already know \( b_1 = 2 \). From (6.1) we have \( b_2 = 1 \) and \( b_3 = 0 \). If \( a > 4 \) then \( b_a \) is either \( -(a/2) + e(a) \) or \( 1 - (a/2) + e(a) \). We set \( D_a = aC_a + (-(a/2) + e(a))f \).

**Theorem 6.1.** - \( D_a \) is not very ample.

In order to prove Theorem 6.1 we need the following.

**Lemma 6.2.** - Let \( X \) be a ruled surface over \( C \). Assume \( e = -1 \). Then there is \( P \in C \) such that \( h^0(2C_0 - Pf) > 1 \).

**Proof.** - We put \( D = 2C_0 - f \). By (0.8) we have \( h^0(D) = h^1(D) \) and \( h^0(2C_0) = h^1(2C_0) = 3 \). By (1.6) \( h^1(2C_0) = 0 \), so \( h^0(2C_0) = 3 \). Now \( h^0(2C_0) = h^0(S^2E) \) so there is a section \( \sigma \) in \( S^2E \) which has some zero, otherwise \( S^2E \) would be trivial which implies \( A^2S^2E = L(3e) \) is trivial which is a contradiction. Then by (1.8) we have \( D[\sigma] = (P) \), i.e. only one point, and \( h^0(2C_0 - Pf) > 1 \). □

**Proof of Theorem 6.1.** - Suppose \( D_a \) very ample. We set \( D_0 = 2C_0 - Pf \).

We have \( D_a \cdot D_0 = 2e(a) \), i.e. \( D_a \cdot D_0 = 1 \) if \( a \) is odd and \( D_a \cdot D_0 = 2 \) if \( a \) is even. In both cases \( D_0 \) is a smooth rational curve (since \( D_0 \) is irreducible) with respect to the embedding provided by \( [D_0] \). But \( \pi|_{D_0} : D_0 \to C \) is a 2:1 map over an elliptic curve, which is a contradiction. □

**Theorem 6.3.** - Let \( D = aC_a + bf \) be divisor on a ruled surface \( X \) over an elliptic curve \( C \). Assume that \( a > 1 \). Then \( D \) is very ample if and only if

\[(6.2) \quad b > \begin{cases} ae + 2 & \text{if } e > 0 \text{ and any } a \text{ or } e = -1 \text{ and } a < 3 \\
1 - (a/2) & \text{if } e = -1 \text{ and } a > 4 . \end{cases}\]
Corollary 6.4. - Let $D$ be as above. Then

$$b_a = \begin{cases} 
  ae + 3 & \text{if } e > 0 \text{ and any } a \text{ or } e = -1 \text{ and } a < 3 \\
  1 - \frac{a}{2} + \varepsilon(a) & \text{if } e = -1 \text{ and } a > 4 
\end{cases}$$

$$\lambda_a = \begin{cases} 
  \frac{1}{2}a(a-1)e + 3a - 2 & \text{if } e > 0 \text{ and any } a \text{ or } e = -1 \text{ and } a < 3 \\
  (a - 1)e(a) + a & \text{if } e = -1 \text{ and } a > 4.
\end{cases}$$

7. - The case $g = 2$.

Let $X$ be a ruled surface over a curve $C$ with $g = g(C) = 2$. Let $D \equiv aC_b + bf$ be a divisor over $X$ with $a > 2$. As for the case $g = 1$, if $e > 0$ we have

$$b_a = ae + 5 \quad (\text{actually it holds also for } a = 1 \text{ and } e < 0).$$

When $e < 0$ we have two cases $e = -1$ and $e = -2$. At first we consider the cases $e = -1$. From (5.4), (5.9) and (4.5) we have

$$\begin{align*}
  b_a &< \begin{cases} 
    -a + 5 & \text{if } a < 7 \\
    -(a/2) + \varepsilon(a) & \text{if } a > 8
  \end{cases} \\
  b_a &> \begin{cases} 
    -(a/2) + 6 + \varepsilon(a) & \text{if } a = 2, 3, 5, 7, 9 \\
    -(a/2) + 3 + \varepsilon(a) & \text{otherwise}
  \end{cases}
\end{align*}$$

Therefore

$$\begin{align*}
  \lambda_a &< \begin{cases} 
    6a - 4 - (a/2)a(a - 1) & \text{if } a < 7 \\
    (a - 1)(\varepsilon(a) - 1) + 2a & \text{if } a > 8
  \end{cases} \\
  \lambda_a &> \begin{cases} 
    6a + 4 + (a - 1)(\varepsilon(a) - 1) & \text{if } a = 2, 3, 5, 7, 9 \\
    5a - 3 + (a - 1)(\varepsilon(a) - 1) & \text{otherwise}
  \end{cases}
\end{align*}$$

Now we consider the case $e = -2$. From (5.4), (5.9) and (4.5) we have

$$\begin{align*}
  b_a &> \begin{cases} 
    -2a + 5 & \text{if } a < 3 \\
    -a + 1 & \text{if } a > 4
  \end{cases} \\
  b_a &< \begin{cases} 
    -a + 5 & \text{if } a < 3 \\
    -a + 4 & \text{if } a > 4.
  \end{cases}
\end{align*}$$
Therefore
\[
\lambda_a = \begin{cases} 
(a - 1)(4 - a) + 2a & \text{if } a < 3 \\
2a & \text{if } a \geq 4
\end{cases}
\]
(7.5)
\[
\lambda_a = \begin{cases} 
6a - 4 & \text{if } a < 3 \\
5a - 3 & \text{if } a \geq 4
\end{cases}
\]

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