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Erratum

k-Symmetric Submanifolds of R^N

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The proof of the connectedness of the group of symmetries given in [2, Sects. 5–7] is wrong by several fatal errors included in it. The reader should disregard the proof entirely.

We present here a correct proof of this statement by a different method.

The notation and definitions will be those of [2]. As in Sect. 5 we assume

$$\sigma_a \notin G_e \text{ (a our chosen point in M)}. \tag{1.1}$$

We put $K = \{g \in G : g(a) = a\}$ so that $M = G/K = G_e/K \cap G_e$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G_e and K_e respectively.

$\mathfrak{g}$ may not be semisimple but it is compact so $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. The action of G_e on M is effective and then

$$\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k} = \{0\}. \tag{1.2}$$

Let $\mathfrak{m}_1$ be a complementary subspace of $\mathfrak{k}$ in $\mathfrak{g}$ such that $\mathrm{Ad}(K)\,\mathfrak{m}_1 \subset \mathfrak{m}_1$. We observe that

$$\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{m}_1 \tag{1.3}$$

and also

$$\mathfrak{k} \subset [\mathfrak{g}, \mathfrak{g}]. \tag{1.4}$$

Let now $\mathfrak{n}$ be the orthogonal complement of $\mathfrak{k}$ in $[\mathfrak{g}, \mathfrak{g}]$ with respect to the Killing form in $[\mathfrak{g}, \mathfrak{g}]$ and put $\mathfrak{m} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{n}$. It is $\mathrm{Ad}(K)$-invariant and so we can take $\mathfrak{m}$ instead of $\mathfrak{m}_1$ in $\mathfrak{g}$. Clearly $\mathfrak{z}(\mathfrak{g})$ and $\mathfrak{n}$ are $\mathrm{Ad}(\sigma_a)$-invariant.

(1.5) **Lemma.** *$M = G_e/K \cap G_e$ is simply connected.*

Proof. We clearly have $K_e \subset G_e$ and then $\varphi : \tilde{M} = G_e/K_e \to M$ is a covering manifold.

Since $M \subset V$, we have an immersion $\varphi : \tilde{M} \to V$ and we can consider its total absolute curvature $\tau(\varphi)$. If, as in [2], we call $T(M)$ the total absolute curvature of M we have $\tau(\varphi) = lT(M)$ where l is the number of sheets of φ i.e., the number of components in $K \cap G_e$.

On the other hand $\chi(\tilde{M}) = l\chi(M)$ and so by [2, (8.3)] we conclude

$$\tau(\varphi) = \chi(\tilde{M})$$

and so our immersion φ is tight. Since $\varphi(\tilde{M}) \subset S(V)$ [1, p. 340 (3.2)] implies that φ is an imbedding and so $K_e = K \cap G_e$.

But $\pi_1(G_e/K_e)$ is abelian and therefore by [2, (8.5)] we have $\pi_1(M) = 0$. □

We have now

(1.6) **Theorem.** $\mathfrak{z}(\mathfrak{g}) = 0$.

Proof. If $\mathfrak{z}(\mathfrak{g}) \neq 0$ then we would have that $\pi_1(M)$ is non-trivial which contradicts (1.5). We have then

(1.7) $$G_e \text{ is semisimple}. \quad \square$$

We may assume then that G_e is semisimple.

Since M is simply connected we have $M = M_1 \times \ldots \times M_h$ where each M_i is a quotient $M_i = G_i/K_i$ with G_i semisimple compact and K_i connected corresponding to a decomposition of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_h, \ \mathfrak{k} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_h$$
$$\mathrm{Ad}(\sigma_a) = \theta = \theta_1 \oplus \ldots \oplus \theta_h \text{ where}$$

$\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{g}_i = \mathfrak{g}_i^{\theta_i}$ and θ_i is an automorphism of odd order $k_i | k$ which does not preserve any proper ideals in $\mathfrak{g}_i$. (k is the least common multiple of $k_1, \ldots, k_h$.)

(1.8) *Remark.* It is important to notice that by our assumption $\sigma_a \notin G_e$ we may think that $\mathrm{Ad}(\sigma_a)$ is outer.

Proof of (1.8). We have some facts:

i) $Z(G_e)$ = center of G_e is finite.

ii) From (i) it follows $[\mathrm{Aut}(\mathfrak{g})]_e = \mathrm{Int}(\mathfrak{g})$.

iii) Let $\chi: G \to \mathrm{Aut}(G)$ be the adjoint representation.

Ker χ = Centralizer of G_e in G.

We shall show that if $\chi(\sigma_a) = \mathrm{Ad}(\sigma_a) \in \mathrm{Int}(\mathfrak{g})$ then our manifold M was already considered in the first part ($\sigma_a \in G_e$) of [2].

Notice that, since σ_a and G_e generate G, we have

$$\chi(G) = \mathrm{Int}(\mathfrak{g}).$$

Let us consider the compact differentiable manifold $U = \mathrm{Int}(\mathfrak{g})/\chi(K)$ and put $t_0 = \chi(\sigma_a)$. Since it is central in $\chi(K)$ we have $t_0[\chi(K)] = [\chi(K)]$. Put $\gamma = I(t_0)$. The automorphism γ satisfies

$$[(\mathrm{Int}(\mathfrak{g}))^\gamma]_e \subset \chi(K) \subset (\mathrm{Int}(\mathfrak{g}))^\gamma$$

which follows from

$$\mathscr{L}((\mathrm{Int}(\mathfrak{g}))^\gamma) = \mathfrak{k} = \mathscr{L}(\chi(\mathrm{K})).$$

Then U. has a regular s-structure of order k and furthermore we have a covering map $\pi: M \to U$ because $Z(G_e)/Z(G_e) \cap K$ acts properly discontinuously on M.

Now the argument of [2, (2.3) (ii)] shows that $\chi(K)$ has maximal rank in $\mathrm{Int}(\mathfrak{g})$ and therefore by [3, p. 95 (4.1)] we have that U *is simply connected*. Then π is a diffeomorphism which we may consider an isometry and also an equivalence of the s-structures.

It follows that

$$Z(G_e) \subset K \cap G_e$$

and therefore $Z(G_e)$ *acts trivially on* M which, since the action is effective, implies

$$\operatorname{Int}(\mathfrak{g}) \overset{\chi}{\cong} G_e .$$

Then there exists $h \in G_e$ such that

$$\chi(h) = t_0 = \chi(\sigma_a)$$

i.e., $(h^{-1}\sigma_a) \in \operatorname{Ker} \chi$

and so we conclude that there is an element $g_0 \in \operatorname{Ker} \chi$ such that

$$\sigma_a = h g_0 = g_0 h .$$

Now in our space V, where we have the imbedding $M \to V$, we may consider the subspace

$$V_1 = \{v \in V : gv = v \ \forall g \in \langle g_0 \rangle\} ,$$

where $\langle g_0 \rangle$ is the subgroup of $\operatorname{Ker} \chi$ generated by g_0. Since the elements in $\langle g_0 \rangle$ commute with G_e we see that V_1 is G_e-invariant and since g_0 acts trivially on M (because $g_{0*}|_a = \mathrm{id}_{M_a}$) we have $M \subset V_1$ and of course $\sigma_a = h$ in V_1. We see then that our extrinsic k-symmetric submanifold M is one of those considered in the first part of [2]. □

Since by (1.8) we have that θ is outer (not inner) we see that at least one of the θ_i's is outer. On the other hand each M_i is extrinsic k_i-symmetric in V (by changing the θ_s with $s \neq i$) and then we may reduce our considerations to M_i.

Put $M = M_i$, $\theta = \theta_i$, $k = k_i$.

We can consider two cases namely

(1.9) *Every power of θ not equal to the identity is outer.*

(1.10) *There is a $j: 1 < j < k$ such that θ^j is inner and θ^r is outer for $r < j$.*

In the first case [3, *I*, (5.5) (5.10)] implies that either $M = L \times L \times \ldots \times L$ with L a compact, simple, simply connected Lie group or $M = L \times \ldots \times L \times B$ (s factors $L_s \geqq 0$) with L as above and B either $\mathrm{Spin}(8)/G_2$ or $\mathrm{Spin}(8)/(SU(3)/Z_3)$.

In the first case we have $H_3(M, R) \neq 0$ and this is also the case if $s > 0$ in the second situation. If, on the other hand, $s = 0$ and $B = \mathrm{Spin}(8)/(SU(3)/Z_3)$ one has $H_2(M, Z) \cong Z_3$. For $B = \mathrm{Spin}(8)/G_2$ the computation of the homology is more complicated but some odd dimensional integral homology group is non zero.

These facts clearly mean that (1.9) is impossible since it would contradict [2, (8.5)].

Let us consider the case (1.10) with $M = G/K$. Again from [3, *V*, (5,2a), (5,2b)] we have $\mathfrak{g} = \mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_r$, ($\mathfrak{F}_i \cong \mathfrak{F} \ \forall i$) and $\theta(b_1, \ldots, b_r) = (\varphi b_r, b_1, \ldots, b_{r-1})$ where $b_i \in \mathfrak{F}_i$ and φ is an automorphism of order s of $\mathfrak{F}_1$. Then $k = \mathrm{l.c.m.}(r, s)$.

If $r>1$ the situation of (1.10) can only happen if θ^{rt} is inner for some t such that $rt<k$. Here $\mathfrak{g}^{\theta}=\{(b,\dots,b):b\in\mathfrak{F}^{\varphi}\}$ and since $\theta^{rt}(b_1,\dots,b_r)=(\varphi^t b_1,\dots,\varphi^t b_r)$ we have

$$\mathfrak{g}^{\theta^{rt}}=\mathfrak{F}^{\varphi^t}\oplus\dots\oplus\mathfrak{F}^{\varphi^t}\supset\mathfrak{g}^{\theta} \tag{1.11}$$

In this case the j of (1.10) is $j=rt$.

Now, for $y\in M$ put $\tau_y=\sigma_y^j$ [j of (1.10)] and let $V_y=F(\tau_y,V)$ and $N_y=(F(\tau_y,M))_y$ (component of y). N_y is a closed totally geodesic submanifold of M and since, by regularity, for each $z\in N_y$ σ_z and τ_y commute we see that N_y *and* V_y are *σ_z-invariant for each* $z\in N_y$.

Put $\beta_z=(\sigma_z|N_y)$ $\forall z\in N_y$. Clearly β_z is a symmetry of N_y at z with the induced metric and $\{\beta_2:z\in N_y\}$ is a *regular s-structure of order j on N_y*.

It is not hard to see that for each $z\in N_y$, $N_z=N_y$ and $V_z=V_y$ and that $\{(\sigma_z|V_y):z\in N_y\}$ *makes N_y an extrinsic j-symmetric submanifold of V_y*.

Now (1.11) shows that for our chosen point $a\in M$ $\mathrm{Ad}(\sigma_a|V_a)$ is an outer automorphism of $\mathfrak{g}^{\theta^j}$ and so are all its powers except the j-th which is the identity. This transforms the case (1.10) in the (1.9) for $N_a\subset V_a$ and so again this case is not possible.

Final Remark. This facts show that, for odd k, we can only have extrinsic k-symmetric submanifolds in the following situations:

i) $\sigma_a\in G_e$ (which implies $G=G_e$)

ii) $\sigma_a\notin G_e$ but $\mathrm{Ad}(\sigma_a)\in\mathrm{Int}(\mathfrak{g})$.

(In this case we can, by reducing the ambient space, write our manifold M as one of those in (i)).

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