

*Correction***Rationality of Moduli Spaces of Stable Bundles**

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I am grateful to S. Ramanan for pointing out to me an error in my paper [1]. The error occurs on lines 17, 18 of p. 257, where it is assumed that $ng - d$ is coprime to d ; this is of course false if $(g, d) \neq 1$. Since the only restrictions on n and d at this point are

$$(n, d) = 1, \quad n(g - 1) < d < ng,$$

this invalidates the proof of the proposition of [1] and hence also that of the main theorem.

There seems to be no simple way of avoiding this problem, but the methods of [1] do give some positive results, as we shall now show. We adopt the notations and conventions of [1] throughout; in particular L is a line bundle of degree d over the complete non-singular algebraic curve X of genus $g \geq 2$, and $S_{n,L}(X)$ is the moduli space of stable bundles of rank n and determinant L over X . Note that, up to isomorphism, $S_{n,L}(X)$ depends only on the residue class of d modulo n ; moreover, if L^* denotes the dual of L , then $S_{n,L^*}(X)$ is isomorphic to $S_{n,L}(X)$. For any bundle E over $S \times X$ and any $s \in S$, we write E_s for the bundle over X obtained by restricting E to $\{s\} \times X$.

Definition. A good (n, L) -family parametrised by a variety S is a bundle E of rank n over $S \times X$ such that

- (i) $\dim S = (n^2 - 1)(g - 1)$;
- (ii) for all $s, t \in S$, $E_s \cong E_t$ if and only if $s = t$;
- (iii) for all $s \in S$, E_s is stable and $\det E_s \cong L$;
- (iv) for all $s \in S$, $H^1(X; E_s) = 0$;
- (v) for all $s \in S$, the infinitesimal deformation map of E at s is injective.

Remark 1. If $(n, d) = 1$, there exists a bundle U over $S_{n,L}(X) \times X$ with the obvious universal property. We shall say that the pair (n, L) is good if $S_{n,L}(X)$ is a rational variety and there exists a Zariski-open subset T of $S_{n,L}(X)$ such that $U|T \times X$ is a good (n, L) -family. Note that $S_{1,L}(X)$ consists of a single point; it therefore follows trivially from Riemann-Roch that $(1, L)$ is good if $d \geq 2g - 1$. [If $n \geq 2$ and $d > n(g - 1)$, one can show that there exists a Zariski-open subset T of $S_{n,L}(X)$ such that $U|T \times X$ is a good (n, L) -family, but we shall not need to use this fact.]

Remark 2. The argument in the first paragraph of [1, Sect. 4] shows that, if there exists a good (n, L) -family parametrised by S , then S is birationally equivalent to $S_{n,L}(X)$; if S is a rational variety, it follows that (n, L) is good. Note in particular that, by [2, Theorem 2], a good (n, L) -family can exist only if $(n, d) = 1$.

The arguments of [1, Sects. 3, 4] now give

Proposition 1. *Suppose that $n(g-1) < d < ng$ and that there exists a good $(ng-d, L)$ -family E' parametrised by T . Then there exists a good (n, L) -family E parametrised by a Zariski-open subset S of $T \times k^p$ for a suitable integer p ; moreover one can choose S so that, for all $s \in S$, the sections of E_s generate a trivial subbundle of E_s .*

Corollary. *Suppose that $n(g-1) < d < ng$, let K denote the canonical line bundle over X , and let M be a line bundle of degree 1 over X such that $H^0(X; M) \neq 0$. Suppose further that either $(ng-d, L)$ or $(d-n(g-1), L^* \otimes K^n \otimes M^n)$ is good. Then (n, L) is good.*

Proof. In the first case, we apply the proposition with T as in Remark 1 and $E' = U|T \times X$; the result then follows from Remark 2. In the second case, the proposition gives us a good $(n, L^* \otimes K^n \otimes M^n)$ -family E parametrised by a rational variety S ; moreover, from the last part of the proposition,

$$H^0(X; E_s \otimes M^*) = 0 \quad \text{for all } s \in S.$$

It follows easily that $E^* \otimes p_X^*(K \otimes M)$ is a good (n, L) -family; hence (n, L) is good by Remark 2.

This corollary allows us to prove that $S_{n,L}(X)$ is rational for certain values of g, n, d . In particular we have

Proposition 2. *$S_{n,L}(X)$ is rational in the following cases:*

- (a) $d \equiv \pm 1 \pmod{n}$;
- (b) $(n, d) = 1$ and g is a prime power;
- (c) $(n, d) = 1$ and the sum of the two smallest distinct prime factors of g is greater than n .

Proof. The proposition is trivial when $n = 1$; it is therefore sufficient to prove it for $n \geq 2$, $n(g-1) < d < ng$. (*)

In this case (a) follows at once from the corollary and the fact that $(1, L')$ is good if $\deg L' \geq 2g-1$; indeed we have the stronger result that (n, L) is good.

For (b) and (c) we again assume (*) and prove that (n, L) is good by induction on n . The essential point is that the hypotheses imply that either $(ng-d, d) = 1$ or $(d-n(g-1), n(2g-1)-d) = 1$; the inductive step now follows easily from the corollary.

Remark. The Corollary to Proposition 1 applies in many other cases (for example $d \equiv \pm 2, g$ odd); the simplest case to which our results do not apply is when $n = 5$, $d \equiv 2$ or 3 and g is divisible by 6.

References

1. Newstead, P.E.: Rationality of moduli spaces of stable bundles. *Math. Ann.* **215**, 251–268 (1975)
2. Ramanan, S.: The moduli spaces of vector bundles over an algebraic curve. *Math. Ann.* **200**, 69–84 (1973)