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Canonical Variables for the Dirac Theory

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Abstract. A new canonical structure for Dirac's theory is proposed. The new configuration space A is a real, four-dimensional subbundle of the spinor bundle. A Lagrangian defined on Q describes a theory equivalent to the Dirac one. In this way we obtain a theory without second-type constraints.

1. Introduction

A new variational formulation for the theory of the bispinor field (Dirac theory) is proposed in this Letter.

The problem of canonical formulation for Dirac's theory is essential for describing the interaction between gravitational and spinor fields. Usually, the Lagrangian L, which describes the theory of the bispinor field, is a linear function of the first derivatives of bispinor variables; cf., [6]. For example, in the Minkowski space, this Lagrangian has a form

$$L = \frac{i}{2} (\overline{\Psi}\gamma^{\mu}\partial_{\mu}\Psi - \partial_{\mu}\overline{\Psi}\gamma^{\mu}\Psi) - m\overline{\Psi}\Psi . \tag{1}$$

The linearity of L implies the following Hamiltonian constraints

$$p^{\mu} := \frac{\partial L}{\partial \overline{\Psi}_{\mu}} = -\frac{i}{2}\gamma^{\mu}\Psi .$$

These are constraints of the second type in Dirac's classification [3]. The quantization of the theory with second-type constraints is obtained by a rather heavy procedure. However, it seems that such constraints result from a false recognition of the canonical structure.

Let us illustrate the problem by the following simple example taken from classical particle mechanics. Our aim will be to show the difference between the standard approach to Dirac's theory and a new procedure proposed later.

We consider a mechanical system with a configuration space $\mathbb{C}^1$ (the variable $z \in \mathbb{C}^1$ is an analog of the bispinor field $\Psi \in \mathbb{C}^4$). The evolution equations are described by a Lagrangian L, which corresponds to the standard Dirac Lagrangian (1):

$$L := \frac{i}{2} (\bar{z}\dot{z} - \dot{\bar{z}}z) - \bar{z}z \tag{2}$$

($\bar{z}$ is a complex conjugate to z and $\dot{z}$ is a time derivative of z).

A complex momentum p is now given by the formula

$$p := \frac{\partial L}{\partial \dot{z}} = -\frac{i}{2} z \tag{3}$$

and the motion equation is given by $i\dot{z} - z = 0$.

In real coordinates

$$z = \frac{1}{\sqrt{2}} (x + iy) ; \qquad p = \frac{1}{\sqrt{2}} (p_x + ip_y) ,$$

formula (3) reads

$$p_x - \tfrac{1}{2}y = 0 , \qquad p_y + \tfrac{1}{2}x = 0 .$$

We deal with second-type constraints since the Poisson bracket does not vanish:

$$\{p_y + \tfrac{1}{2}x, p_x - \tfrac{1}{2}y\} = 1 .$$

The equations of motion are

$$\dot{x} - y = 0 , \qquad \dot{y} + x = 0$$

and, thus, we obtain the following real equation of motion, $\ddot{x} + x = 0$. The theory described by Lagrangian (2) is, therefore, a theory of the harmonic oscillator. It admits a simple formulation starting with $x = \mathrm{Re}\, z$ as a configuration and $y = \mathrm{Im}\, z$ as a momentum conjugate to x. To explain the role of the unusual Lagrangian (2) let us rewrite it in terms of x and y:

$$L(x, y, \dot{x}, \dot{y}) = \tfrac{1}{2}(y\dot{x} - \dot{y}x - x^2 - y^2) = y\dot{x} - \tfrac{1}{2}[x^2 + y^2 + (xy)^{\bullet}] .$$

We see that up to a complete time derivative, Lagrangian (2) is equivalent to

$$\tilde{L}(x, y, \dot{x}, \dot{y}) = y\dot{x} - \tfrac{1}{2}(x^2 + y^2) = y\dot{x} - H(x, y) .$$

The numerical value of the latter is precisely equal to that of the standard Lagrangian for the harmonic oscillator which we obtained from the Hamiltonian H via the Legendre transformation. The next step of the Legendre transformation consists of expressing momentum y in terms of velocity $\dot{x}$. This way we obtain the standard Lagrangian for the harmonic oscillator. To pass from two degrees of freedom (z or (x, y)) to one degree of freedom (x) is an important simplification. The price we must pay for this simplification is that the symmetry of the first Lagrangian (2) with respect to the transformation

$$z' = z\, e^{-i\varphi} ; \quad \varphi \in R^1$$

cannot be described on a Lagrangian level. This is essentially the Hamiltonian symmetry and not a Noether symmetry.

The aim of this Letter is to show that an analogical simplification of the canonical structure is also possible in Dirac's theory. One obtains the theory without any of the second-type constraints. The real and imaginary parts (in the Majorana representation) of a bispinor Ψ are interpreted as new canonical variables. The advantage of such a division has been pointed out by J. Schwinger [9], who also analyzed some physical interpretations of this trick. However, the approach presented here gives a new insight into Schwinger's ideas. The division of variables into 'configurations' and 'momenta' enables a variational formulation of Dirac's theory. It is interesting to note that due to the linearity assumption, one obtains a unique (up to gauge) division of Ψ.

2. Notation

Let us denote by (M, g) the spacetime – a four-dimensional, smooth manifold M with a metric tensor g. The signature of g is $(+, -, -, -)$. Nonholonomic (tetrad) coordinates are denoted by $a, b, c, \ldots$ and the holonomic coordinates by $\lambda, \mu, \nu, \ldots$. $\{e_a\}_0^3$ is a field of orthonormal tetrads on M. Partial differentiation is denoted by ∂ and covariant differentiation by ∇. A section of the bispinor bundle S (with the fibre isomorphic $\mathbb{C}^4$) is called a bispinor field Ψ. Dirac matrices are denoted by γ^a and $\gamma^{ab} := \frac{1}{2}(\gamma^a\gamma^b - \gamma^b\gamma^a)$. A complex conjugate of h is denoted by h^*, a transposition of h by h^t, $h^+ := h^{*t}$ is a Hermitian conjugate of h. Matrices A, B, C are assumed to satisfy the conditions

$$\begin{aligned} &A\gamma^a A^{-1} = \gamma^{a+} \quad &&\text{and} \quad A = A^+ \,, \\ &B\gamma^a B^{-1} = -\gamma^{at} \quad &&\text{and} \quad B = -B^t \,, \\ &C^{-1}\gamma^a C = -\gamma^{a*} \quad &&\text{and} \quad C^{-1} = C^* \end{aligned} \tag{4}$$

(compare [1]).

We work in the system of units where $\hbar = c = 1$ and the Einstein summations convention is used everywhere.

3. Construction of Configurations Variables

The Dirac Lagrangian L is a function of a bispinor field and its derivatives $\Psi_\mu := \partial_\mu\Psi$. We use the following notation derived from the theory of analytical functions

$$\mathrm{d}L =: \frac{\partial L}{\partial\Psi}\,\mathrm{d}\Psi + \mathrm{d}\overline{\Psi}\,\frac{\partial L}{\partial\overline{\Psi}} + \frac{\partial L}{\partial\Psi_\mu}\,\mathrm{d}\Psi_\mu + \mathrm{d}\overline{\Psi}_\mu\,\frac{\partial L}{\partial\overline{\Psi}_\mu}\,,$$

where, as usual, $\overline{\Psi} := \Psi^+ A$. More precisely, we should write

$$\sum_A \frac{\partial L}{\partial\Psi_A}\,\mathrm{d}\Psi_A$$

(where Ψ_A are coordinates of Ψ) instead of $(\partial L/\partial\Psi)\,\mathrm{d}\Psi$.

We shall follow the formalism of symplectic relations proposed in [7]. If $\theta^\mu := \bar{p}^\mu\,\mathrm{d}\Psi + \mathrm{d}\overline{\Psi}p^\mu$ then the equation $\mathrm{d}L = \partial_\mu\theta^\mu$ is equivalent to the Euler–Lagrange equations

$$\begin{aligned} p^{\mu} &:= \frac{\partial L}{\partial \overline{\Psi}_{\mu}}, \qquad \bar{p}^{\mu} := \frac{\partial L}{\partial \Psi_{\mu}} \\ \partial_{\mu} p^{\mu} &= \frac{\partial L}{\partial \overline{\Psi}}, \qquad \partial_{\mu} \bar{p}^{\mu} = \frac{\partial L}{\partial \Psi}. \end{aligned} \tag{5}$$

Following [7], we define a symplectic structure in the space of Cauchy data; i.e., the space of field sections (Ψ, p^{μ}) on a fixed hypersurface Σ:

$$\omega^{\Sigma} := \int_{\Sigma} \mathrm{d}\theta^{\mu} \otimes \mathrm{d}n_{\mu},$$

where we denote by $\mathrm{d}n_{\mu}$ an oriented surface element on Σ. As usual, any physical quantity f on Σ determines a vector field X_f by the equality $\mathrm{d}f =: X_f \lrcorner \omega^{\Sigma}$. The Poisson bracket of two smooth quantities f, g on Σ is given by

$$\{f, g\}_{\Sigma} := X_f(g). \tag{6}$$

The explicit form of the Lagrangian L for Dirac's theory is

$$L = \tfrac{1}{2}\sqrt{-g}\,(\overline{\Psi}\gamma^{\mu}\nabla_{\mu}\Psi - \overline{\nabla_{\mu}\Psi}\gamma^{\mu}\Psi) - m\overline{\Psi}\Psi$$

where $\gamma^{\mu} := e^{\mu}_{a}\gamma^{a}$ and $\nabla_{\mu}\Psi$ denotes a covariant derivative in the spinor bundle S:

$$\nabla_{\mu}\Psi := \partial_{\mu}\Psi - \omega_{ab\mu}\gamma^{ab}\Psi, \qquad \omega_{ab\mu} := \tfrac{1}{4}g(\nabla_{\mu}e_{a}, e_{b}),$$

where ∇_{μ} denotes a metric connection on M; cf. [2]. The Euler–Lagrange equations (5) now have the form

$$p^{\mu} = -\frac{i}{2}\sqrt{-g}\,\gamma^{\mu}\Psi \tag{7a}$$

$$\partial_{\mu}p^{\mu} = \sqrt{-g}\left(\frac{i}{2}\gamma^{\mu}\nabla_{\mu}\Psi - m\Psi\right). \tag{7b}$$

Hence we obtain the Dirac equation

$$i\gamma^{\mu}\nabla_{\mu}\Psi - m\Psi = 0. \tag{8}$$

Formula (7a) defines second-type constraints in a phase space; see [5]. Notice that on a constraints space we have

$$\omega^{\mu} := \mathrm{d}\theta^{\mu} = i\sqrt{-g}\,\mathrm{d}\overline{\Psi}\gamma^{\mu} \wedge \mathrm{d}\Psi. \tag{9}$$

Let us consider the charge conjugation operator $\mathscr{C}(\Psi) := C\Psi^{*}$ (C is defined by formula (4)) and its eigenspaces corresponding to ± 1 eigenvalue

$$C_{\pm} := \{\Psi \in S;\ \mathscr{C}(\Psi) = \pm\Psi\}.$$

A spinor bundle S is the sum of subbundles C_+ and C_-. Now

$$\Psi = q + ip \tag{10}$$

where

$$q := \frac{1}{2}(\Psi + \mathscr{C}(\Psi)), \qquad p := \frac{1}{2i}(\Psi - \mathscr{C}(\Psi)),$$

and formula (9) reads as

$$\omega^\mu = 2\sqrt{-g}\, \mathrm{d}p^t B\gamma^\mu \wedge \mathrm{d}q \tag{11}$$

and, therefore, the Poisson brackets of the q variables vanish on every hypersurface Σ in M: $\{q_A, q_B\}_\Sigma = 0$. This observation motivates the choice of $Q := C_+$ (the space parametrized by variables $q = (q_A)$) as a new configuration space.

The new momenta p^μ can be found from the canonical form of ω^μ on Q, i.e.,

$$\omega^\mu = \mathrm{d}(p^{\mu t}\,\mathrm{d}q) = \mathrm{d}p^{\mu t} \wedge \mathrm{d}q\,.$$

So, we have the equality

$$\mathrm{d}p^{\mu t} \wedge \mathrm{d}q = 2\sqrt{-g}\,\mathrm{d}p^t B\gamma^\mu \wedge \mathrm{d}q\,.$$

This identity implies that new momenta p^μ conjugate to configurations q are equal,

$$p^\mu = 2\sqrt{-g}\, B\gamma^\mu p\,. \tag{12}$$

Hence, variables p parametrize the space of momenta.

4. The Noninteracting, Half-spin Particle Case

It is seen after simple computations that the Dirac equation (8) in (q, p) variables (i.e., when Ψ is given by (10)) has the form

$$i\gamma^\mu \nabla_\mu q - mq = 0\,, \tag{13a}$$

$$i\gamma^\mu \nabla_\mu p - mp = 0\,. \tag{13b}$$

Equation (13a) implies that the new theory based on q variables as configurations has Lagrangian constraints.

THEOREM 1. *The theory of the noninteracting half-spin particle is a theory obtained from the Lagrangian which vanishes on the Lagrangian constraints* (13a). *By using the multipliers p, L can be written as follows*

$$L = 2\sqrt{-g}\, p^t B(\gamma^\mu \nabla_\mu q + imq)\,. \tag{14}$$

A variation of L on constraints (13a) *gives* (13b). *A bispinor Ψ describing the particle is now given by* (10).

Proof. It is sufficient to show that a variation of Lagrangian (14) leads to (13b), but notice that

$$\frac{1}{2}\frac{\delta L}{\delta q} = \sqrt{-g}\, p^t B(im + \gamma^\mu \gamma^{ab} \omega_{ab\mu}) - \partial_\mu(\sqrt{-g}\, p^t B \gamma^\mu) = 0\,.$$

This easily implies that

$$\nabla_\mu(\sqrt{-g}\, \gamma^\mu p) + \sqrt{-g}\, imp = 0\,.$$

Due to $\nabla \gamma^\mu = 0$ (see [2]) we have $\gamma^\mu \nabla_\mu p + imp = 0$. □

5. The Half-spin, Charged Particle Case

A bispinor field Ψ which describes a half-spin, charged particle fulfils the equation

$$i(\gamma^\mu \nabla_\mu \Psi + ieA_\mu \gamma^\mu \Psi) = m\Psi\,, \tag{15}$$

where e denotes the electric charge of a particle and (A_μ) is the potential of an external electromagnetic field. The above equation in (q, p) variables takes the form

$$\gamma^\mu \nabla_\mu q - eA_\mu \gamma^\mu p + imq = 0 \tag{16a}$$

$$\gamma^\mu \nabla_\mu p + eA_\mu \gamma^\mu q + imp = 0\,. \tag{16b}$$

Let us denote

$$\tilde{p}(q) := \frac{eA_\mu \gamma^\mu}{A^2}\,(\gamma^\nu \nabla_\nu q + imq) \tag{17}$$

where we have assumed that $A^2 := g^{\mu\nu} A_\mu A_\nu \neq 0$.

It is implied by (16a) that in an electromagnetic gauge with $A^2 = 0$, one deals with the Lagrangian constraints

$$A_\mu \gamma^\mu(\gamma^\nu \nabla_\nu q + imq) = 0\,.$$

Correspondingly, the momentum conjugate to q is not uniquely determined by q and its derivatives.

THEOREM 2. *A theory of the charged, half-spin particle interacting with an external electromagnetic field can be obtained from the Lagrangian*

$$L_s = -e\sqrt{-g}\, A_\nu \left[q^t B \gamma^\nu q + \frac{1}{A^2}\,(\gamma^\lambda \nabla_\lambda q + imq)^t B \gamma^\nu (\gamma^\mu \nabla_\mu q + imq) \right]. \tag{18}$$

A variation of this Lagrangian with respect to q variables leads to four (real) second-rank equations

$$\gamma^\mu \nabla_\mu \tilde{p} + eA_\mu \gamma^\mu q + im\tilde{p} = 0\,. \tag{19}$$

If q satisfies (19), *then a bispinor* $\Psi := q + i\tilde{p}(q)$ *describes the particle, i.e., satisfies Equation* (15).

Proof. We have

$$p^\mu = \frac{\partial L_s}{\partial q^t_\mu} = -2\,\frac{e}{A^2}\,\sqrt{-g}\,A_\nu \gamma^{\mu t} B \gamma^\nu (\gamma^\lambda \nabla_\lambda q + imq)$$

$$= 2\,\frac{e}{A^2}\,\sqrt{-g}\,A_\nu B \gamma^\mu \gamma^\nu (\gamma^\lambda \nabla_\lambda q + imq) = 2\sqrt{-g}\,B\gamma^\mu \tilde{p}$$

(compare formula (12)) and

$$\partial_\mu p^\mu = \frac{\partial L_s}{\partial q^t}$$

$$= -2e\sqrt{-g}\,A_\nu\left\{B\gamma^\nu q + \frac{1}{A^2}[-\omega_{ab\mu}(\gamma^\mu\gamma^{ab})^t + im]B(\gamma^\lambda\nabla_\lambda q + imq)\right\}$$

$$= -2\sqrt{-g}\,B(eA_\nu\gamma^\nu q - \omega_{ab\mu}\gamma^{ab}\gamma^\mu\tilde{p} + im\tilde{p})\,.$$

Hence,

$$\nabla_\mu(\sqrt{-g}\,\gamma^\mu\tilde{p}) + \sqrt{-g}(eA_\mu\gamma^\mu q - \omega_{ab\mu}\gamma^{ab}\gamma^\mu\tilde{p} + im\tilde{p}) = 0\,,$$

which easily transforms to

$$\nabla_\mu(\sqrt{-g}\,\gamma^\mu\tilde{p}) + \sqrt{-g}(eA_\nu\gamma^\nu q + im\tilde{p}) = 0$$

and

$$\gamma^\mu\nabla_\mu\tilde{p} + eA_\mu\gamma^\mu q + im\tilde{p} = 0\,.$$

After observing that definition (17) is equivalent to

$$\gamma^\mu\nabla_\mu q + imq - eA_\mu\gamma^\mu\tilde{p} = 0\,,$$

we see that the bispinor $\Psi = q + i\tilde{p}$ fulfils Equation (16) and, thus, also (15). □

As is known, Lagrangian (1) and Equation (15) are invariant with respect to the electromagnetic gauge

$$\Psi' = \Psi\,\mathrm{e}^{-i\varphi}\,, \qquad A'_\mu = A_\mu + e\,\partial_\mu\varphi\,.$$

The above transformation has the following form in (q, p) variables

$$q' = q\cos\varphi + p\sin\varphi\,, \qquad p' = -q\sin\varphi + p\cos\varphi\,,$$

and A_μ transforms as above.

Of course, Equations (16) are invariant with respect to this transformation, but now it is a transformation of the entire phase space and not of the configuration space. It is possible to consider this transformation as being reduced to the velocity space or to exactly the space of the jets of q:

$$q' = q\cos\varphi + \tilde{p}\sin\varphi\,, \qquad \tilde{p}' = e\,\frac{A'_\mu\gamma^\mu}{A'^2}\,(\gamma^\nu\nabla_\nu q' + imq')\,, \qquad A'_\mu = A_\mu + e\,\partial_\mu\varphi\,.$$

But now Lagrangian (18) is not invariant with respect to such a transformation.

This situation is similar to that in the mechanical example from the Introduction. The Hamiltonian theory of the oscillator is invariant with respect to gauge of the first type:

$$x' = x\cos\varphi + y\sin\varphi,$$

$$y' = -x\sin\varphi + y\cos\varphi.$$

Of course, we have $H(x', y') = H(x, y)$.

It is obvious that the standard Lagrangian theory of the harmonic oscillator is not invariant with respect to the gauge

$$x' = x\cos\varphi + \dot{x}\sin\varphi,$$

but the invariance of the Hamiltonian theory implies an invariance of the Lagrangian theory with the unusual Lagrangian (2), which is defined on the phase space.

The invariance of the standard Dirac theory in the Lagrangian approach is closely connected with the invariance of the new formulation of Dirac's theory in the Hamilton approach. The standard Dirac Lagrangian is more a Hamiltonian than a Lagrangian.

6. The Theory of the Self-interacting Dirac and Maxwell Fields Case

Take the following equations

$$i\gamma^\mu(\nabla_\mu + ieA_\mu)\Psi = m\Psi \tag{20a}$$

$$\partial_\mu(\sqrt{-g}\, g^{\mu\lambda}g^{\nu\sigma}f_{\lambda\sigma}) = e\sqrt{-g}\,\overline{\Psi}\gamma^\nu\Psi \tag{20b}$$

where $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

THEOREM 3. *The Lagrangian*

$$L_{se} := L_s + L_e, \tag{21}$$

where L_s *is given by* (18) *and* $L_e = -\frac{1}{4}\sqrt{-g}\, g^{\mu\nu}g^{\lambda\sigma}f_{\mu\lambda}f_{\nu\sigma}$, *leads to the following equations*

$$\gamma^\mu\nabla_\mu\bar{p} + eA_\mu\gamma^\mu q + im\bar{p} = 0, \tag{22a}$$

$$\partial_\mu(\sqrt{-g}\, g^{\mu\lambda}g^{\nu\sigma}f_{\lambda\sigma}) = e\sqrt{-g}(q^t B\gamma^\nu q + \bar{p}^t B\gamma^\nu\bar{p}). \tag{22b}$$

The above equations are equivalent to (20) *with* $\Psi := q + i\bar{p}$.

Proof. Due to Theorem 2, Equation (22a) is satisfied. Let us check to see if

$$\partial_\mu(\sqrt{-g}\, g^{\mu\lambda}g^{\nu\sigma}f_{\lambda\sigma}) = -\frac{\partial L_{se}}{\partial A_\nu} = e\sqrt{-g}(q^t B\gamma^\nu q + \bar{p}^t B\gamma^\nu\bar{p}).$$

Denote $K := \gamma^\mu\nabla_\mu q + imq$ and notice that

$$-\frac{\partial L_{se}}{\partial A_\nu} = e\sqrt{-g}\left(q^t B\gamma^\nu q + \frac{1}{A^2}K^t B\gamma^\nu K - \frac{2}{A^4}A^\nu A_\mu K^t B\gamma^\mu K\right)$$

and

$$\bar{p}^t B\gamma^\nu \bar{p} = -\frac{1}{A^4} A_\lambda A_\sigma K^t B\gamma^\lambda \gamma^\nu \gamma^\sigma K = -\frac{2}{A^4} A_\lambda A^\nu K^t B\gamma^\lambda K + \frac{1}{A^2} K^t B\gamma^\nu K .$$

Thus, (22b) holds.

From the equality $\overline{\Psi} = (q^t - i\bar{p}^t)B$, we have

$$\overline{\Psi}\gamma^\nu\Psi = q^t B\gamma^\nu q + \bar{p}^t B\gamma^\nu \bar{p} + i(q^t B\gamma^\nu \bar{p} - \bar{p}^t B\gamma^\nu q) .$$

But

$$q^t B\gamma^\nu \bar{p} - \bar{p}^t B\gamma^\nu q = 0$$

and, hence,

$$j^\nu = e\sqrt{-g}\,\overline{\Psi}\gamma^\nu\Psi = e\sqrt{-g}\,(q^t B\gamma^\nu q + \bar{p}^t B\gamma^\nu \bar{p}) .$$

This identity shows that Equations (20b) and (22b) are equivalent. □

In this way we again obtained the theory of bispinor fields without second-type constraints.

7. Uniqueness of the Construction

Now the following problem arises: does there exist a possibility of choosing another configuration space leading to a bispinor theory without second-type constraints? If so, then the new configuration variables (q'_A) should have vanishing Poisson brackets on every hypersurface Σ in M:

$$\{q'_A, q'_B\}_\Sigma = 0 .$$

But from formula (6) we have

$$\begin{aligned} 0 = \{q'_A, q'_B\}_\Sigma &= \langle X_{q'_A}; \mathrm{d}q'_B \rangle \\ &= \langle X_{q'_A}; X_{q'_B} \lrcorner\, \omega^\Sigma \rangle = \langle X_{q'_B}, X_{q'_A}; \omega^\Sigma \rangle . \end{aligned}$$

The equality means that vectors $\{X_{q'_A}\}_1^4$ form a four-dimensional, isotropic subspace for all forms $\{\omega^\mu\}_0^3$:

$$\forall A, B \quad \forall \mu \quad \langle X_{q'_A}, X_{q'_B}; \omega^\mu \rangle = 0$$

where $\{\omega^\mu\}_0^3$ are given by formula (11).

LEMMA 1. *The intersection of the isotropic subspaces of* $\{\omega^\mu\}_0^3$ *is spanned by vectors*

$$X_{q'_A} = -\sin\varphi\,\frac{\partial}{\partial q_A} + \cos\varphi\,\frac{\partial}{\partial p_A}\,, \quad A = 1, 2, 3, 4$$

Proof. The proof will be given in the Majorana representation (see [1]). In this representation bispinors q and p are real; $q, p \in R^4$. Let us denote by $\{X_A\}$ linearly independent vectors

$$X_A = \alpha_A \frac{\partial}{\partial q} + \beta_A \frac{\partial}{\partial p}, \quad A = 1, 2, 3, 4,$$

where $\alpha_A, \beta_A \in R^4$.

We shall look for X_A satisfying the condition

$$\forall A, B \quad \forall a \quad \langle X_A, X_B; \omega^a \rangle = 0 .$$

Hence,

$$\forall A, B \quad \forall a \quad \alpha_A^t B \gamma^a \beta_B - \alpha_B^t B \gamma^a \beta_A = 0 . \tag{23}$$

Let us assume that bispinors α_1, α_2 are linearly independent. One easily checks that condition (23) is SU(2)-invariant. Hence, the bispinors $(g \cdot \alpha_A, g \cdot \beta_A)$ (where $g \in \mathrm{SU}(2)$) also satisfy (23). Thus, we can work with

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \alpha_2 = \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix}, \quad \text{where} \quad a^2 + b^2 + c^2 \neq 0 .$$

If α_3 is now linearly dependent from α_1, α_2 we can put $\alpha_3 = 0$ without loss of generality.

Now, from (23) we have

$$\forall a \quad \alpha_1^t B \gamma^a \beta_3 = 0 \quad \text{and} \quad \alpha_2^t B \gamma^a \beta_3 = 0 .$$

Hence β_3 vanishes and also the vector $X_3 = 0$. We obtain a contradiction, hence bispinors α_1, α_2, α_3 are linearly independent. Similarly, we can show that α_4 is linearly independent from α_1, α_2, α_3 and thus we can choose these bispinors in the form

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

After easy computations, we finally obtain from (23)

$$\exists \mu \in R, \quad \forall A \quad \beta_A = \mu \alpha_A .$$

This completes the proof. □

Dirac's theory with a bispinor Ψ as a configuration is linear with respect to the bispinor field Ψ. If we do not want to lose the linearity of the new theories, we must assume that new variables q' are linearly dependent on the bispinor Ψ. Under this assumption we have

THEOREM 4. *The only choice of new configurations is given by*

$$q' = g \cdot (q \cos \varphi + p \sin \varphi)\,, \quad \textit{where } \varphi \in R,\ g \in \mathrm{Gl}(4, R)\,.$$

Proof. New configuration variables

$$q'_A = \sum_B (\lambda_A^B q_B + \mu_A^B p_B)$$

should vanish on vectors $\{X_B\}_1^4$:

$$\forall A, B \quad X_B(q'_A) = 0\,.$$

The only solutions of these equations are given in the thesis of the theorem. □

It can be seen that the only nontrivial freedom choice for new configurations

$$q' = q \cos \varphi + p \sin \varphi \tag{24}$$

is related to the following bispinor field gauge

$$\Psi' = \Psi\, e^{-i\varphi}\,.$$

In Dirac's theory without electromagnetic interactions variable $\varphi \in R^1$ is constant and in the Dirac–Maxwell theory variable φ is a real function on spacetime M. Simultaneously with the change of configurations given by (24), the electromagnetic potential must be changed

$$A'_\mu = A_\mu + e\, \partial_\mu \varphi\,.$$

8. Remarks

1. The variable q may be represented by a spinor field $\varkappa \in \mathbb{C}^2$.

It is known (see [8]) that from a geometrical point of view a spinor field $\varkappa$ can be described as a simple, null two-vector $X^{\mu\nu} \in \Lambda^2 TM$) called a flag and it is possible to express Lagrangian (21) only in terms of the flag ($X^{\mu\nu}$) and potential (A_μ). A formulation of the Maxwell–Dirac theory in this language will be published elsewhere.

2. The problem of describing the interaction between gravitational, spinor, and electromagnetic fields in the framework of the unified theory of gravitation and electromagnetism (compare [4]) is investigated in a forthcoming paper. In the case of the metric, torsionless connection, the proposed theory is equivalent to the standard one.

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