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3D Derivations of Static Plate Theories

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Synonyms

Dimensional reduction; First-order shear deformation theory; Plate model or theory; Reissner or Hencky or Reissner-Mindlin thick plate theory

Definitions

Thin plate model: A model where the only kinematic d.o.f. is the transverse deflection. It neglects the shear energy.

Thick plate model: A model including also two in-plane rotation d.o.f. and including shear deflection.

Introduction

Plates are three-dimensional structures with a small dimension compared to the other two dimensions. Numerous approaches were suggested in order to replace the three-dimensional problem by a two-dimensional problem while guaranteeing the accuracy of the reconstructed three-dimensional fields. Turning the 3D problem into a 2D plate model is known as dimensional reduction.

The approaches for deriving a plate model from 3D elasticity may be separated in two main categories: axiomatic and asymptotic approaches. Axiomatic approaches start with ad hoc assumptions on the 3D field representation of the plate, separating the out-of-plane coordinate from the in-plane coordinates. The limitation of these approaches comes from the educated guess for the 3D field distribution. Asymptotic approaches come often after axiomatic approaches. They are based on the explicit introduction of the plate thickness, which is assumed to go to 0, in the equations of the 3D problem. Following a rather well-established procedure, they enable the derivation of plate models, often justifying a posteriori axiomatic approaches, and are the basis of a convergence result.

The very first and simplest model is the Kirchhoff-Love plate model or thin-plate model (Kirchhoff 1850; Love 1888), where the out-of-plane deflection is the only kinematic degree of freedom. In this model, it is assumed that the fiber normal to the plate mid-surface remains normal during the motion. In order to take into account the influence of shear energy on the deflection, several thick plate models were suggested almost simultaneously (Reissner 1944; Hencky 1947; Bollé 1947). In these models, gathered here under the common denomination Reissner-Hencky models, two in-plane rotations are added

H. Altenbach, A. Öchsner (eds.), *Encyclopedia of Continuum Mechanics*,
https://doi.org/10.1007/978-3-662-53605-6_132-1

to the kinematics. Note that the denomination Reissner-Mindlin is also very common in the literature. It comes from Mindlin's contribution based on dynamic considerations (Mindlin 1951). Whereas all these models were historically derived axiomatically, they also have close relations with asymptotic considerations.

This chapter is dedicated to the case of a homogeneous and linear elastic plate with static loading which was the foundation of many extensions to heterogeneous plates. It recalls in detail the derivation of the thick plate model from Hencky (1947) as well as the one from Reissner (1944). Both approaches are related but yield different plate models. This choice is motivated by the following considerations. First, Reissner-Hencky models are the most widely used plate models in engineering applications. Indeed, their boundary conditions seem more natural than those of the Kirchhoff-Love plate model. They also relax the higher regularity of the Kirchhoff-Love displacement required for finite elements implementations. Second, the Kirchhoff-Love model may be directly retrieved from these models by means of the Kirchhoff kinematic restriction ▸ "Direct Derivation of Plate Theories".

Two modifications are made with respect to the historical contributions. First, the membrane model is also included in the present derivation at very little price. Second, the applied load is a body force uniformly distributed through the thickness instead of a force per unit surface applied only on the upper face of the plate. This choice leads to a more compact derivation and removes a higher-order coupling between the membrane and bending problems widely ignored in the historical literature. Finally, all mathematical developments are purely formal and the reader is referred to (▸ "Mathematical Justification of Plate Models" and Ciarlet 1997) for rigorous justifications.

The 3D Problem

The plate is the cylindrical body $\Omega = \omega \times \mathcal{T}$ where ω denotes the midplane surface of the plate and $\mathcal{T} = \left[-\frac{h}{2}, \frac{h}{2}\right]$ is the transverse coordinate range. The boundary, $\partial\Omega$, is decomposed into three parts (Fig. 1):

$$\partial\Omega = \partial\Omega_{\text{lat}} \cup \partial\Omega_3^+ \cup \partial\Omega_3^-, \quad (1)$$

with:

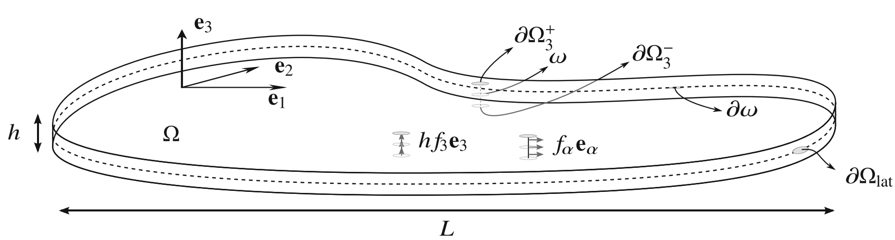
$$\partial\Omega_{\text{lat}} = \partial\omega \times \mathcal{T} \text{ and } \partial\Omega_3^{\pm} = \omega \times \left\{\pm\frac{h}{2}\right\}. \quad (2)$$

It is assumed that the plate follows a prescribed displacement $\mathbf{u}^d$ on its lateral boundary, $\partial\Omega_{\text{lat}}$, and is subjected to body forces $\mathbf{f}(\mathbf{x})$ in Ω of the form:

$$\mathbf{f} = (f_1(x_1, x_2), f_2(x_1, x_2), hf_3(x_1, x_2)). \quad (3)$$

In the third component of the body force, the thickness is factored out in order to follow the usual scaling of applied forces when the thickness goes to 0 and will be motivated in the following. It is assumed that the fourth-order stiffness tensor $\mathbf{C}$ characterizing the elastic properties of the constituent material at every point $\mathbf{x} = (x_1, x_2, x_3)$ of Ω is uniform on the whole body. The tensor $\mathbf{C}$ follows the classical minor and major symmetries of linear elasticity and is positive definite. In addition, monoclinic symmetry with respect to a plan of normal $\mathbf{e}_3$ is assumed:

$$C_{3\alpha\beta\gamma} = C_{\alpha 333} = 0, \quad (4)$$



3D Derivations of Static Plate Theories, Fig. 1 The 3D problem for a homogeneous plate where it is recalled that for Greek indices α, β, $\gamma \ldots = 1, 2$ (in-plane components), while for Latin indices $i, j, k \ldots = 1, 2, 3$ (in-plane and out-of-plane components).

Thus, the constitutive equation writes:

$$\begin{cases} \sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\varepsilon_{\delta\gamma} + C_{\alpha\beta 33}\varepsilon_{33}, \\ \sigma_{\alpha 3} = 2C_{\alpha 3\beta 3}\varepsilon_{3\beta}, \\ \sigma_{33} = C_{33\alpha\beta}\varepsilon_{\beta\alpha} + C_{3333}\sigma_{33}, \end{cases} \tag{5}$$

or conversely,

$$\begin{cases} \varepsilon_{\alpha\beta} = S_{\alpha\beta\gamma\delta}\sigma_{\delta\gamma} + S_{\alpha\beta 33}\sigma_{33}, \\ \varepsilon_{\alpha 3} = 2S_{\alpha 3\beta 3}\sigma_{3\beta}, \\ \varepsilon_{33} = S_{33\alpha\beta}\sigma_{\beta\alpha} + S_{3333}\sigma_{33}, \end{cases} \tag{6}$$

where $\sigma = (\sigma_{ij})$ is the stress tensor, $\varepsilon = (\varepsilon_{ij})$ is the strain tensor, and $\mathbf{S}$ is the inverse of $\mathbf{C}$ and has the same properties (4) as the tensor $\mathbf{C}$. In the above equation and in the remainder of this chapter, Einstein's summation convention is used.

The full 3D linear elastic problem, $\mathcal{P}^{3D}$, is to find in Ω a displacement vector field $\mathbf{u}^{3D}$, a strain tensor field ε^{3D}, and a stress tensor field σ^{3D} such that the static conditions:

$$SC^{3D}: \begin{cases} \sigma_{\alpha j,j} + f_\alpha = 0 \quad \text{on} \quad \Omega, & (7) \\ \sigma_{3j,j} + hf_3 = 0 \quad \text{on} \quad \Omega, & (8) \\ \sigma_{i3} = 0 \quad \text{on} \quad \partial\Omega_3^{\pm}, & (9) \end{cases}$$

for regular enough σ, the kinematic conditions:

$$KC^{3D}: \begin{cases} 2\varepsilon_{ij} = u_{i,j} + u_{j,i} \quad \text{on} \quad \Omega, & (10) \\ u_i = u_i^d \quad \text{on} \quad \partial\Omega_{\text{lat}}, & (11) \end{cases}$$

for regular enough $\mathbf{u}$, and the constitutive law (5) are satisfied.

Kinematic Derivation of Hencky's Plate Model

In this section, the kinematic derivation of a thick plate model from Hencky (1947) and Bollé (1947) is presented. It delivers the correct plate generalized variables. However, the constitutive equations are incorrect. It starts with the assumption of a 3D kinematically compatible displacement field. The plate model is derived from the application of the minimum potential energy principle.

Plate Kinematics

The following 3D kinematics is assumed for the plate:

$$\begin{aligned} u_\alpha^H(x_i) &= U_\alpha(x_\eta) + x_3\phi_\alpha(x_\eta) \quad \text{and} \\ u_3^H(x_i) &= U_3(x_\eta). \end{aligned} \tag{12}$$

Here, U_α is the membrane in-plane displacement, U_3 is the out-of-plane displacement, and ϕ_α is the material inclination of the fiber normal to the midplane of the plate. The corresponding in-plane rotation vector is $\boldsymbol{\theta}$, where $\theta_1 = -\phi_2$ and $\theta_2 = \phi_1$.

With proper scaling, it may be demonstrated that this kinematics is related to the asymptotic expansion of the 3D displacement solution of $\mathcal{P}^{3D}$ with respect to the thickness of the plate h. The membrane displacement and the out-of-plane displacement are the leading-order terms of the expansion. The material rotation is related to the next-order term of the expansion (Ciarlet and Destuynder 1979).

As a consequence of this choice of kinematics, it must be assumed in this section that the prescribed 3D displacement $\mathbf{u}^d$ on the boundary is as follows:

$$\begin{aligned} u_\alpha^d &= U_\alpha^d + x_3\phi_\alpha^d \text{ and} \\ u_3^d &= U_3^d \text{ on } \partial\Omega_{\text{lat}}, \end{aligned} \tag{13}$$

where U_α^d, ϕ_α^d, and U_3^d are prescribed generalized displacements on the boundary.

The corresponding plate boundary conditions are:

$$\begin{aligned} U_\alpha &= U_\alpha^d, \phi_\alpha = \phi_\alpha^d \text{ and} \\ U_3 &= U_3^d \text{ on } \partial\omega. \end{aligned} \tag{14}$$

The 3D strain derived from Hencky's kinematics writes as:

$$\begin{aligned} \varepsilon^H_{\alpha\beta} &= E_{\alpha\beta} + x_3 \chi_{\alpha\beta}, \\ 2\varepsilon^H_{\alpha 3} &= \gamma_\alpha \text{ and } \varepsilon^H_{33} = 0. \end{aligned} \tag{15}$$

Here the plate generalized strains are defined as follows:

$$\begin{aligned} 2E_{\alpha\beta} &= U_{\alpha,\beta} + U_{\beta,\alpha}, \\ 2\chi_{\alpha\beta} &= \phi_{\alpha,\beta} + \phi_{\beta,\alpha}, \\ \gamma_\alpha &= \phi_\alpha + U_{3,\alpha}. \end{aligned} \tag{16}$$

The symmetric second-order tensor $E_{\alpha\beta}$ is the membrane strain. The symmetric second-order tensor $\chi_{\alpha\beta}$ is the material curvature. The in-plane vector γ_α is the generalized shear strain. It measures the difference between the normal to the deformed plate mid-surface $U_{3,\alpha}$ and the material inclination ϕ_α (Fig. 2).

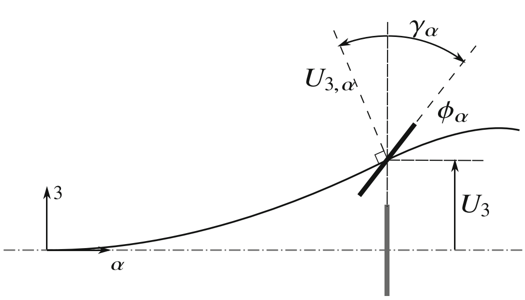
Finally the plate kinematically compatible fields are gathered as:

$$KC^{\mathrm{P}} = \{(U_\alpha, \phi_\alpha, U_3) \text{ s.t. } (16) \text{ and } (14)\}. \tag{17}$$

Formulation of Hencky's Plate Model

Minimum Potential Energy

The minimum potential energy principle states that the strain solution ε^{3D} of $\mathcal{P}^{3D}$ is the one



3D Derivations of Static Plate Theories, Fig. 2 Hencky's kinematics

that minimizes the potential energy among all kinematically compatible strain fields:

$$\boldsymbol{\varepsilon}^{3D} = \underset{\boldsymbol{\varepsilon}\in KC^{3D}}{\arg\min} \left\{ \int_\Omega \frac{1}{2} \boldsymbol{\varepsilon}:\mathbf{C}:\boldsymbol{\varepsilon} - h f_3 u_3 - f_\alpha u_\alpha \, d\Omega \right\}, \tag{18}$$

The stationarity condition – also known as principle of virtual work – writes for the solution as:

$$\forall \hat{\mathbf{u}} \in KC^{3D,0},$$
$$\int_\Omega \boldsymbol{\sigma}\left(\mathbf{u}^{3D}\right) : \boldsymbol{\varepsilon}(\hat{\mathbf{u}}) - h f_3 \hat{\mathbf{u}}_3 - f_\alpha \hat{u}_\alpha \, d\Omega = 0, \tag{19}$$

where $KC^{3D,0}$ is the set of 3D kinematically compatible fields with vanishing prescribed displacement.

Plate Generalized Stresses

Specifying (19) for Hencky's kinematics yields $\forall \left(\hat{U}_\alpha, \hat{\phi}_\alpha, \hat{U}_3\right) \in KC^{P,0}$

$$\int_\omega \langle\sigma_{\alpha\beta}\rangle \hat{E}_{\alpha\beta} + \langle x_3\sigma_{\alpha\beta}\rangle \hat{\chi}_{\alpha\beta} + \langle\sigma_{\alpha 3}\rangle \hat{\gamma}_a - h^2 f_3 \hat{U}_3 - h f_\alpha \hat{U}_\alpha d\omega = 0, \tag{20}$$

where integration through the thickness is denoted: $\langle f(x_3)\rangle = \int_{-\frac{h}{2}}^{\frac{h}{2}} f(x_3) dx_3$

This suggests the following definition of plate generalized stresses:

$$\begin{aligned} N_{\alpha\beta} &= \langle\sigma_{\alpha\beta}\rangle, \\ M_{\alpha\beta} &= \langle x_3\sigma_{\alpha\beta}\rangle, \\ Q_\alpha &= \langle\sigma_{\alpha 3}\rangle. \end{aligned} \tag{21}$$

$N_{\alpha\beta}$ is the membrane stress in duality with the membrane strain $E_{\alpha\beta}$, $M_{\alpha\beta}$ is the bending moment tensor in duality with the material curvature $\chi_{\alpha\beta}$, and Q_α is the shear force in duality with the generalized shear strain γ_α.

Plate Equilibrium

Integrating by parts Eq. (20) over the domain ω and taking into account plate boundary conditions yields:

$$\int_\omega -\left(N_{\alpha\beta,\beta} + hf_\alpha\right)\hat{U}_\alpha - \left(Q_{\alpha,\alpha} + h^2 f_3\right)\hat{U}_3 - \left(M_{\alpha\beta,\beta} - Q_a\right)\hat{\phi}_\alpha \mathrm{d}\omega = 0. \qquad (22)$$

This leads to the following plate equilibrium equations:

$$sc^{\mathrm{P}} = \begin{cases} N_{\alpha\beta,\beta} + hf_\alpha = 0 \text{ on } \omega, & (23) \\ Q_\alpha - M_{\alpha\beta,\beta} = 0 \text{ on } \omega, & (24) \\ Q_{\alpha,\alpha} + h^2 f_3 = 0 \text{ on } \omega, & (25) \end{cases}$$

Equation (23) is the in-plane or membrane equilibrium equation. Equation (25) is the out-of-plane equilibrium equation. Equation (24) is the bending equilibrium equation.

These equilibrium equations are almost identical to those obtained from the direct derivation ► "Direct Derivation of Plate Theories". Here the drilling moment vanishes, and the membrane stress and bending moment tensors are symmetric because the plate is originally assumed as a 3D Cauchy medium.

Natural Scaling of Stresses in Plates

Because the upper and lower face of the plate are actually free of stress, there is a natural scaling of stresses when, for a fixed in-plane dimension L, the thickness h goes to 0.

Indeed, from the out-of-plane part of the 3D equilibrium equation (7), it appears that the normal stress scales like $\sigma_{33} \sim h^2 f_3$ and that the transverse shear stress scales like $\sigma_{\alpha 3} \sim Lhf_3$. Furthermore, the bending equilibrium equation (24) ensures the following relation between the in-plane stress and the transverse shear stress: $\sigma_{\alpha 3} \sim \frac{h}{L}\sigma_{\alpha\beta}$. Hence $\sigma_{\alpha\beta} \sim L^2 f_3$ is of order h^0. This is also in agreement with the in-plane equilibrium which yields $\sigma_{\alpha\beta} \sim Lf_\alpha$ also of order h^0 and motivates the initial scaling of the load.

Finally, the in-plane stresses are of order h^0, the transverse shear stresses are of order h^1, and the normal stress is of order h^2. A direct consequence of this observation is that at leading order in h, the plate is in a state of plane-*stress*.

Constitutive Equations

Once plate kinematically as well as statically compatible fields are derived, there remains to establish plate constitutive equations. This is usually performed, integrating through the thickness the strain energy related to the approximation of strains (15). However, whereas Hencky's kinematics is correct asymptotically when h goes to 0, the corresponding strain field is not the leading order of the expansion of the 3D solution. Indeed, $\varepsilon^{\mathrm{H}}_{33} = 0$ corresponds to a plane-*strain* state. It is in contradiction with the natural scaling of stresses in the plate and does not satisfy the free boundary conditions on the upper and lower face of the plate. A small out-of-plane displacement is required to release out-of-plane Poisson's effect (see Braess et al. 2010 among others).

In most textbooks, it is arbitrarily assumed at this stage that the correct constitutive equations are those derived in a previous work from Reissner following static considerations and detailed below.

Static Derivation of Reissner Plate Model

In this derivation, Hencky's kinematics relating plate generalized displacements and 3D displacement is dropped and another interpretation of the plate kinematics will be derived. Reissner's model is obtained from the derivation of a statically compatible 3D stress distribution and the application of the minimum complementary energy principle.

Derivation of a Statically Compatible Stress Field

While the out-of-plane components of the strain $\varepsilon^{\mathrm{H}}_{i3}$ are not correct asymptotically, the in-plane components of the strain $\varepsilon^{\mathrm{H}}_{\alpha\beta}$ are correct. They are taken as the starting point for deriving an approximation of the in-plane stress:

$$\varepsilon_{\alpha\beta}(x_1,x_2,x_3) = E_{\alpha\beta}(x_1,x_2) + x_3\chi_{\alpha\beta}(x_1,x_2), \quad (26)$$

Using plane-*stress* constitutive equation, the in-plane stress writes as:

$$\sigma_{\alpha\beta} = C^{\sigma}_{\alpha\beta\gamma\delta}\left(E_{\delta\gamma} + x_3\chi_{\delta\gamma}\right), \quad (27)$$

where $C^{\sigma}_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma\delta} - C_{\alpha\beta 33}C_{33\gamma\delta}/C_{3333} = (S_{\alpha\beta\gamma\delta})^{-1}$ is the plane-stress stiffness tensor.

Computing the membrane stress and the bending moment for this stress distribution leads to the following leading-order relations between generalized stress and strains:

$$\begin{aligned} N_{\alpha\beta} &= \langle\sigma_{\alpha\beta}\rangle = hC^{\sigma}_{\alpha\beta\gamma\delta}E_{\delta\gamma}, \text{ and} \\ M_{\alpha\beta} &= \langle x_3\sigma_{\alpha\beta}\rangle = \frac{h^3}{12}C^{\sigma}_{\alpha\beta\gamma\delta}\chi_{\delta\gamma}. \end{aligned} \quad (28)$$

These equations are leading-order plate constitutive equations. The exact ones will be derived in the following. From these relations and Eq. (27), it is found:

$$\sigma_{\alpha\beta} = \frac{1}{h}N_{\alpha\beta} + \frac{12x_3}{h^3}M_{\alpha\beta}. \quad (29)$$

Note that Reissner (1944) started directly from this in-plane stress distribution. However, (29) would not hold in case the plate is heterogeneous, whereas the in-plane strain distribution (26) remains true at leading order ▶ “Homogenization of Thin Periodic Plates”.

From this in-plane stress distribution, a complete statically compatible stress distribution is now derived by successively integrating through the thickness of the 3D equilibrium equation (7).

The transverse shear distribution is derived by integrating with respect to x_3 the 3D equilibrium equations $\sigma_{\alpha\beta,\beta} + \sigma_{\alpha 3,3} + f_\alpha = 0$ and by taking into account the free boundary conditions (9) as well as plate equilibrium equations (23) and, (24). This yields:

$$\sigma_{\alpha 3} = \frac{3}{2h}\left(1 - \frac{4x_3^2}{h^2}\right)Q_\alpha. \quad (30)$$

Note that both upper and lower free boundary conditions are satisfied simultaneously.

Similarly, the normal traction σ_{33} is derived by integrating with respect to x_3 the 3D equilibrium equations $\sigma_{\alpha 3,\alpha} + \sigma_{33,3} + hf_3 = 0$ and by taking into account the free boundary conditions (9) as well as the out-of-plane equilibrium equation (25). This yields:

$$\sigma_{33} = f_3\frac{hx_3}{2}\left(1 - \frac{4x_3^2}{h^2}\right). \quad (31)$$

Again, upper and lower free boundary conditions are satisfied simultaneously.

Finally, the following Reissner stress distribution was derived:

$$\boldsymbol{\sigma}^{\mathrm{R}}(\mathbf{N},\mathbf{M},\mathbf{Q};f_3) = \begin{cases} \sigma^{\mathrm{R}}_{\alpha\beta} = \frac{1}{h}N_{\alpha\beta} + \frac{12x_3}{h^3}M_{\alpha\beta}, \\ \sigma^{\mathrm{R}}_{\alpha 3} = \frac{3}{2h}\left(1 - \frac{4x_3^2}{h^2}\right)Q_\alpha, \\ \sigma^{\mathrm{R}}_{33} = f_3\frac{hx_3}{2}\left(1 - \frac{4x_3^2}{h^2}\right). \end{cases} \quad (32)$$

This stress distribution is in the set of 3D statically compatible stress fields SC^{3D} iff ($\mathbf{N}$, $\mathbf{M}$, $\mathbf{Q}$) satisfy the plate equilibrium equations (23), (24), and (25). It delivers a much better approximation of 3D stress fields than the one obtained from the 3D constitutive equation and the strains (15).

Formulation of the Reissner Plate Model

Minimum of the Complementary Energy

The minimum complementary energy principle states that the stress solution σ^{3D} of the 3D problem $\mathcal{P}^{3D}$ is the one that minimizes the complementary energy among all statically compatible stress fields:

$$\boldsymbol{\sigma}^{3D} = \underset{\boldsymbol{\sigma}\in SC^{3D}}{\arg\min} \left\{ \int_\Omega \frac{1}{2}\boldsymbol{\sigma} : \mathbf{S} : \boldsymbol{\sigma}\ \mathrm{d}\Omega - \int_{\partial\Omega_{\text{lat}}} (\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{u}^d\,\mathrm{d}S \right\} \tag{33}$$

where $\mathbf{n}$ is the outer normal to $\partial\Omega_{\text{lat}}$.

Inserting stress distributions of the form (32) following plate equilibrium equations (23), (24), and (25) in the principle of minimum complementary energy yields the following minimization problem:

$$(\mathbf{N},\mathbf{M},\mathbf{Q})^{\text{R}} = \underset{(\mathbf{N},\mathbf{M},\mathbf{Q})\in SC^{\text{P}}}{\arg\min} \left\{ P^{*\text{R}}(\mathbf{N},\mathbf{M},\mathbf{Q}) \right\}, \tag{34}$$

where the plate complementary energy writes as:

$$P^{*\text{R}}(\mathbf{N},\mathbf{M},\mathbf{Q}) = \int_\omega w^{*\text{R}}(\mathbf{N},\mathbf{M},\mathbf{Q}; f_3)\,\mathrm{d}S - \int_{\partial\omega} U^d_\alpha N_{\alpha\beta}n_\beta + \phi^d_\alpha M_{\alpha\beta}n_\beta + U^d_3 Q_\alpha n_\alpha\ \mathrm{d}s. \tag{35}$$

The generalized stress energy density is:

$$w^{*\text{R}} = \left\langle \frac{1}{2}\boldsymbol{\sigma}^{\text{R}} : \mathbf{S} : \boldsymbol{\sigma}^{\text{R}} \right\rangle, \tag{36}$$

and the generalized displacements on the boundary are defined as:

$$U^d_\alpha = \frac{1}{h}\left\langle u^d_\alpha \right\rangle, \phi^d_\alpha = \left\langle \frac{12x_3}{h^3}u^d_\alpha \right\rangle \quad \text{and} \quad U^d_3 = \left\langle \frac{3}{2h}\left(1-\frac{4x_3^2}{h^2}\right)u^d_3 \right\rangle \tag{37}$$

This minimization problem fully determines Reissner's plate theory, and the corresponding details may be found in ▶ "Direct Derivation of Plate Theories". The plate kinematically compatible and statically compatible fields found from this formulation are the same as those obtained from Hencky's derivation ((23), (24), (25), and (17)). However, the constitutive equations are different and the plate kinematics may be interpreted differently.

Plate Kinematics

The definition of the generalized displacement on the boundary (37) encourages the following interpretation of the plate kinematics as projections of the 3D displacement:

$$U_\alpha \approx \frac{1}{h}\langle u_\alpha \rangle,\ \phi_\alpha \approx \left\langle \frac{12x_3}{h^3}u_\alpha \right\rangle \quad \text{and} \quad U_3 \approx \left\langle \frac{3}{2h}\left(1-\frac{4x_3^2}{h^2}\right)u_3 \right\rangle. \tag{38}$$

The membrane displacement U_α is the average through the thickness of the plate of the in-plane displacement. The material inclination ϕ_α is the odd part of the in-plane displacement. Finally, the plate deflection U_3 appears as a weighted average of the out-of-plane displacement.

Remarkably, Hencky's kinematics (12) is in agreement with the projections (38).

Constitutive Equations

From the plate stress energy density (36), the following constitutive equations are derived:

$$\begin{aligned} E_{\alpha\beta} &= \frac{1}{h}S_{\alpha\beta\gamma\delta}N_{\delta\gamma}, \\ \chi_{\alpha\beta} &= \frac{12}{h^3}S_{\alpha\beta\gamma\delta}M_{\delta\gamma} + p_{\alpha\beta}hf_3, \\ \gamma_\alpha &= f_{\alpha\beta}Q_\beta, \end{aligned} \tag{39}$$

where the shear force compliance $\mathbf{f}$ is:

$$f_{\alpha\beta} = \frac{6}{5h}4S_{\alpha3\beta3}, \tag{40}$$

and $\mathbf{p}$ is a prescribed curvature related to the applied load:

$$p_{\alpha\beta} = -\frac{S_{\alpha\beta33}}{5}. \tag{41}$$

It must be noted that **p** depends on the way the load is applied on the 3D body. Indeed, loading the plate on the upper and lower faces leads to a different value of **p**.

These equations may be inverted and lead to the following constitutive equations:

$$\mathbf{N}=\mathbf{A}:\mathbf{E},\quad \mathbf{M}=\mathbf{D}:(\boldsymbol{\chi}-\mathbf{p}h\,f_3)\quad \text{and} \quad \mathbf{Q}=\mathbf{F}\cdot\boldsymbol{\gamma}, \tag{42}$$

where:

$$\begin{aligned} A_{\alpha\beta\gamma\delta} &= hC^{\sigma}_{\alpha\beta\gamma\delta}, \\ D_{\alpha\beta\gamma\delta} &= \frac{h^3}{12}C^{\sigma}_{\alpha\beta\gamma\delta} \quad \text{and} \\ F_{\alpha\beta} &= \frac{5h}{6}C_{\alpha3\beta3}. \end{aligned} \tag{43}$$

Note that the membrane problem for U_α generalized displacements is fully uncoupled from the bending problem for U_3 and ϕ_α generalized displacements. This is because of the monoclinic symmetry assumed in (4) and the mirror symmetry with respect to the midplane of the plate. For heterogeneous plates, this uncoupling is not always true ▶ "Anisotropic and Refined Plate Theories".

Finally, the so-called "shear correction factor" 5/6 taking into account the nonuniform distribution of the out-of-plane shear stress was obtained. Note that, when dealing with heterogeneous plates, this definition is meaningless since several shear stiffness moduli may be involved in the shear force constitutive equation.

Static Boundary Conditions

It is also possible to enforce static boundary conditions on the lateral boundary of the plate. However, whereas kinematic boundary conditions are satisfied weakly on the boundary, the static derivation from Reissner requires that static boundary conditions are satisfied strongly on the boundary. Hence, considering the form of the stress approximation (32), only stress distributions through the thickness of the following form:

$$\begin{aligned} T^d_\alpha &= \frac{1}{h}N^d_\alpha + \frac{12x_3}{h^3}M^d_\alpha \quad \text{and} \\ T^d_3 &= \frac{3}{2h}\left(1-\frac{4x_3^2}{h^2}\right)Q^d, \end{aligned} \tag{44}$$

may be enforced on the lateral boundary $\partial\omega^\sigma \times \mathcal{T}$. Here, $\partial\omega^\sigma$ denotes the portion of the boundary where static conditions may apply. They correspond to the following plate static boundary conditions:

$$\begin{aligned} &N_{\alpha\beta}n_\beta = N^d_\alpha,\, M_{\alpha\beta}n_\beta = M^d_\alpha \quad \text{and} \\ &Q_\alpha n_\alpha = Q^d \text{ on } \partial\omega^\sigma, \end{aligned} \tag{45}$$

where N^d_α is an in-plane traction, M^d_α is an in-plane couple, and Q^d is a shear force enforced on the boundary. A direct consequence is that traction free boundary conditions are strongly satisfied with Reissner plate model.

Conclusion

The approaches from Hencky and Reissner for deriving a thick plate theory are often confused in the literature. Whereas they are closely related, they actually yield different plate models which suffer from different limitations.

The kinematic derivation from Hencky is probably the most straightforward but leads to incorrect estimates of the local stresses as well as the plate's constitutive equations. The constitutive equations derived by Reissner are commonly used to correct Hencky's model.

The extension of this model to the case of laminated plates was early performed (Yang et al. 1966). This approach is referred to as first-order shear deformation theory and suffers even more critically from the inconsistencies encountered for homogeneous plates. The advantage of this approach is that its kinematics may be extended to large displacements and rotations.

A natural strategy for solving these inconsistencies is to enrich the plate kinematics so that it can accommodate free boundaries at the upper and lower face of the plate. This is the main concept behind hierarchical models (Babuška and Li 1992; Paumier and Raoult 1997; Alessandrini et al. 1999) where the 3D displacement is assumed as a polynomial of the out-of-plane coordinate and each monomial is multiplied by an in-plane function being a generalized plate displacement. However, this requires more plate kinematic degrees of freedom than those of Reissner-Hencky models.

The static derivation from Reissner leads to a very accurate model in the framework of static linear elasticity and it was observed empirically that it converges faster than Kirchhoff-Love model in some specific configurations (Lebée and Sab 2017b). However, its rigorous extension to laminated plates requires the introduction of numerous additional plate degrees of freedom and is impractical for engineering applications (Lebée and Sab 2017a).

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