

TcT: Tyrolean Complexity Tool

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Abstract. In this paper we present TcT v3.0, the latest version of our fully automated complexity analyser. TcT implements our framework for automated complexity analysis and focuses on extensibility and automation. TcT is open with respect to the input problem under investigation and the resource metric in question. It is the most powerful tool in the realm of automated complexity analysis of term rewrite systems. Moreover it provides an expressive problem-independent strategy language that facilitates proof search. We give insights about design choices, the implementation of the framework and report different case studies where we have applied TcT successfully.

1 Introduction

Automatically checking programs for correctness has attracted the attention of the computer science research community since the birth of the discipline. Properties of interest are not necessarily functional, however, and among the non-functional ones, noticeable cases are bounds on the amount of resources (like time, memory and power) programs need, when executed. A variety of verification techniques have been employed in this context, like abstract interpretations, model checking, type systems, program logics, or interactive theorem provers; see [1–3, 12–16, 21, 25, 27] for some pointers.

In this paper, we present TcT v3.0, the latest version of our fully automated complexity analyser. TcT is open source, released under the BSD3 license, and available at

<http://cl-informatik.uibk.ac.at/software/tct/>.

TcT features a standard command line interface, an interactive interface, and a web interface. In the setup of the complexity analyser, TcT provides a *transformational approach*, depicted in Fig. 1. First, the input program in relation to the resource of interest is *transformed* to an *abstract representation*. We refer to

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the result of applying such a transformation as *abstract program*. It has to be guaranteed that the employed transformations are *complexity reflecting*, that is, the resource bound on the obtained abstract program reflects upon the resource usage of the input program. More precisely, the complexity analysis deals with a general *complexity problem* that consists of a program together with the resource metric of interest as input. Second, we employ problem specific techniques to derive bounds on the given problem and finally, the result of the analysis, i.e. a complexity bound or a notice of failure, is relayed to the original program. We emphasise that TCT does not make use of a *unique* abstract representation, but is designed to employ a *variety* of different representations. Moreover, different representations may interact with each other. This improves *modularity* of the approach and provides scalability and precision of the overall analysis. For now we make use of *integer transition systems* (*ITSs* for short) or various forms of *term rewrite systems* (*TRSs* for short), not necessarily first-order. Currently, we are in the process of developing dedicated techniques for the analysis of *higher-order rewrite systems* (*HRSs* for short) that once should become another abstraction subject to resource analysis (depicted as `tct-hrs` in the figure). Concretising this abstract setup, TCT currently provides a fully automated runtime complexity analysis of pure OCaml programs as well as a runtime analysis of object-oriented bytecode programs. Furthermore the tool provides runtime and size analysis of ITSs as well as complexity analysis of first-order rewrite systems. With respect to the latter application, TCT is the most powerful complexity analyser of its kind.¹ The latest version is a complete reimplementa- tion of the tool that takes full advantage of the abstract *complexity framework* [6,7] introduced by the first and second author. TCT is open with respect to the complexity problem under investigation and problem specific techniques for the resource analysis. Moreover it provides an expressive problem independent *strategy language* that facilitates *proof search*. In this paper, we give insights about design choices, the implementation of the framework and report different case studies where we have applied TCT successfully.

Development Cycle. TCT was envisioned as a dedicated tool for the automated complexity analysis of first-order term rewrite systems. The first version was made available in 2008. Since then, TCT has successfully taken part in the complexity categories of TERMCOMP. The competition results have shown that TCT is the most powerful complexity solver for TRSs. The previous version [5] conceptually corresponds now to the `tct-trs` component depicted in Fig. 1. The reimplementa- tion of TCT was mainly motivated by the following observations:

- automated resource analysis of programming languages is typically done by establishing complexity reflecting abstractions to formal systems
- the complexity framework is general enough to integrate those abstractions as transformations of the original program
- modularity and decomposition can be represented independently of the analysed complexity problem

¹ See for example the results of TCT at this year’s TERMCOMP, available from http://termination-portal.org/wiki/Termination_Competition_2015/.

We have rewritten the tool from scratch to integrate and extend all the ideas that were collected and implemented in previous versions in a clean and structured way. The new tool builds upon a small core (`tct-core`) that provides an expressive strategy language with a clearly defined semantics, and is, as envisioned, open with respect to the type of the complexity problem.

Structure. The remainder of the paper is structured as follows. In the next section, we provide an overview on the design choices of the resource analysis in TcT , that is, we inspect the middle part of Fig. 1. In Sect. 3 we revisit our abstract complexity framework, which is the theoretical foundation of the core of TcT (`tct-core`). Section 4 provides details about the implementation of the complexity framework and Sect. 5 presents four different use cases that show how the complexity framework can be instantiated. Among them the instantiation for higher-order programs (`tct-hoca`), as well as the instantiation to complexity analysis of TRSs (`tct-trs`). Finally we conclude in Sect. 6.

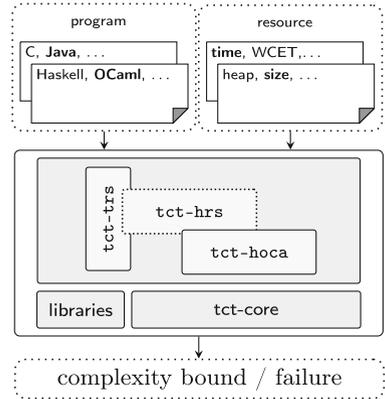


Fig. 1. Complexity Analyser TcT .

2 Architectural Overview

In this section we give an overview of the architecture of our complexity analyser. All components of TcT are written in the strongly typed, lazy functional programming language *Haskell* and released *open source* under BSD3. Our current code base consists of approximately 12.000 lines of code, excluding external libraries. The core constitutes roughly 17% of our code base, 78% of the code is dedicated to complexity techniques. The remaining 5% attribute to interfaces to external tools, such as `CeTA`² and SMT solvers, and common utility functions.

As depicted in Fig. 1, the implementation of TcT is divided into separate components for the different program kinds and abstractions thereof supported. These separate components are no islands however. Rather, they *instantiate* our abstract *complexity framework* for complexity analysis [6], from which TcT derives its power and modularity. In short, in this framework complexity techniques are modelled as *complexity processors* that give rise to a set of inferences over *complexity proofs*. From a completed complexity proof, a complexity bound can be inferred. The theoretical foundations of this framework are given in Sect. 3.

The abstract complexity framework is implemented in TcT 's *core library*, termed `tct-core`, which is depicted in Fig. 2 at the bottom layer. Central, it provides a common notion of a proof state, viz *proof trees*, and an interface for specifying processors. Furthermore, `tct-core` complements the framework with

² See <http://cl-informatik.uibk.ac.at/software/ceta/>.

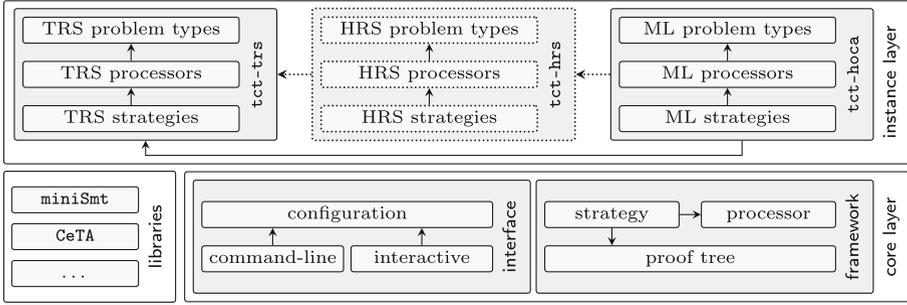


Fig. 2. Architectural Overview of \mathcal{TCT} .

a simple but powerful *strategy language*. Strategies play the role of tactics in interactive theorem provers like *Isabelle* or *Coq*. They allow us to turn a set of processors into a sophisticated complexity analyser. The implementation details of the core library are provided in Sect. 4.

The complexity framework implemented in our core library leaves the type of *complexity problem*, consisting of the analysed program together with the resource metric of interest, abstract. Rather, concrete complexity problems are provided by concrete *instances*, such as the two *instances* `tct-hoca` and `tct-trs` depicted in Fig. 2. We will look at some particular instances in detail in Sect. 5. Instances implement complexity techniques on defined problem types in the form of complexity processors, possibly relying on external libraries and tools such as e.g. SMT solvers. Optionally, instances may also specify strategies that compose the provided processors. Bridges between instances are easily specified as processors that implement conversions between problem types defined in different instances. For example our instance `tct-hoca`, which deals with the runtime analysis of pure *OCaml* programs, makes use of the instance `tct-trs`. Thus our system is open to the seamless integration of alternative problem types through the specification of new instances. Exemplarily, we mention the envisioned instance `tct-hrs` (see Fig. 1), which should incorporate dedicated techniques for the analysis of HRSs. We intend to use `tct-hrs` in future versions for the analysis of functional programs.

3 A Formal Framework for Complexity Analysis

We now briefly outline the theoretical framework upon which our complexity analyser \mathcal{TCT} is based. As mentioned before, both the input language (e.g. *Java*, *OCaml*, ...) as well as the resource under consideration (e.g. execution time, heap usage, ...) is kept abstract in our framework. That is, we assume that we are dealing with an abstract class of *complexity problems*, where however, each complexity problem \mathcal{P} from this class is associated with a *complexity function* $cp_{\mathcal{P}} : D \rightarrow D$, for a *complexity domain* D . Usually, the complexity domain D will be the set of natural numbers \mathbb{N} , however, more sophisticated choices of

complexity functions, such as e.g. those proposed by Danner et al. [11], fall into the realm of our framework.

In a concrete setting, the complexity problem \mathcal{P} could denote, for instance, a Java program. If we are interested in heap usage, then $D = \mathbb{N}$ and $\text{cp}_{\mathcal{P}} : \mathbb{N} \rightarrow \mathbb{N}$ denotes the function that describes the maximal heap usage of \mathcal{P} in the sizes of the program inputs. As indicated in the introduction, any transformational solver converts concrete programs into abstract ones, if not already interfaced with an abstract program. Based on the possible abstracted complexity problem \mathcal{P} the analysis continues using a set of *complexity techniques*. In particular, a reasonable solver will also integrate some form of *decomposition techniques*, transforming an intermediate problem into various smaller *sub-problems*, and analyse these sub-problems separately, either again by some form of decomposition method, or eventually by some *base technique* which infers a suitable resource bound. Of course, at any stage in this *transformation chain*, a solver needs to keep track of computed complexity-bounds, and relay these back to the initial problem.

To support this kind of reasoning, it is convenient to formalise the internals of a complexity analyser as an inference system over *complexity judgements*. In our framework, a complexity judgement has the shape $\vdash \mathcal{P} : B$, where \mathcal{P} is a complexity problem and B is a set of *bounding functions* $f : D \rightarrow D$ for a complexity domain D . Such a judgement is *valid* if the complexity function of \mathcal{P} lies in B , that is, $\text{cp}_{\mathcal{P}} \in B$. Complexity techniques are modelled as *processors* in our framework. A processor defines a transformation of the *input problem* \mathcal{P} into a list of sub-problems $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ (if any), and it relates the complexity of the obtained sub-problems to the complexity of the input problem. Processors are given as inferences

$$\frac{\text{Pre}(\mathcal{P}) \quad \vdash \mathcal{Q}_1 : B_1 \quad \dots \quad \vdash \mathcal{Q}_n : B_n}{\vdash \mathcal{P} : B},$$

where $\text{Pre}(\mathcal{P})$ indicates some *pre-conditions* on \mathcal{P} . The processor is *sound* if under $\text{Pre}(\mathcal{P})$ the validity of judgements is preserved, i.e.

$$\text{Pre}(\mathcal{P}) \wedge \text{cp}_{\mathcal{Q}_1} \in B_1 \wedge \dots \wedge \text{cp}_{\mathcal{Q}_n} \in B_n \implies \text{cp}_{\mathcal{P}} \in B.$$

Dual, it is called *complete* if under the assumptions $\text{Pre}(\mathcal{P})$, validity of the judgement $\vdash \mathcal{P} : B$ implies validity of the judgements $\vdash \mathcal{Q}_i : B_i$.

A *proof* of a judgement $\vdash \mathcal{P} : B$ from the *assumptions* $\vdash \mathcal{Q}_1 : B_1, \dots, \vdash \mathcal{Q}_n : B_n$ is a deduction using sound processors only. The proof is *closed* if its set of assumptions is empty. Soundness of processors guarantees that our formal system is correct. Application of complete processors on a valid judgement ensures that no invalid assumptions are derived. In this sense, the application of a complete processor is always safe.

Proposition 1. *If there exists a closed complexity proof $\vdash \mathcal{P} : B$, then the judgement $\vdash \mathcal{P} : B$ is valid.*

4 Implementing the Complexity Framework

The formal complexity framework described in the last section is implemented in the core library, termed `tct-core`. In the following we outline the two central components of this library: (i) the generation of complexity proofs, and (ii) common facilities for instantiating the framework to concrete tools, see Fig. 2.

4.1 Proof Trees, Processors, and Strategies

The library `tct-core` provides the verification of a valid complexity judgement $\vdash \mathcal{P} : B$ from a given input problem \mathcal{P} . More precise, the library provides the environment to construct a complexity proof witnessing the validity of $\vdash \mathcal{P} : B$.

Since the class B of bounding-functions is a result of the analysis, and not an input, the complexity proof can only be constructed once the analysis finished successfully. For this reason, proofs are not directly represented as trees over complexity judgements. Rather, the library features *proof trees*. Conceptually, a proof tree is a tree whose leaves are labelled by *open complexity problems*, that is, problems which remain to be analysed, and whose internal nodes represent successful applications of processors. The complexity analysis of a problem \mathcal{P} then amounts to the *expansion* of the proof tree whose single node is labelled by the open problem \mathcal{P} . *Processors* implement a single expansion step. To facilitate the expansion of proof trees, `tct-core` features a rich *strategy language*, similar to tactics in interactive theorem provers like `Isabelle` or `Coq`. Once a proof tree has been completely expanded, a complexity judgement for \mathcal{P} together with the witnessing complexity proof can be computed from the proof tree.

In the following, we detail the central notions of proof tree, processor and strategy, and elaborate on important design issues.

Proof Trees: The first design issue we face is the representation of complexity problems. In earlier versions of `TCT`, we used a concrete problem type `Problem` that captured various notions of complexity problems, but all were based on term rewriting. With the addition of new kinds of complexity problem, such as runtime of functional or heap size of imperative programs, this approach became soon infeasible. In the present reimplementaion, we therefore abstract over problem types, at the cost of slightly complicating central definitions. This allows concrete instantiations to precisely specify which problem types are supported. Consequently, proof trees are parameterised in the type of complexity problems.

The corresponding (generalised) algebraic data-type `ProofTree` α (from module `Tct.Core.Data.ProofTree`) is depicted in Fig. 3. A constructor `Open` represents a leaf labelled by an open problem of type α . The ternary constructor `Success` represents the successful application of a processor of type β . Its first argument, a value of type `ProofNode` β , carries the applied processor, the current complexity problem under investigation as well as a proof-object of type `ProofObject` β . This information is useful for proof analysis, and allows a detailed textual representation of proof trees. Note that `ProofObject` is a type-level function, the concrete

```

data ProofTree  $\alpha$  where
  Open    ::  $\alpha \rightarrow$  ProofTree  $\alpha$       -- open proof node
  Success :: Processor  $\beta \Rightarrow$           -- successful application
           ProofNode  $\beta \rightarrow$  CertFn  $\rightarrow$  [ProofTree  $\alpha$ ]  $\rightarrow$  ProofTree  $\alpha$ 
  Failure :: Reason  $\rightarrow$  ProofTree  $\alpha$       -- failed application

```

Fig. 3. Data-type declaration of proof trees in `tct-core`.

representation of a proof-object thus depends on the type of the applied processor. The second argument to `Success` is a *certificate-function*

```
type CertFn = [Certificate]  $\rightarrow$  Certificate ,
```

which is used to relate the estimated complexity of generated sub-problems to the analysed complexity problem. Thus currently, the set of bounding-functions B occurring in the final complexity proof is fixed to those expressed by the data-type `Certificate` (module `Tct.Core.Data.Certificate`). `Certificate` includes various representations of complexity classes, such as the class of polynomials, exponential, primitive and multiple recursive functions, but also the more fine grained classes of bounding-functions $\mathcal{O}(n^k)$ for all $k \in \mathbb{N}$. The remaining argument to the constructor `Success` is a forest of proof trees, each individual proof tree representing the continuation of the analysis of a corresponding sub-problem generated by the applied processor. Finally, the constructor `Failure` indicates that the analysis failed. It results for example from the application of a processor to an open problem which does not satisfy the pre-conditions of the processor. The argument of type `Reason` allows a textual representation of the failure-condition. The analysis will always abort on proof trees containing such a *failure node*.

Processors. The interface for processors is specified by the type-class `Processor`, which is defined in module `Tct.Core.Data.Processor` and depicted in Fig. 4. The type of input problem and generated sub-problems are defined for processors on an individual basis, through the type-level functions `In` and `Out`, respectively. This eliminates the need for a global problem type, and facilitates the seamless combination of different instantiations of the core library. Each processor instance specifies additionally the type of proof-objects `ProofObject α` – the meta information provided in case of a successful application. The proof-object is constrained to instances of `ProofData`, which beside others, ensures that a textual representation can be obtained. Each instance of `Processor` has to implement a method `execute`, which given an input problem of type `In α` , evaluates to a `TctM` action that produces a value of type `Return α` . The monad `TctM` (defined in module `Tct.Core.Data.TctM`) extends the `IO` monad with access to runtime information, such as command line parameters and execution time. The data-type `Return α` specifies the result of the application of a processor to its given input problem. In case of a successful application, the return value carries the proof-object, a value of type `CertFn`, which relates complexity-bounds

```

data Return  $\alpha$  =
  NoProgress Reason
  | Progress (ProofObject  $\alpha$ ) CertFn [ProofTree (Out  $\alpha$ )]

class (ProofData (ProofObject  $\alpha$ ))  $\Rightarrow$  Processor  $\alpha$  where
  type In  $\alpha$  -- type of input problem
  type Out  $\alpha$  -- type of output problems
  type ProofObject  $\alpha$  -- meta information
  execute ::  $\alpha \rightarrow$  In  $\alpha \rightarrow$  TctM (Return  $\alpha$ ) -- implementation

-- application of processor to a problem, resulting in a proof tree
apply :: Processor  $\alpha \Rightarrow \alpha \rightarrow$  In  $\alpha \rightarrow$  TctM (ProofTree (Out  $\alpha$ ))
apply p i = toProofTree <$> (execute p i 'catchError' handler)
  where
    toProofTree (NoProgress r) = Failure r
    toProofTree (Progress ob cf ts) = Success (ProofNode p i ob) cf ts
    handler err = return (NoProgress (IOError err))

```

Fig. 4. Data-type and class definitions related to processors in `tct-core`.

on sub-problems to bounds on the input-problem and the list of generated sub-problems. In fact the type is slightly more liberal and allows for each generated sub-problem a, possibly open, proof tree. This generalisation is useful in certain contexts, for example, when the processor makes use of a second processor.

Strategies. To facilitate the expansion of a proof tree, `tct-core` features a simple but expressive *strategy language*. The strategy language is *deeply embedded*, via the generalised algebraic data-type `Strategy $\alpha \beta$` defined in Fig. 5. Semantics over strategies are given by the function

$$\text{evaluate} :: \text{Strategy } \alpha \beta \rightarrow \text{ProofTree } \alpha \rightarrow \text{TctM } (\text{ProofTree } \beta) ,$$

defined in module `Tct.Core.Data.Strategy`. A strategy of type `Strategy $\alpha \beta$` thus translates a proof tree with open problems of type α to one with open problems of type β .

The first four primitives defined in Fig. 5 constitute our tool box for modelling sequential application of processors. The strategy `Id` is implemented by the identity function on proof trees. The remaining three primitives traverse the given proof tree in-order, acting on all the open proof-nodes. The strategy `Apply p` replaces the given open proof-node with the proof tree resulting from an application of `p`. The strategy `Abort` signals that the computation should be aborted, replacing the given proof-node by a failure node. Finally, the strategy `Cond predicate s1 s2 s3` implements a very specific conditional. It sequences the application of strategies `s1` and `s2`, provided the proof tree computed by `s1` satisfies the predicate `predicate`. For the case where the predicate is not satisfied, the conditional acts like the third strategy `s3`.

In Fig. 6 we showcase the definition of derived sequential strategy combinators. Sequencing `s1 \ggg s2` of strategies `s1` and `s2` as well as a (left-biased)

```

data Strategy  $\alpha$   $\beta$  where
  -- primitives for sequential processor application
  Id      :: Strategy  $\alpha$   $\alpha$ 
  Apply   :: (Processor  $\gamma$ )  $\Rightarrow$   $\gamma \rightarrow$  Strategy (In  $\gamma$ ) (Out  $\gamma$ )
  Abort   :: Strategy  $\alpha$   $\beta$ 
  Cond    :: (ProofTree  $\beta \rightarrow$  Bool)  $\rightarrow$  Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\beta$   $\gamma$ 
            $\rightarrow$  Strategy  $\alpha$   $\gamma \rightarrow$  Strategy  $\alpha$   $\gamma$ 
  -- primitives for parallel processor application
  Par     :: Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta$ 
  Race    :: Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta$ 
  Better  :: (ProofTree  $\beta \rightarrow$  ProofTree  $\beta \rightarrow$  Ordering)
            $\rightarrow$  Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta$ 
  -- control operators
  Timeout :: Time  $\rightarrow$  Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta$ 
  WithStatus :: (TcTStatus  $\alpha \rightarrow$  Strategy  $\alpha$   $\beta$ )  $\rightarrow$  Strategy  $\alpha$   $\beta$ 

```

Fig. 5. Deep Embedding of our strategy language in tct-core.

```

-- auxiliary predicates on proof trees
nonFailing, progress :: ProofTree  $\alpha \rightarrow$  Bool
nonFailing t         = null [ Failure {}  $\leftarrow$  subTrees t ]
progress Open{}     = False
progress _           = True
-- choice
(<|>) :: Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta$ 
s1 <|> s2 = Cond nonFailing s1 Id s2
-- composition
(>>>) :: Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\beta$   $\gamma \rightarrow$  Strategy  $\alpha$   $\gamma$ 
s1 >>> s2 = Cond nonFailing s1 s2 Abort
-- backtracking
try :: Strategy  $\alpha$   $\alpha \rightarrow$  Strategy  $\alpha$   $\alpha$ 
try s = s <|> Id
force :: Strategy  $\alpha$   $\beta \rightarrow$  Strategy  $\alpha$   $\beta$ 
force s = Cond progress s Id Abort
-- iteration
exhaustive, exhaustive+ :: Strategy  $\alpha$   $\alpha \rightarrow$  Strategy  $\alpha$   $\alpha$ 
exhaustive s = try (exhaustive+ s)
exhaustive+ s = force s >>> exhaustive s

```

Fig. 6. Derived sequential strategy combinators.

choice operator $s_1 <|> s_2$ are derived from the conditional primitive `Cond`. The strategy `try s` behaves like `s`, except when `s` fails then `try s` behaves as an identity. The combinator `force` complements the combinator `try`: the strategy `force s` enforces that strategy `s` produces a new proof-node. The combinator `try` brings *backtracking* to our strategy language, i.e. the strategy `try s1 >>> s2` first applies strategy `s1`, backtracks in case of failure, and applies `s2` afterwards. Finally, the strategies `exhaustive s` applies `s` zero or more times, until

$$\begin{array}{ll}
s1 \gg (s2 \gg s3) \equiv (s1 \gg s2) \gg s3 & (\gg \text{ is associative}) \\
s1 \langle | \rangle (s2 \langle | \rangle s3) \equiv (s1 \langle | \rangle s2) \langle | \rangle s3 & (\langle | \rangle \text{ is associative}) \\
s \gg \text{Id} \equiv s \equiv \text{Id} \gg s & (\text{Id identity element of } \gg) \\
s \langle | \rangle \text{Abort} \equiv s \equiv \text{Abort} \langle | \rangle s & (\text{Abort identity element of } \langle | \rangle) \\
s1 \gg (s2 \langle | \rangle s3) \equiv (s1 \gg s2) \langle | \rangle (s1 \gg s3) & (\gg \text{ distributes over } \langle | \rangle) \\
(s1 \langle | \rangle s2) \gg s3 \equiv (s1 \gg s3) \langle | \rangle (s2 \gg s3) & \\
\text{force } (s1 \langle | \rangle s2) \equiv \text{force } s1 \langle | \rangle \text{force } s2 & (\text{force distributes over } \langle | \rangle) \\
\text{force } (\text{try } s) \equiv \text{force } s & (\text{force eliminates try})
\end{array}$$

Fig. 7. Some laws obeyed by the derived operators.

strategy s fails. The combinator `exhaustive+` behaves similarly, but applies the given strategy at least once. The obtained combinators satisfy the expected laws, compare Fig. 7 for an excerpt.

Our strategy language features also three dedicated types for parallel proof search. The strategy `Par s` implements a form of data level parallelism, applying strategy s to all open problems in the given proof tree in parallel. In contrast, the strategies `Race s1 s2` and `Better comp s1 s2` apply to each open problem the strategies $s1$ and $s2$ concurrently, and can be seen as parallel version of our choice operator. Whereas `Race s1 s2` simply returns the (non-failing) proof tree of whichever strategy returns first, `Better comp s1 s2` uses the provided comparison-function `comp` to decide which proof tree to return.

The final two strategies depicted in Fig. 5 implement timeouts, and the dynamic creation of strategies depending on the current `TctStatus`. `TctStatus` includes global state, such as command line flags and the execution time, but also proof relevant state such as the current problem under investigation.

4.2 From the Core to Executables

The framework is instantiated by providing a set of sound processors, together with their corresponding input and output types. At the end of the day the complexity framework has to give rise to an executable tool, which, given an initial problem, possibly provides a complexity certificate.

To ease the generation of such an executable, `tct-core` provides a default implementation of the `main` function, controlled by a `TctConfig` record (see module `Tct.Core.Main`). A minimal definition of `TctConfig` just requires the specification of a default strategy, and a parser for the initial complexity problem. Optionally, one can for example specify additional command line parameters, or a list of *declarations* for custom strategies, which allow the user to control the proof search. Strategy declarations wrap strategies with additional *meta information*, such as a *name*, a *description*, and a list of *parameters*. Firstly, this information is used for documentary purposes. If we call the default implementation with the command line flag `--list-strategies` it will present a documentation of the available processors and strategies to the user. Secondly, declarations

facilitate the parser generation for custom strategies. It is noteworthy to mention that declarations and the generated parsers are type safe and are checked during compile-time. Declarations, together with usage information, are defined in module `Tct.Core.Data.Declaration`. Given a path pointing to the file holding the initial complexity problem, the generated executable will perform the following actions in order:

1. Parse the command line options given to the executable, and reflect these in the aforementioned `TctStatus`.
2. Parse the given file according to the parser specified in the `TctConfig`.
3. Select a strategy based on the command line flags, and apply the selected strategy on the parsed input problem.
4. Should the analysis succeed, a textual representation of the obtained complexity judgement and corresponding proof tree is printed to the console; in case the analysis fails, the uncompleted proof tree, including the `Reason` for failure is printed to the console.

Interactive. The library provides an *interactive mode* via the `GHCi` interpreter, similar to the one provided in `TcT` v2 [5]. The interactive mode is invoked via the command line flag `--interactive`. The implementation keeps track of a *proof state*, a list of proof trees that represents the history of the interactive session. We provide an interface to inspect and manipulate the proof state. Most noteworthy, the user can select individual sub-problems and apply strategies on them. The proof state is updated accordingly.

5 Case Studies

In this section we discuss several instantiations of the framework that have been established up to now. We keep the descriptions of the complexity problems informal and focus on the big picture. In the discussion we group abstract programs in contrast to real world programs.

5.1 Abstract Programs

Currently `TcT` provides *first-order term rewrite systems* and *integer transition systems* as abstract representations. As mentioned above, the system is open to the seamless integration of alternative abstractions.

Term Rewrite Systems. Term rewriting forms an abstract model of computation, which underlies much of declarative programming. Our results on pure `OCaml`, see below, show how we can make practical use of the clarity of the model. The `tct-trs` instance provides automated resource analysis of *first-order term rewrite systems* (*TRSs* for short) [8, 26]. Complexity analysis of TRSs has received significant attention in the last decade, see [19] for details. A TRS consists of a set of rewrite rules, i.e. directed equations that can be applied from left to right. Computation is performed by *normalisation*, i.e. by successively

```

WORST_CASE(? ,0(n^2))
*** 1 Progress [(? ,0(n^2))] ***
Considered Problem:
  Strict TRS Rules:
    mult(0(),y)  -> 0()
    mult(s(x),y) -> plus(y,mult(x,y))
    plus(x,0())  -> x
    plus(s(x),y) -> s(plus(x,y))
    square(x)    -> mult(x,x)
  Signature:
    {mult/2,plus/2,square/1} / {0/0,s/1}
  Obligation: runtime innermost

Applied Processor:
  NaturalPI {shape = Mixed 2}

Proof:
  Polynomial Interpretation:
    p(0) = 1
    p(mult) = 3*x1 + 2*x1*x2 + 2*x2
    p(plus) = 2 + 2*x1 + x2
    p(s) = 1 + x1
    p(square) = 6 + 7*x1 + 2*x1^2

Orientation:
  mult(0(),y) = 3 + 4*y
                > 1
                = 0()
  mult(s(x),y) = 3 + 3*x + 2*x*y + 4*y
                > 2 + 3*x + 2*x*y + 4*y
                = plus(y,mult(x,y))
  plus(x,0()) = 3 + 2*x
               > x
               = x
  plus(s(x),y) = 4 + 2*x + y
               > 3 + 2*x + y
               = s(plus(x,y))
  square(x) = 6 + 7*x + 2*x^2
            > 5*x + 2*x^2
            = mult(x,x)

```

Fig. 8. Polynomial Interpretation Proof.

applying rewrite rules until no more rules apply. As an example, consider the following TRS \mathcal{R}_{sq} , which computes the squaring function on natural numbers in unary notation.

$$\begin{array}{lll}
 \text{sq}(x) \rightarrow x * x & x * 0 \rightarrow 0 & x + 0 \rightarrow x \\
 s(x) * y \rightarrow y + (x * y) & & s(x) + y \rightarrow s(x + y).
 \end{array}$$

The *runtime complexity* of a TRS is naturally expressed as a function that measures the length of the longest reduction, in the sizes of (normalised) starting terms. Figure 8 depicts the proof output of `tct-trs` when applying a polynomial interpretation [18] processor with maximum degree 2 on \mathcal{R}_{sq} . The resulting proof tree consists of a single progress node and returns the (optimal) quadratic asymptotic upper bound on the runtime complexity of \mathcal{R}_{sq} . The success of \mathcal{TCT} as a complexity analyser, and in particular the strength of `tct-trs` instance is apparent from its performance at TERMCOMP.³ It is noteworthy to mention that at this year’s competition \mathcal{TCT} not only won the combined ranking, but also the certified category. Here only those techniques are admissible that have been machine checked, so that soundness of the obtained resource bound is almost without doubt, cf. [7]. The `tct-trs` instance has many advantages in comparison to its predecessors. Many of them are subtle and are due to the redesign of the architecture and reimplementations of the framework. However, the practical consequences are clear: the instance `tct-trs` is more powerful than its predecessor, cf. the last year’s TERMCOMP, where both the old and new version competed against each other. Furthermore, the actual strength of the

³ See <http://termination-portal.org/wiki/Termination.Competition.2015/>.

latest version of TCT shows when combining different modules into bigger ones, as we are going to show in the sequent case studies.

Integer Transition Systems. The `tct-its` module deals with the analysis of *integer transition systems* (ITSs for short). An ITS can be seen as a TRS over terms $f(x_1, \dots, x_n)$ where the variables x_i range over integers, and where rules are additionally equipped with a guard $\llbracket \cdot \rrbracket$ that determines if a rule triggers. The notion of runtime complexity extends straight forward from TRSs to ITSs. ITSs naturally arise from imperative programs using loops, conditionals and integer operations only, but can also be obtained from programs with user-defined data structures using suitable size-abstractions (see e.g. [22]). Consider the following program, that computes the remainder of a natural number m with respect to n .

```
int rem(int m, int n) { while (n > 0 && m > n) { m = m - n }; return m; }
```

This program is represented as the following ITS:

$$r(m, n) \rightarrow r(m - n, n) \llbracket n > 0 \wedge m > n \rrbracket \quad r(m, n) \rightarrow e(m, n) \llbracket \neg(n > 0 \wedge m > n) \rrbracket.$$

It is not difficult to see that the runtime complexity of the ITS, i.e. the maximal length of a computation starting from $r(m, n)$, is linear in m and n . The linear asymptotic bound is automatically derived by `tct-its`, in a fraction of a second. The complexity analysis of ITSs implemented by `tct-its` follows closely the approach by Brockschmidt et al. [10].

5.2 Real World Programs

One major motivation for the complexity analysis of abstract programs is that these models are well equipped to abstract over real-world programs whilst remaining conceptually simple.

Pure OCaml. For the case of higher-order functional programs, a successful application of this has been demonstrated in recent work by the first and second author in collaboration with Dal Lago [4]. In [4], we study the runtime complexity of pure OCaml programs. A suitable adaption of Reynold’s *defunctionalisation* [24] technique translates the given program into a slight generalisation of TRSs, an *applicative term rewrite system* (ATRS for short). In ATRSs closures are explicitly represented as first-order structures. Evaluation of these closures is defined via a global apply function (denoted by $\textcircled{\@}$).

The structure of the defunctionalised program is necessarily intricate, even for simple programs. However, in conjunction with a sequence of sophisticated and in particular complexity reflecting transformations one can bring the defunctionalised program in a form which can be effectively analysed by first-order complexity provers such as the `tct-trs` instance; see [4] for the details. An example run is depicted in Fig. 9. All of this has been implemented in a prototype implementation, termed HoCA.⁴ We have integrated the functionality

⁴ See <http://cbr.uibk.ac.at/tools/hoca/>.

(a) Reversing a list, taken from Bird’s textbook on functional programming [9].

```
let rec fold_left f acc = function
  [] → acc
  | x::xs → fold_left f (f acc x) xs ;;
let rev l = fold_left (fun xs x → x::xs) [] l ;;
```

(b) Defunctionalised applicative rewrite system.

$$\begin{array}{ll}
 \text{main}(x_0) \rightarrow m_1(x_0) \textcircled{f} & r(x_0) \textcircled{x_1} \rightarrow x_0 \textcircled{r_1} \textcircled{[]} \textcircled{x_1} \\
 m_1(x_0) \textcircled{x_1} \rightarrow m_2(x_0) \textcircled{r(x_1)} & r_1 \textcircled{x_0} \rightarrow r_2(x_0) \\
 m_2(x_0) \textcircled{x_1} \rightarrow x_1 \textcircled{x_0} & r_2(x_0) \textcircled{x_1} \rightarrow x_1 :: x_0 \\
 f \textcircled{x_0} \rightarrow f_1 \textcircled{x_0} & f_3(x_0, x_1) \textcircled{x_2} \rightarrow f_4(x_2, x_0, x_1) \\
 f_1 \textcircled{x_1} \rightarrow f_2(x_1) & f_4([], x_0, x_1) \rightarrow x_1 \\
 f_2(x_1) \textcircled{x_2} \rightarrow f_3(x_1, x_2) & f_4(x_0 :: x_1, x_2, x_3) \rightarrow f \textcircled{x_1} \textcircled{(x_2 \textcircled{x_3} \textcircled{x_0})} \textcircled{x_2}
 \end{array}$$

(c) Simplified first-order term rewrite system.

$$\text{main}(x_0) \rightarrow f([], x_0) \quad f(x_0, []) \rightarrow x_0 \quad f(x_0, x_1 :: x_2) \rightarrow f(x_1 :: x_0, x_2)$$

Fig. 9. Example run of the HoCA prototype on a OCaml program.

```
hoca :: Maybe String → Strategy ML TrsProblem
hoca name = mlToAtrs name >>> atrsToTrs >>> toTctProblem

mlToAtrs :: Maybe String → Strategy ML ATRS
mlToAtrs name = mlToPcf name >>> defunctionalise >>> try simplifyAtrs

atrsToTrs :: Strategy ATRS TRS
atrsToTrs = try cfa >>> uncurryAtrs >>> try simplifyTrs
```

Fig. 10. HoCA transformation pipeline modelled in tct-hoca.

of HoCA in the instance `tct-hoca`. The individual transformations underlying this tool are seamlessly modelled as processors, its transformation pipeline is naturally expressed in our strategy language. The corresponding strategy, termed `hoca`, is depicted in Fig. 10. It takes an OCaml source fragment, of type `ML`, and turns it into a term rewrite system as follows. First, via `mlToAtrs` the source code is parsed and desugared, the resulting abstract syntax tree is turned into an expression of a typed λ -calculus with constants and fixpoints, akin to Plotkin’s PCF [23]. All these steps are implemented via the strategy `mlToPcf :: Maybe String → Strategy ML TypedPCF`. The given parameter, an optional function name, can be used to select the analysed function. With `defunctionalise :: Strategy TypedPCF ATRS` this program is then turned into an ATRS, which is simplified via the strategy `simplifyAtrs :: Strategy ATRS ATRS` modelling the heuristics implemented in HoCA. Second, the strategy `atrsToTrs` uses the control-flow analysis provided by HoCA to *instantiate* occurrences of higher-order

```

jbc :: Strategy ITS () → Strategy TRS () → Strategy JBC ()
jbc its trs = toCTRS >>> Race (toIts >>> its) (toTrs >>> trs)

```

Fig. 11. `jat` transformation pipeline modelled in `tct-jbc`.

variables [4]. The instantiated ATRS is then translated into a first-order rewrite system by *uncurrying* all function calls. Further simplifications, as foreseen by the HoCA prototype at this stage of the pipeline, are performed via the strategy `simplifyTrs :: Strategy TRS TRS`.

Currently, all involved processors are implemented via calls to the library shipped with the HoCA prototype, and operate on exported data-types. The final strategy in the pipeline, `toTctProblem :: Strategy TRS TrsProblem`, converts HoCA’s representation of a TRS to a complexity problem understood by `tct-trs`. Due to the open structure of \mathcal{TCT} , the integration of the HoCA prototype worked like a charm and was finalised in a couple of hours. Furthermore, essentially by construction the strength of `tct-hoca` equals the strength of the dedicated prototype. An extensive experimental assessment can be found in [4].

Object-Oriented Bytecode Programs. The `tct-jbc` instance provides automated complexity analysis of object-oriented bytecode programs, in particular *Jinja bytecode* (JBC for short) programs [17]. Given a JBC program, we measure the maximal number of bytecode instructions executed in any evaluation of the program. We suitably employ techniques from data-flow analysis and abstract interpretation to obtain a *term based* abstraction of JBC programs in terms of *constraint term rewrite systems* (*cTRSs* for short) [20]. CTRSs are a generalisation of TRSs and ITSs. More importantly, given a cTRS obtained from a JBC program, we can extract a TRS or ITS fragment. All these abstractions are complexity reflecting. We have implemented this transformation in a dedicated tool termed `jat` and have integrated its functionality in `tct-jbc` in a similar way we have integrated the functionality of HoCA in `tct-hoca`. The corresponding strategy, termed `jbc`, is depicted in Fig. 11. We then can use `tct-trs` and `tct-its` to analyse the resulting problems. Our framework is expressive enough to analyse the thus obtained problems in parallel. Note that `Race s1 s2` requires that `s1` and `s2` have the same output problem type. We can model this with transformations to a dummy problem `()`. Nevertheless, as intended any witness that is obtained by an successful application of `its` or `trs` will be relayed back.

6 Conclusion

In this paper we have presented \mathcal{TCT} v3.0, the latest version of our fully automated complexity analyser. \mathcal{TCT} is open source, released under the BSD3 license. All components of \mathcal{TCT} are written in *Haskell*. \mathcal{TCT} is open with respect to the complexity problem under investigation and problem specific techniques. It is the most powerful tool in the realm of automated complexity analysis of term rewrite systems, as for example verified at this year’s TERMCOMP. Moreover

it provides an expressive problem independent strategy language that facilitates the proof search, extensibility and automation.

Further work will be concerned with the finalisation of the envisioned instance `tct-hrs`, as well as the integration of current and future developments in the resource analysis of ITSs.

References

1. Albert, E., Arenas, P., Genaim, S., Puebla, G., Román-Díez, G.: Conditional termination of loops over heap-allocated data. *SCP* **92**, 2–24 (2014)
2. Aspinall, D., Beringer, L., Hofmann, M., Loidl, H.W., Momigliano, A.: A program logic for resources. *TCS* **389**(3), 411–445 (2007)
3. Atkey, R.: Amortised resource analysis with separation logic. *LMCS* **7**(2), 1–33 (2011)
4. Avanzini, M., Dal Lago, U., Moser, G.: Analysing the complexity of functional programs: higher-order meets first-order. In: *Proceeding of the 20th ICFP*, pp. 152–164. ACM (2015)
5. Avanzini, M., Moser, G.: Tyrolean complexity tool: features and usage. In: *Proceeding of the 24th RTA, LIPIcs*, vol. 21, pp. 71–80 (2013)
6. Avanzini, M., Moser, G.: A combination framework for complexity. *IC* (to appear, 2016)
7. Avanzini, M., Sternagel, C., Thiemann, R.: Certification of complexity proofs using CeTA. In: *Proceeding of the 26th RTA, LIPIcs*, vol. 36, pp. 23–39 (2015)
8. Baader, F., Nipkow, T.: *Term Rewriting and All That*. Cambridge University Press, Cambridge (1998)
9. Bird, R.: *Introduction to Functional Programming using Haskell*, 2nd edn. Prentice Hall, Upper Saddle River (1998)
10. Brockschmidt, M., Emmes, F., Falke, S., Fuhs, C., Giesl, J.: Alternating runtime and size complexity analysis of integer programs. In: Ábrahám, E., Havelund, K. (eds.) *TACAS 2014 (ETAPS)*. LNCS, vol. 8413, pp. 140–155. Springer, Heidelberg (2014)
11. Danner, N., Paykin, J., Royer, J.S.: A static cost analysis for a higher-order language. In: *Proceeding of the 7th PLPV*, pp. 25–34. ACM (2013)
12. Gimenez, S., Moser, G.: The complexity of interaction. In: *Proceeding of the 40th POPL* (to appear, 2016)
13. Hirokawa, N., Moser, G.: Automated complexity analysis based on the dependency pair method. In: Armando, A., Baumgartner, P., Dowek, G. (eds.) *IJCAR 2008*. LNCS (LNAI), vol. 5195, pp. 364–379. Springer, Heidelberg (2008)
14. Hoffmann, J., Aehlig, K., Hofmann, M.: Multivariate amortized resource analysis. *TOPLAS* **34**(3), 14 (2012)
15. Hofmann, M., Moser, G.: Multivariate amortised resource analysis for term rewrite systems. In: *Proceeding of the 13th TLCA in LIPIcs* Vol. 38, pp. 241–256 (2015)
16. Jost, S., Hammond, K., Loidl, H.W., Hofmann, M.: Static determination of quantitative resource usage for higher-order programs. In: *Proceeding of the 37th POPL*, pp. 223–236. ACM (2010)
17. Klein, G., Nipkow, T.: A machine-checked model for a java-like language, virtual machine, and compiler. *TOPLAS* **28**(4), 619–695 (2006)
18. Lankford, D.: *On Proving Term Rewriting Systems are Noetherian*. Technical Report MTP-3. Louisiana Technical University (1979)

19. Moser, G.: Proof Theory at Work: Complexity Analysis of Term Rewrite Systems. CoRR abs/0907.5527, Habilitation Thesis (2009)
20. Moser, G., Schaper, M.: A Complexity Preserving Transformation from Jinja Bytecode to Rewrite Systems (2012). CoRR, cs/PL/1204.1568, last revision: 6 May 2014
21. Noschinski, L., Emmes, F., Giesl, J.: Analyzing innermost runtime complexity of term rewriting by dependency pairs. *JAR* **51**(1), 27–56 (2013)
22. Hill, P.M., Payet, E., Spoto, F.: Path-length analysis of object-oriented programs. In: Proceeding of the 1st EAAI. Elsevier (2006)
23. Plotkin, G.D.: LCF considered as a programming language. *TCS* **5**(3), 223–255 (1977)
24. Reynolds, J.C.: Definitional interpreters for higher-order programming languages. *HOSC* **11**(4), 363–397 (1998)
25. Sinn, M., Zuleger, F., Veith, H.: A simple and scalable static analysis for bound analysis and amortized complexity analysis. In: Biere, A., Bloem, R. (eds.) CAV 2014. LNCS, vol. 8559, pp. 745–761. Springer, Heidelberg (2014)
26. TeReSe: Term Rewriting Systems, Cambridge Tracks in Theoretical Computer Science, vol. 55. Cambridge University Press, Cambridge (2003)
27. Wilhelm, R., Engblom, J., Ermedahl, A., Holsti, N., Thesing, S., Whalley, D., Bernat, G., Ferdinand, C., Heckmann, R., Mitra, T., Mueller, F., Puaut, I., Puschner, P., Staschulat, J., Stenstrom, P.: The worst case execution time problem - overview of methods and survey of tools. *TECS* **7**(3), 1–53 (2008)