Length Estimation for Exponential Parameterization and ε-Uniform Samplings

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Abstract. This paper discusses the problem of estimating the length of the unknown curve γ in Euclidean space, from ε-uniformly (for $\varepsilon \geq 0$) sampled reduced data $Q_m = \{q_i\}_{i=0}^m$, where $\gamma(t_i) = q_i$. The interpolation knots $\{t_i\}_{i=0}^m$ are assumed here to be unknown (yielding the so-called non-parametric interpolation). We fit Q_m with the piecewise-quadratic interpolant $\hat{\gamma}_2$ combined with the so-called *exponential parameterization* (characterized by the parameter $\lambda \in [0,1]$). Such parameterization (applied e.g. in computer graphics for curve modeling [1], [2]) uses estimates of the missing knots $\{t_i\}_{i=0}^m \approx \{\hat{t}_i\}_{i=0}^m$. The asymptotic orders $\beta_\varepsilon(\lambda)$ for length estimation $d(\gamma) \approx d(\hat{\gamma}_2)$ in case of $\lambda = 0$ (uniformly guessed knots) read as $\beta_\varepsilon(0) = \min\{4, 4\varepsilon\}$ (for $\varepsilon > 0$) - see [3]. On the other hand $\lambda = 1$ (cumulative chords) renders $\beta_\varepsilon(1) = \min\{4, 3+\varepsilon\}$ (see [4]). A recent result [5] proves that for all $\lambda \in [0,1)$ and ε-uniform samplings, the respective orders amount to $\beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$. As such $\beta_\varepsilon(\lambda)$ are independent of $\lambda \in [0,1)$. In addition, the latter renders a discontinuity in asymptotic orders $\beta_\varepsilon(\lambda)$ at $\lambda = 1$. In this paper we verify experimentally the above mentioned theoretical results established in [5].

Keywords: Length estimation, interpolation, numerical analysis, computer graphics and vision.

1 Introduction

In classical non-parametric interpolation (see e.g. [6]) the sampled data points $Q_m = \{q_i\}_{i=0}^m$ satisfying $\gamma(t_i) = q_i \in \mathbb{R}^n$ yield the following pair $(\{t_i\}_{i=0}^m, Q_m)$ commonly known as *non-reduced data*. For the need of this paper, we also stipulate that $t_i < t_{i+1}$, $q_i \neq q_{i+1}$ and that $\gamma : [0,T] \to \mathbb{R}^n$ (with $0 < T < \infty$) is a sufficiently smooth (specified later) regular curve $\dot{\gamma}(t) \neq \mathbf{0}$. Recall that the length of the curve γ is defined as:

$$d(\gamma) = \int_0^T \|\dot{\gamma}(t)\| dt. \tag{1}$$

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In order to estimate the length $d(\gamma)$ of the unknown curve γ we first fit the available data (both non-reduced and reduced) with a specific interpolant $\bar{\gamma} : [0,T] \to \mathbb{R}^n$, chosen here as a piecewise-r-degree Lagrange polynomial - see [6]. The length $d(\bar{\gamma})$ is in turn used in this paper to estimate the searched length $d(\gamma)$ of γ. Consequently, two questions arise. Namely, first whether the approximation $d(\gamma) \approx d(\bar{\gamma})$ holds at all and then, if so how quickly it eventuates i.e. what the respective asymptotic orders are for the estimation of $d(\gamma)$.

To secure any kind of convergence for $d(\gamma) \approx d(\bar{\gamma})$ it is necessary to stipulate that sampling $\{t_i\}_{i=0}^m$ satisfies the so-called *admissibility condition*:

$$\lim_{m\to\infty} \delta_m = 0, \quad \text{where} \quad \delta_m = \max_{0\leq i\leq m-1} (t_{i+1} - t_i). \tag{2}$$

This class of samplings is denoted here by $\in V_G^m$ (see [7]). From now on, the subscript m in δ_m is omitted by setting $\delta = \delta_m$. We consider here two substantial subfamilies of V_G^m.

The *first one* $V_{mol}^m \subset V_G^m$ refers to the so-called *more-or-less uniform samplings* [7], [8]:

$$\beta\delta \leq t_{i+1} - t_i \leq \delta, \tag{3}$$

for some $\beta \in (0,1]$. The left inequality in (3) excludes samplings with distance between consecutive knots less then $\beta\delta$. The right inequality of (3) follows from (2). Condition (3), as shown in [7], can be replaced by the equivalent condition (4) holding for each $i = 0, 1, \ldots m-1$ and some constants $0 < K_1 \leq K_2$:

$$\frac{K_1}{m} \leq t_{i+1} - t_i \leq \frac{K_2}{m}. \tag{4}$$

The *second subfamily* $V_\varepsilon^m \subset V_G^m$ includes the so-called ε*-uniform samplings* [3]:

$$t_i = \phi(\frac{iT}{m}) + O(\frac{1}{m^{1+\varepsilon}}), \tag{5}$$

where $\varepsilon \geqslant 0$, $\phi : [0,T] \to [0,T]$ is smooth and $\dot{\phi} > 0$ (so that $t_i < t_{i+1}$). Clearly, the smaller ε gets, the bigger distortion of uniform distribution occurs (modulo mapping ϕ). The case when $\varepsilon = 0$ requires special care enforcing the inequality $t_i < t_{i+1}$ to hold. The latter is asymptotically guaranteed for all $\varepsilon > 0$. We remark here that each ε-uniform sampling with $\varepsilon > 0$ is also more-or-less uniform [7]. Note also that the second term in (5) can be substituted by $O(\delta^{1+\varepsilon})$.

2 Problem Formulation and Motivation

A standard result for *non-reduced data* $(\{t_i\}_{i=0}^m, Q_m)$ (handling general admissible samplings (2)) and for piecewise-r-degree polynomial $\bar{\gamma} = \tilde{\gamma}_r$ reads (see e.g. [6] or [7]):

Theorem 1. *Let $\gamma \in C^{r+1}$ be a regular curve $\gamma : [0,T] \to \mathbb{R}^n$ with knot parameters $\{t_i\}_{i=0}^m \in V_G^m$ given and $m\delta = O(1)$. Then a piecewise-r-degree Lagrange* *polynomial interpolation* $\tilde{\gamma}_r$ *used with* $\{t_i\}_{i=0}^m$ *given, yields a sharp length's estimate:*

$$d(\tilde{\gamma}_r) - d(\gamma) = O(\delta^{r+1}). \tag{6}$$

By (6) piecewise-quadratics (-cubics) $\tilde{\gamma}_2$ ($\tilde{\gamma}_3$) secure *cubic* (*quartic*) error orders in length approximation, respectively.

The claim of Th. 1 can be improved once ε-uniform samplings, r set as even and higher degree of smoothness of γ are admitted (see [7] or [9]):

Theorem 2. *Let* $\gamma \in C^{r+2}$ *be a regular curve* $\gamma : [0, T] \to \mathbb{R}^n$ *with knot parameters* $\{t_i\}_{i=0}^m \in V_\varepsilon^m$ *given (here* $\varepsilon \geq 0$*). Then a piecewise-r-degree Lagrange polynomial interpolation* $\tilde{\gamma}_r$ *used with* $\{t_i\}_{i=0}^m$ *known, renders a sharp length's estimate:*

$$d(\tilde{\gamma}_r) - d(\gamma) = \begin{cases} O(\delta^{r+1}) & \textit{for } r \geq 1 \textit{ odd,} \\ O(\delta^{\min\{r+2, r+1+\varepsilon\}}) & \textit{for } r \geq 2 \textit{ even.} \end{cases} \tag{7}$$

Visibly, upon inspecting (6) and (7) for $r = 2$ (i.e. piecewise-quadratic $\tilde{\gamma}_2$) the accelerated orders $3 + \varepsilon$ range from 3 to 4 (once $0 \leq \varepsilon \leq 1$), accordingly.

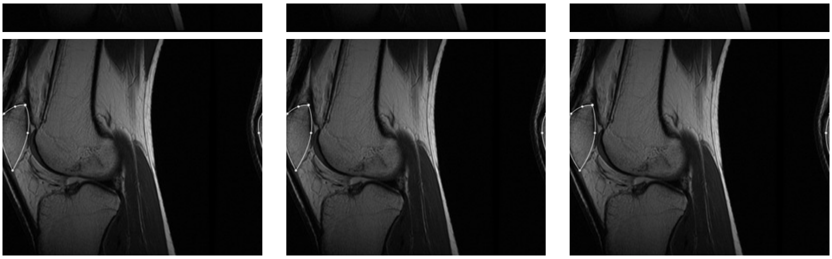
In many applications in computer graphics and vision, engineering or physics, one deals exclusively with the *reduced data* Q_m (see e.g. [1], [2], [10] or [11]). Here the corresponding interpolation knots $\{t_i\}_{i=0}^m$ are not available and as such they need first to be estimated somehow. A family of the so-called *exponential parameterization* $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$ is often invoked then and applied e.g. for curve modeling [2], [12]:

$$\hat{t}_0 = 0, \qquad \hat{t}_{i+1} = \hat{t}_i + \|q_{i+1} - q_i\|^\lambda, \tag{8}$$

where $0 \leq \lambda \leq 1$ and $i = 0, 1, \ldots, m-1$. The special cases when $\lambda \in \{1, 0.5, 0\}$, yield *cumulative chords*, *centripetal* or *uniform parameterizations*, respectively.

We call a piecewise-r-degree polynomial based on (8) and Q_m as $\bar{\gamma} = \hat{\gamma}_r : [0, \hat{T}] \to \mathbb{R}^n$, where $\hat{T} = \sum_{i=0}^{m-1} \|q_{i+1} - q_i\|^\lambda$. Note that in case of any reduced data Q_m for asymptotic estimation of γ (or of $d(\gamma)$) by $\hat{\gamma}_r$ (or by $d(\hat{\gamma}_r)$), *a re-parameterization* $\psi : [0, T] \to [0, \hat{T}]$ synchronizing both domains of γ and $\hat{\gamma}_r$, needs to be defined (see e.g. [7]). Although both $\hat{\gamma}_r$ and $d(\hat{\gamma}_r)$ are perfectly calculable over an external domain $[0, \hat{T}]$, the underpinning analysis to compare them with γ and $d(\gamma)$ (both defined over internal domain $[0, T]$) relies on finding some ψ as a re-parameterization.

Example 1. This example demonstrates the influence of knots selection for the specific application in computer vision. Figure 1 shows the image of the same knee joint section. One of the goal here is to isolate the kneecap and to estimate the length of its boundary. The interpolation points Q_m positioned on the boundary are selected e.g. by the physician (here $m = 5$) - see Figure 1. Evidently, the internal parameterization of the kneecap boundary (i.e. some curve γ) is a priori unknown. Upon fitting reduced data Q_m with different $\hat{\gamma}_2^\lambda$ in accordance to (8) (in fact here with three quadratic segments) one finds different boundaries $\hat{\gamma}_2^\lambda$ approximating the unknown curve γ (see also [13]). Then, the corresponding lengths $d(\hat{\gamma}_2^\lambda)$ estimating $d(\gamma)$ are computed. Namely for $\lambda \in \{0, 1/4, 1/2, 3/4, 1\}$ the following $d(\gamma) \approx d(\hat{\gamma}_2^\lambda) \in \{301.963, 301.198, 300.942, 301.082, 301.606\}$ (in pixel's side length) holds, respectively. Obviously the data Q_5 used here are sparse (since m is not large). Still the asymptotic analysis determining the respective convergence order for different $\lambda \in [0, 1]$ can be applied. Namely, by default it refers evidently to any admissible dense data Q_m. However, if only sparse Q_m forms the data, the resulting highest convergence order for $\lambda \in [0, 1]$ (see next section) usually renders a better approximation of length. □



a) b) c)

Fig. 1. Isolating the kneecap with $\hat{\gamma}_2$, for a) $\lambda = 0$, b) $\lambda = 0.5$, c) $\lambda = 1$

In Example 3 (see Figure 2) the specific reduced data Q_{22} (generated by specially sampled spiral γ_{sp}) are fitted with $\hat{\gamma}_2$ for either $\lambda = 0$ or $\lambda = 1$ rendering a pair of noticeably different interpolants having two unequal lengths. The latter reiterates the importance of the knot selection $\{\hat{t}_i\}_{i=0}^m \approx \{t_i\}_{i=0}^m$ in estimating both trajectory of γ and its length $d(\gamma)$.

More *real data examples* emphasizing the importance of the knots' selection for a given interpolation scheme in *computer graphics* (light-source motion estimation or image rendering), *computer vision* (image segmentation or video compression), *geometry* (trajectory, curvature or area estimation by e.g. resorting to the Green's Th.) or in *engineering and physics* (fast particles' motion estimation) can be found among all in [1].

We pass now to the parametric interpolation based on exponential parameterization.

2.1 Uniform Parameterization - $\lambda = 0$

The case when $\lambda = 0$, yields in (8) *a uniform* knots' guesses $\hat{t}_i = i$. No account is taken here for the geometrical distribution of Q_m. For $r = 2$ and $\lambda = 0$ in (8) the following result holds (see [3] or [7]):

Theorem 3. *Let the unknown* $\{t_i\}_{i=0}^m$ *be sampled* ε*-uniformly, where* $\varepsilon > 0$ *and* $\gamma \in C^4$*. Then there is a uniform piecewise-quadratic Lagrange interpolant* $\hat{\gamma}_2 : [0, \hat{T} = m] \to \mathbb{R}^n$*, calculable in terms of* Q_m *(with* $\hat{t}_i = i$*) and piecewise* C^∞ *re-parameterization* $\psi : [0, T] \to [0, \hat{T}]$ *such that the following estimates hold:*

$$d(\hat{\gamma}_2) - d(\gamma) = O(\delta^{\min\{4, 4\varepsilon\}}). \tag{9}$$

Note that Th. 3 with $\beta_{\varepsilon>0}(0) = \min\{4, 4\varepsilon\}$ extends to $\varepsilon = 0$ with $\beta_0(0) = 0$ provided ψ is a re-parameterization, the sampling $\{t_i\}_{i=0}^m$ satisfies $t_i < t_{i+1}$ and falls also into more-or-less uniformity (3) - see [7]. Thus convergence versus divergence duality may occur for $\varepsilon = 0$ - see [7]. An inspection of (9) reveals also that for ε-uniform samplings the respective orders in length estimation vary from $\beta_{\varepsilon=0}(0) = 0$ via $\beta_{0<\varepsilon<1}(0) = 4\varepsilon$ to $\beta_{\varepsilon\geq 1}(0) = 4$. The above characteristics are verified experimentally in [3] or [7]. However, it should be underlined here that the conducted numerical tests do not confirm the sharpness of (9). In fact, the numerically computed rates $\bar{\beta}_{0<\varepsilon<1}(0)$ are faster than orders in (9). Nevertheless, the latter is still consistent with the claim of Th. 3 (which only sets the lower bounds for the convergence orders in length estimation).

2.2 Cumulative Chords - $\lambda = 1$

The case when $\lambda = 1$ in (8) yields *cumulative chords* [2], [11]. Such estimates $\{\hat{t}_i\}_{i=0}^m$ of the unknown knots $\{t_i\}_{i=0}^m$ incorporate the geometry of Q_m and as such render much better asymptotic orders for length estimation (at least for $r = 2, 3$) as opposed to $\lambda = 0$ and (9) (see [4] and [7]). Indeed we have:

Theorem 4. *Let* γ *be a regular* C^k *curve in* $\mathbb{R}^n$*, where* $k \geqslant r + 1$ *and* $r = 2, 3$ *sampled according to (2). Let* $\hat{\gamma}_r : [0, \hat{T} = \sum_{i=0}^{m-1} \|q_{i+1} - q_i\|] \to \mathbb{R}^n$ *be the cumulative chord piecewise-quadratic(-cubic) interpolant defined by* Q_m *and* $\lambda = 1$ *in (8). Assume also that* $m\delta = O(1)$*. Then there is a piecewise-*C^r *re-parameterization* $\psi : [0, T] \to [0, \hat{T}]$*, with sharp orders*

$$d(\hat{\gamma}_r) - d(\gamma) = O(\delta^{r+1}). \tag{10}$$

In addition, if $k \geq 4$*,* $r = 2$ *and* $\{t_i\}_{i=0}^m$ *is* ε*-uniform (with* $\varepsilon \geq 0$*) then the following sharp estimate holds:*

$$d(\hat{\gamma}_2) - d(\gamma) = O(\delta^{\min\{4, 3+\varepsilon\}}). \tag{11}$$

Interestingly, contrary to Th. 1, any increment of $r \geq 4$ does not improve the convergence orders from (10). Visibly, for $r = 2$ formula (10) yields the cubic order 3 which not only improves (9) but also matches the non-reduced data case (6) (with $r = 2$). Finally, (11) yields accelerated convergence orders $\beta_\varepsilon(1) = \min\{4, 3+\varepsilon\}$) which again coincide with non-reduced data case claimed by (7). The relevant numerical tests confirming the sharpness of (10) and (11) are conducted in [4] and [7].

2.3 Exponential Parameterization - $\lambda \in [0, 1]$

Recent research by [5] extends the results for $r = 2$ claimed by Th. 3 (where $\lambda = 0$) and by Th. 4 (where $\lambda = 1$) to the remaining cases of exponential parameterization (8) i.e. to all $\lambda \in [0, 1]$. Indeed the following result holds:

Theorem 5. *Suppose γ is a regular C^4 curve in $\mathbb{R}^n$ sampled ε-uniformly (5) (with $\varepsilon > 0$). Let $\hat{\gamma}_2 : [0, \hat{T} = \sum_{i=0}^{m-1} \|q_{i+1} - q_i\|^\lambda] \to \mathbb{R}^n$ be the piecewise-quadratic interpolant defined by Q_m and (8) (with $\lambda \in [0, 1]$). Then there is a piecewise-C^∞ re-parameterization $\psi : [0, T] \to [0, \hat{T}]$, such that:*

$$d(\hat{\gamma}_2) - d(\gamma) = \begin{cases} O(\delta^{\min\{4,4\varepsilon\}}) & \text{for } \lambda \in [0, 1), \\ O(\delta^{\min\{4,3+\varepsilon\}}) & \text{for } \lambda = 1. \end{cases} \tag{12}$$

For general class of admissible samplings (2) (e.g. when also $\varepsilon = 0$) and $\lambda = 1$ with $m\delta = O(1)$ the length is approximated modulo $O(\delta^3)$ term. In addition, for $\{t_i\}_{i=0}^m$ uniform (i.e. when in (5) $\phi \equiv id$ and the term $O(m^{1+\varepsilon})$ vanishes) we have:

$$d(\hat{\gamma}_2) - d(\gamma) = O(\delta^4). \tag{13}$$

We remark here that for each $\varepsilon > 0$ and $\lambda \in [0, 1)$ the piecewise-quadratics $\psi : [0, T] \to [0, \hat{T}]$ introduced in Th. 5 define genuine re-parameterizations (see [14]). As already indicated in this paper, the latter is independently established for $\lambda = 1$ (in fact for general admissible samplings (2) - see [4]), and for $\lambda = 0$ (see [3]) and finally for uniform samplings (see [7]).

However, in case of $\lambda \in [0, 1)$ the latter does not always hold for those 0-uniform samplings which are also more-or-less uniform - see [15]. The pertinent sufficient conditions guaranteeing ψ to define a genuine re-parameterization for such samplings are formulated in [5]. The proof of Th. 5 is then extendable to $\varepsilon = 0$ (for all $\lambda \in [0, 1)$ with samplings (3)) and yields in (12) the order $O(1)$ for length estimation (which renders a possible convergence versus divergence duality).

An inspection of Th. 5 underlines also two important underpinning features occurring for ε-uniform samplings and exponential parameterization, namely:

a) Firstly, the resulting *discontinuity* in convergence orders $\beta_\varepsilon(\lambda)$ at $\lambda = 1$ for all $\varepsilon \in [0, 1)$. The respective orders for length approximation jump here from 4ε to $3 + \varepsilon$.
b) Secondly, the orders $\beta_\varepsilon(\lambda)$ claimed by Th. 5 are *independent from parameter* $\lambda \in [0, 1)$ and remain merely as a function of ε.

This situation is similar to the behavior of asymptotic estimates established for the trajectory approximation, where the equation $\hat{\gamma}_2 \circ \psi - \gamma = O(\delta^{\alpha(\lambda)})$ is examined based on reduced data Q_m, exponential parameterization (8) and on either ε-uniform (see [14]) or on more-or-less uniform samplings (see [15]). More specifically, if $\{t_i\}_{i=0}^m \in V_{mol}^m$ then as proved in [15] we have $\alpha(\lambda) = 1$ (for $\lambda \in [0, 1)$) and $\alpha(1) = 3$ (in fact again $\lambda = 1$ admits general samplings

(2)). In addition, by [14] for $\{t_i\}_{i=0}^m \in V_\varepsilon^m$ the following orders prevail $\alpha_\varepsilon(\lambda) = \min\{3, 1+2\varepsilon\}$ for $\lambda \in [0,1)$ and $\alpha_\varepsilon(1) = 3$ (here if $\varepsilon = 0$, the function ψ must be a re-parameterization). The orders for trajectory estimation are proved theoretically (see [14] and [15]) and independently experimentally confirmed (see [13]) to be sharp.

2.4 Aim of This Research

In this paper we verify experimentally both properties *a)* and *b)* specified in the preceding Subsection 2.3. Additionally, we also test the following:

c) the sharpness of the asymptotics derived for length estimation in Th. 5.

Recall that by *asymptotic sharpness* one understands the existence of at least one curve $\gamma \in C^r([0,T])$ (with r set accordingly) sampled with some $\{t_i\}_{i=0}^m$ from a prescribed subfamily of V_G^m for which the asymptotic estimates in question are exactly matched (i.e. no faster asymptotics eventuates).

The tests performed in this paper include merely some planar and spatial curves (i.e. when $n = 2, 3$). However, it should be stressed at that point, that all presented here Theorems 1-5 admit any regular and sufficiently smooth curve γ in arbitrary euclidean space $\gamma : [0,T] \to \mathbb{R}^n$.

This paper includes also illustrative examples for real application of interpolating two dimensional reduced data. More real-life applications referring to n-dimensional reduced data Q_m fitted with piecewise-quadratics $\hat{\gamma}_2$ or any other interpolation schemes based on exponential parameterization (8) can be found e.g. in [1], [2] or [11].

3 Experiments

The tests conducted in this paper are performed in *Mathematica 9.0* (see [16]) and are run on a 2.4GHZ Intel Core 2 Duo computer with 8GB RAM. Note that since $T = \sum_{i=1}^m (t_{i+1} - t_i) \leq m\delta$ the following holds $m^{-\beta} = O(\delta^\beta)$, for $\beta > 0$. Thus, the verification of any asymptotics expressed in terms of $O(\delta^\beta)$ can be accomplished by inspecting first the claims of Th. 5 in terms of $O(1/m^\beta)$ asymptotics.

Recall that for a parametric regular curve $\gamma : [0,T] \to \mathbb{R}^n$, $\lambda \in [0,1]$ with m varying between $m_{min} \leq m \leq m_{max}$ the i-th component of the error for $d(\gamma)$ estimation (over $[t_i, t_{i+2}]$) reads as:

$$E_m^i = \int_{\hat{t}_i}^{\hat{t}_{i+2}} \|\hat{\gamma}_2^i(\hat{s})\| d\hat{s} - \int_{t_i}^{t_{i+2}} \|\dot{\gamma}(s)\| ds = d(\hat{\gamma}_2^i) - d(\gamma|_{[t_i, t_{i+2}]}),$$

where $\hat{\gamma}_2^i : [t_i, t_{i+2}] \to \mathbb{R}^n$ is a Lagrange quadratic satisfying $\hat{\gamma}_2|_{[t_i,t_{i+2}]} = \hat{\gamma}_2^i$. We implicitly assume here that the sufficient conditions for Th. 5 guaranteeing ψ to be a re-parameterization (see [5]) are fulfilled. The latter is automatically satisfied by [14] for $\varepsilon > 0$ and $\lambda \in [0,1]$ or by [4] for $\lambda = 1$ and samplings

(2). Obviously, the quantity E_m defined as a sum of E_m^i represents the searched error $d(\hat{\gamma}_2) - d(\gamma) = O(\delta^{\beta(\lambda)})$ in length approximation of curve γ. From the set of *absolute errors* $\{E_m\}_{m=m_{min}}^{m_{max}}$ the numerical estimate $\bar{\beta}(\lambda)$ of genuine order $\beta(\lambda)$ is next computed by using *a linear regression* applied to the pair of points $\mathcal{A} = \{(\log(m), -\log(E_m))\}_{m=m_{min}}^{m_{max}}$ (see also [7]). Since piecewisely $deg(\hat{\gamma}_2) = 2$ the number of interpolation points $\{q_i\}_{i=0}^m$ is odd i.e. $m = 2k$ is even as indexing runs over $0 \leq i \leq m$. The *Mathematica* built-in functions *LinearModelFit* yields the coefficient $\bar{\beta}(\lambda)$ from the computed regression line $y(x) = \bar{\beta}(\lambda)x + b$ based on $\mathcal{A}$. Two special collections of ε-uniform samplings are used here for our experimentation. Namely, the first one reads as

$$t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{m^{1+\varepsilon}}. \tag{14}$$

The second one is defined as follows:

$$t_i = \begin{cases} \frac{i}{m} & \text{if } i \text{ even,} \\ \frac{i}{m} + \frac{1}{2m^{1+\varepsilon}} & \text{if } i = 4k+1, \\ \frac{i}{m} - \frac{1}{2m^{1+\varepsilon}} & \text{if } i = 4k+3. \end{cases} \tag{15}$$

For both (14) and (15) we set $t_0 = 0$ and $t_m = 1$, and hence $t_i \in [0,1]$. The examples of some $\{\gamma(t_i)\}_{i=0}^m$ distribution are presented later in Figures 2 and 3. Recall that for all $\varepsilon > 0$ (and $\lambda \in [0,1)$) or $\lambda = 1$ or $\{t_i\}_{i=0}^m$ uniform function ψ from Th. 5 is a re-parameterization and therefore both (12) and (13) apply. The case of $\varepsilon = 0$ renders both (14) and (15) as more-or-less uniform samplings (4) with either $K_1 = 1/3$ and $K_2 = 5/3$ or $K_1 = 1/2$ and $K_2 = 3/2$, accordingly. Sufficient conditions for ψ_i to be a re-parameterization are formulated in [5]. The latter enables to test the validity of Th. 5 also for $\varepsilon = 0$.

We pass now to the experiments designed to verify the convergence orders claimed by (12) and (13). It should however, be emphasized that since both these formulas have asymptotic character, they are only relevant for sufficiently large $m \geq m_0$. Here m_0 is unknown unless non-trivial analysis revisiting the proof of Th. 5 is supplemented. Consequently, the lower bound m_{min} cannot be selected as too small (in fact we ought to have $m_0 \leq m_{min}$). On the other hand, if m_{max} is too large machine errors may distort the entire computation. In particular, in an effort of avoiding machine errors it is quite feasible that inequality $m_{max} < m_0$ is reached. The latter naturally impacts on any reliable interpretation of the computed lengths' estimates. Hence the selection of both bounds (m_{min}, m_{max}) should be made with special care. One possible approach is to plot the set $\mathcal{A}$ in the preliminary step and to confirm whether its points are well concentrated along a certain line. Given above, it is evident therefore that a linear regression applied to $\mathcal{A}$ and $m_{min} \leq m \leq m_{max}$ to verify *a)-c)* should be treated cautiously and should serve rather as a numerical guidance complementing merely a solid mathematical proof for *a)-c)*.

3.1 Length Estimation for Reduced Data from Planar Curves

The first test is performed for length estimation of the cubic curve in $\mathbb{R}^2$.

Example 2. Consider now the following regular *cubic curve* $\gamma_c : [0,1] \to \mathbb{R}^2$: $\gamma_c(t) = (\pi t, (\pi t + 1)^3(\pi + 1)^{-3})$, sampled according to either (14) or (15). For the first sampling we set $m_{min} = 40$ and $m_{max} = 200$ whereas for the second one $m_{min} = 100$ and $m_{max} = 120$. The corresponding length of γ_c reads as $d(\gamma_c) = 3.452$. The linear regression applied to $m_{min} \leq m \leq m_{max}$ renders computed $\bar{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ ($\varepsilon \geq 0$), which are listed in Table 1 and Table 2.

Table 1. Estimated $\bar{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ for γ_c and sampling (14) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in [0,2]$

λ	$\varepsilon = 0.0$	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$	$\varepsilon = 2.0$
$\beta_\varepsilon(\lambda)$	0.000	0.400	1.320	2.000	2.800	3.600	4.000	4.000
0.00	-0.048	2.597	2.800	3.064	3.456	3.857	4.043	4.095
0.10	-0.057	2.601	4.977	3.686	4.033	3.981	4.058	4.001
0.33	2.003	2.183	2.640	2.664	3.333	3.702	3.963	3.890
0.50	1.996	2.196	2.646	2.971	3.346	3.730	3.992	3.909
0.70	1.996	2.196	2.644	2.969	3.340	3.718	3.982	3.902
0.90	1.900	2.194	2.629	2.936	3.265	3.363	4.139	3.814
$\beta_\varepsilon(1)$	3.000	3.100	3.330	3.500	3.700	3.900	4.000	4.000
1.00	2.992	3.111	3.364	3.541	3.749	3.954	4.056	7.070

Table 2. Estimated $\bar{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ for γ_c and sampling (15) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in [0,2]$

λ	$\varepsilon = 0.0$	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$	$\varepsilon = 2.0$
$\beta_\varepsilon(\lambda)$	0.000	0.400	1.320	2.000	2.800	3.600	4.000	4.000
0.00	1.787	2.379	2.743	3.051	3.449	3.865	4.067	4.044
0.10	2.018	2.555	1.196	2.862	3.287	3.480	3.374	4.001
0.33	2.018	2.216	2.283	3.026	3.431	3.835	4.025	4.037
0.50	2.018	2.221	2.684	3.027	3.432	3.837	4.029	4.037
0.70	2.018	2.221	2.684	3.028	3.431	3.836	4.027	4.036
0.90	2.018	2.223	2.684	2.025	3.426	3.823	4.008	4.032
$\beta_\varepsilon(1)$	3.000	3.100	3.330	3.500	3.700	3.900	4.000	4.000
1.00	4.046	4.157	4.404	4.578	4.773	4.971	5.078	3.974

Visibly, *the sharpness* of Th. 5 for either $\lambda = 1$ with $\varepsilon \geq 0$ or for $\lambda \in [0,1)$ with $\varepsilon = 1$ is confirmed in Table 1 (see the last row and the last column, respectively). In addition, an inspection of the second column of Table 1 demonstrates the divergence versus convergence duality (see negative and positive $\bar{\beta}_0(\lambda)$) which is expected to occur for $\varepsilon = 0$. Such duality is not transparent in Table 2. On the other hand, Table 2 shows more clearly the discontinuity in convergence orders $\beta_\varepsilon(\lambda)$ at $\lambda = 1$ for $\varepsilon \in [0, 1)$, predicted by Th. 5. Table 2 underlines also (see each column) the expected independence of $\beta_\varepsilon(\lambda)$ on λ once ε is fixed. Both Tables 1 and 2 show also that for λ fixed, increasing ε from 0 to 1 makes $\bar{\beta}_\varepsilon(\lambda)$ bigger and closer to 4. Upon satisfying $\varepsilon \geq 1$ the quartic orders in convergence are reached. The latter coincides with the asymptotics held by $\{t_i\}_{i=0}^m$ uniform. However, the results obtained in both Tables 1 and 2 for $\lambda \in [0, 1)$ suggest faster convergence rates $\beta_\varepsilon(\lambda)$ as compared to (12). The latter even if indeed true, does not stand in contradiction with Th. 5, which merely determines the slowest possible rates $\beta_\varepsilon(\lambda)$ of convergence in length estimation of γ with the aid of $\hat{\gamma}_2$ and exponential parametrization (8). In that sense all results from Tables 1 and 2 are consistent with Th. 5. In case of $\{t_i\}_{i=0}^m$ uniform the computed convergence orders $\bar{\beta} \approx 4$ (see (13)) read for $\lambda \in \{0, 0.1, 0.33, 0.5, 0.7, 0.9, 1\}$ as $\{4.036, 4.044, 4.036, 4.037, 4.037, 4.039, 4.074\}$, respectively. Evidently the sharpness of (13) is experimentally confirmed. One final remark should also be made here having in mind the asymptotic character of Th. 5. Evidently, during the computation procedure one may try to adjust experimentally (m_{min}, m_{max}) accordingly to each choice of (ε, λ) in order to illustrate more precisely *a)-c)*. □

The next example refers again to the spiral curve in $\mathbb{R}^2$.

Example 3. i) Let *a planar regular convex spiral* $\gamma_{sp} : [0, 5\pi] \to \mathbb{R}^2$: $\gamma_{sp}(t) = ((6\pi - t)\cos(t), (6\pi - t)\sin(t))$ be sampled according to either (14) (rescaled by factor 5π) with $t_0 = 0$ and $t_m = 5\pi$. Figure 2 illustrates γ_{sp} (a dashed line) and $\hat{\gamma}_2$ (a continuous line) coupled with (14), for $\varepsilon = 0.33$, $m = 22$ and $\lambda \in \{0, 1\}$. The difference between γ_{sp} and $\hat{\gamma}_2^\lambda$ on reduced data Q_{22} is transparent (at least for $\lambda = 0$). The length of γ_{sp} amounts to $d(\gamma_{sp}) = 173.608$ which is approximated here by $d(\hat{\gamma}_2^{\lambda=0}) = 173.109$ and $d(\hat{\gamma}_2^{\lambda=1}) = 172.900$, respectively. This shows the impact on trajectory and length estimation upon selecting various $\lambda \in [0, 1]$ for parametric interpolation $\hat{\gamma}_2$ based on reduced data. Note that as shown on sparse data worse trajectory can give a better length approximation.

ii) Consider another *planar regular spiral* $\gamma_{sp1} : [0, 1] \to \mathbb{R}^2$: $\gamma_{sp1}(t) = ((t + 0.2)\cos(\pi(1 - t)), (t + 0.2)\sin(\pi(1 - t)))$. To estimate $\beta_\varepsilon(\lambda)$ a linear regression is applied again to $100 = m_{min} \leq m \leq m_{max} = 120$ and to ε-uniform samplings (15). The pertinent numerical results for $\bar{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda)$ are listed in Table 3. Again all issues raised and confronted in Example 2 are also inferable from Table 3. In particular, the computed orders $\bar{\beta}_\varepsilon(\lambda)$ exceed (for $\lambda \in [0, 1]$ and $\varepsilon \in [0, 1)$) the convergence rates claimed by Th. 5, which again does not provide an argument for sharpness of (12) - at least for $\lambda \in [0, 1)$. However, still the inequality $\bar{\beta}_\varepsilon(\lambda) \geq \beta_\varepsilon(\lambda)$ deems the results from Table 3 as consistent with (12). On the other hand, the sharpness is confirmed for $\lambda = 1$ with $\varepsilon \geq 0$ or for $\lambda \in [0, 1)$ with $\varepsilon \geq 1$. Visibly, each column (i.e. with fixed ε) and $\lambda \in (0, 1)$ shows almost equal $\bar{\beta}_\varepsilon(\lambda)$. Similarly, each row of Table 3 indicates the increasing tendency in values of $\bar{\beta}_\varepsilon(\lambda)$ while varying ε from 0 to 1. Once $\varepsilon = 1$ is reached, the orders $\bar{\beta}_1(\lambda) \approx 4$ are attained. In addition, the expected discontinuity of $\beta_\varepsilon(\lambda)$ at $\lambda = 1$ is also manifested upon inspecting

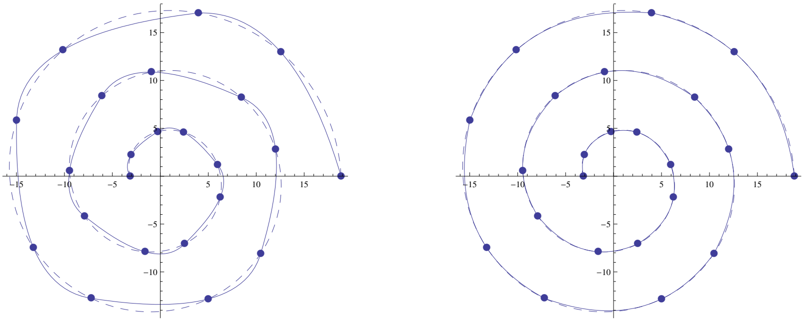


Fig. 2. The plot of the spiral γ_{sp} sampled as in (14) (a dashed line) and interpolant $\hat{\gamma}_2^{\lambda}$ (a continuous line), for $m = 22$ and $\varepsilon = 0.33$ with either a) $\lambda = 0$ or b) $\lambda = 1$

the last three rows of Table 3. Finally, the case when $\{t_i\}_{i=0}^{m}$ is uniform renders $\bar{\beta} \approx 4$ (see (13)) for different $\lambda \in \{0, 0.1, 0.33, 0.5, 0.7, 0.9, 1\}$ equal to $\{4.0352, 4.0351, 4.0350, 4.0349, 4.0348, 4.0348, 4.0348\}$, respectively. The sharpness of (13) is again experimentally confirmed. $\square$

Table 3. Estimated $\bar{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ for γ_{sp1} and sampling (15) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0,1]$ and $\varepsilon \in [0,2]$

λ	$\varepsilon = 0.0$	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$	$\varepsilon = 2.0$
$\beta_\varepsilon(\lambda)$	0.000	0.400	1.320	2.000	2.800	3.600	4.000	4.000
0.00	1.631	2.368	2.704	2.914	3.054	4.110	4.044	4.035
0.10	1.994	2.522	2.827	3.618	3.987	4.039	4.041	4.035
0.33	2.022	2.220	2.706	3.103	3.639	3.992	4.037	4.035
0.50	2.019	2.224	2.701	3.089	3.611	3.983	4.036	4.035
0.70	2.019	2.224	2.704	3.095	3.623	3.987	4.037	4.035
0.90	2.023	2.233	2.728	3.160	3.729	4.011	4.038	4.035
$\beta_\varepsilon(1)$	3.000	3.100	3.330	3.500	3.700	3.900	4.000	4.000
1.00	4.026	4.102	4.100	4.071	4.052	4.043	4.040	4.035

3.2 Length Estimation for Reduced Data from Spatial Curves

The last example copes with the reduced data Q_m generated by sampling a regular spatial curve in $\mathbb{R}^3$.

Example 4. i) Finally, we test the claims of Th. 5 for *a quadratic elliptical helix*: $\gamma_{h1}(t) = (2\cos(t), \sin(t), t^2)$, with $t \in [0, 2\pi]$ and sampled ε-uniformly according

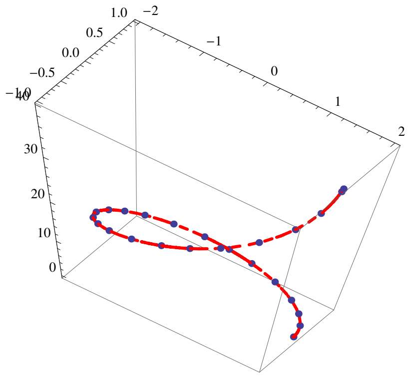


Fig. 3. The plot of the helix γ_{h1} sampled as in (15), for $m = 22$ and $\varepsilon = 0.5$

to (15) (again rescaled by factor 2π). Figure 3 illustrates the plot of γ_{h1} sampled by (15) for $\varepsilon = 0.5$ and $m = 22$.

ii) Consider now *an elliptical helix*: $\gamma_{h2}(t) = ((3/2)\cos(t), \sin(t), t/4)$, with $t \in [0, 2\pi]$ and sampled ε-uniformly according to (14) (similarly rescaled by factor 2π). The corresponding length of γ_{h2} reads as $d(\gamma_{2h}) = 8.090$. In order to approximate $\beta_\varepsilon(\lambda)$, the linear regression is applied with $m_{min} = 100 \leq m \leq m_{max} = 120$. The respective computed estimates $\bar{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \min\{4, 4\varepsilon\}$ are presented in Table 4. The results are consistent with the asymptotics from Th. 5, though the examined sharpness of (12) is again not experimentally confirmed (for $\lambda \in [0, 1)$). All other aspects raised in previous examples (including *a)-b)*) are again positively verified. Note that convergence versus divergence duality for $\varepsilon = 0$ reappears here as in Example 2. □

Table 4. Estimated $\bar{\beta}_\varepsilon(\lambda) \approx \beta_\varepsilon(\lambda) = \{4, 4\varepsilon\}$ for γ_{h2} and sampling (14) interpolated by $\hat{\gamma}_2$ with $\lambda \in [0, 1]$ and $\varepsilon \in [0, 2]$

λ	$\varepsilon = 0.0$	$\varepsilon = 0.1$	$\varepsilon = 0.33$	$\varepsilon = 0.5$	$\varepsilon = 0.7$	$\varepsilon = 0.9$	$\varepsilon = 1.0$	$\varepsilon = 2.0$
$\beta_\varepsilon(\lambda)$	0.000	0.400	1.320	2.000	2.800	3.600	4.000	4.000
0.00	-0.011	2.516	2.685	2.755	5.980	4.066	4.037	4.034
0.10	0.033	2.527	3.021	3.749	4.002	4.033	4.034	4.034
0.33	2.018	2.191	2.694	3.128	3.706	4.001	4.031	4.033
0.50	1.990	2.197	2.686	3.104	3.672	3.994	4.031	4.033
0.70	1.990	2.197	2.689	3.114	3.687	3.997	4.031	4.033
0.90	1.995	2.212	2.731	3.217	3.803	4.015	4.034	4.033
$\beta_\varepsilon(1)$	3.000	3.100	3.330	3.500	3.700	3.900	4.000	4.000
1.00	3.929	4.056	4.079	4.050	4.037	4.034	4.034	4.033

4 Conclusions

In this paper we discuss the problem of length estimation of the unknown curve γ sampled ε-uniformly (5) by using piecewise-quadratic interpolation $\hat{\gamma}_2$ based on reduced data Q_m. The latter is combined here with the application of the exponential parameterization (8) which depends on parameter $\lambda \in [0, 1]$.

Reduced data coupled with exponential parameterization are often invoked in computer graphics for curve modeling - see e.g. [2], [6], [10] or [11]. Special cases of (8) with $\lambda = 0$ (see e.g. [3]) or $\lambda = 1$ (see e.g. [4] or [7]) were earlier studied in the context of examining the asymptotics for trajectory and length estimation. Recent results by [14] and [15] with full mathematical proofs guarantee sharp asymptotics for the trajectory estimation covering the missing cases of $\lambda \in (0, 1)$.

In addition, the very recent work by [5] establishes (also by mathematical means) the corresponding asymptotics for length estimation $d(\gamma)$ of the unknown curve γ - see Th. 5 (holding for all $\lambda \in [0, 1]$). In this paper we experimentally verify the claim of (12) and (13) (including their sharpness) established in [5]. In particular, we address here three issues raised in Subsections 2.3 and 2.4 of this paper and listed as items *a)-c)*.

Various experiments conducted for the purpose of this research (see Section 3) seem to experimentally verify in affirmative both conjectures *a)-b)*. The last hypothesis *c)*, i.e. the sharpness of either (12) or (13) is only partially confirmed. Indeed, the cases when either $\lambda = 1$ or $\lambda \in [0, 1)$ with $\varepsilon \geq 1$ or $\{t_i\}_{i=0}^m$ is uniform yield $\bar{\beta}_\varepsilon(\lambda)$ very close to $\beta_\varepsilon(\lambda)$. On the other hand when $\lambda \in [0, 1)$ and $\varepsilon \in [0, 1)$ we obtain faster convergence rates, i.e. $\bar{\beta}_\varepsilon(\lambda) > \beta_\varepsilon(\lambda)$. Nevertheless, given the asymptotic character of Th. 5, all estimates $\bar{\beta}_\varepsilon(\lambda)$ derived in Section 3 for curves both in $\mathbb{R}^2$ or $\mathbb{R}^3$ are consistent with the asymptotics established in Th. 5.

In consequence, it remains still *an open question* whether indeed (12) is sharp or if not what the sharp or improved estimates for length approximation are.

The answer to the above question is equally important for dense and sparse data Q_m (with m either large or small, respectively). The asymptotic character of (12) applies by default to m large and thus also naturally to any dense data. However, a high convergence order usually yields also in practice a satisfactory approximation on sparse data. Hence the choice of cumulative chords (with λ in (8) set to 1) yielding quartic orders in (12) deems to be the most appropriate, unless other criteria like shape modeling are also accounted for.

A possible extension of this work is to study other smooth interpolation schemes [6] combined with reduced data Q_m and exponential parameterization (8) - see [2]. Certain clues may be given in [17], where complete C^2 splines are dealt with for $\lambda = 1$, to obtain the fourth orders of convergence in length estimation. The analysis of C^1 interpolation for reduced data with cumulative chords (i.e. again with $\lambda = 1$) can additionally be found in [7] or [18]. More discussion on *applications* (including *real data* examples - see [1]) and *theory* of *non-parametric interpolation* can be found e.g. in [2] [7], [10] or [11].

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