On the Non-additive Sets of Uniqueness in a Finite Grid

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Abstract. In Discrete Tomography there is a wide literature concerning (weakly) bad configurations. These occur in dealing with several questions concerning the important issues of uniqueness and additivity. Discrete lattice sets which are additive with respect to a given set S of lattice directions are uniquely determined by X-rays in the direction of S. These sets are characterized by the absence of weakly bad configurations for S. On the other side, if a set has a bad configuration with respect to S, then it is not uniquely determined by the X-rays in the directions of S, and consequently it is also non-additive. Between these two opposite situations there are also the non-additive sets of uniqueness, which deserve interest in Discrete Tomography, since their unique reconstruction cannot be derived via the additivity property. In this paper we wish to investigate possible interplays among such notions in a given lattice grid $\mathcal{A}$, under X-rays taken in directions belonging to a set S of four lattice directions.

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1 Introduction

In Discrete Tomography the usual line integrals employed in Computerized Tomography are replaced simply by the discrete X-rays, counting the number of points on each line parallel to given directions, so providing the so-called Discrete Radon Transform (DRT). The inversion of DRT aims to deduce the local atomic structure from the collected counting data. The original motivation came from High-Resolution Transmission Electron Microscopy (HRTEM) which is able to obtain images with atomic resolution and provides quantitative information on

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the number of atoms that lie in single atomic columns in crystals choosing main X-ray directions such as $(0,1),\ (1,0),\ (1,1),\ (1,2),\ldots$ to be resolvable by the microscopy (see [18–20]). The high energies required to produce the discrete X-rays of a crystal mean that only a small number of X-rays can be taken before the crystal is damaged. Therefore, DT focuses on the reconstruction of images with few different grey levels, and, in particular, on the reconstruction of binary images from a small number of X-rays. It is worth mentioning that this problem was considered in its pure mathematical form even before its connection with electron microscopy ([6]). Atoms are modeled by lattice points and so crystals by finite sets of lattice points. The tomographic grid is a finite set $\mathcal{G}$ of lattice points which are intersections of lines parallel to the X-ray directions corresponding to nonzero X-ray and feasible solutions of the reconstruction problem are subsets of $\mathcal{G}$ [15]. If there is only one solution the lattice set is $\mathcal{G}$-unique, or simply unique. On this regard, a special class of geometric objects, called *additive* sets, has been studied in considerable depth (see Section 2 for the formal definition). It was shown in [6] that a finite subset F of $\mathbb{Z}^2$ is uniquely determined by its X-rays in the coordinate directions if and only if F is additive. The sufficient condition was later extended to any dimension, pointing out that notions of additivity and uniqueness are equivalent when two directions are employed, whereas, for three or more directions, additivity is more demanding than uniqueness. Actually, every additive set is uniquely determined, but there are non-additive sets of uniqueness [7]. Further generalizations have been considered in [8], where the notion of additivity has been extended to n-dimension, with respect to a set of linear manifolds. The literature suggests that, without the additivity property, it may be quite difficult to decide whether a lattice set is uniquely determined by its X-rays taken in a set of more than three directions. In fact, the inversion of DRT is generally NP-hard ([17]), so that any reconstruction algorithm must consist of exponentially many steps in the size of F. One related problem is to find suitable sub-classes of lattice sets that can be reconstructed in polynomial time (see, for instance [2, 3]), or to provide uniqueness results from the a priori knowledge of the geometric features of the class (see [9, 10]). In general, uniqueness is not a property of the set S of X-ray directions, as for each S there exists a lattice set which is not uniquely determined by S. On the contrary if we restrict to bounded sets in a given rectangular grid $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \ :\ 0 \le i < m,\ 0 \le j < n\}$, there are whole families of lattice directions which uniquely determine all the finite subsets (so called bounded sets) of $\mathcal{A}$ [4]. Unless the tomographic grid $\mathcal{G}$ is contained in $\mathcal{A}$, uniqueness in $\mathcal{A}$ does not imply uniqueness in $\mathcal{G}$. Therefore it is interesting to try to understand which bounded sets uniquely determined by S in $\mathcal{A}$ are also $\mathcal{G}$-unique and/or additive. In some sense, roughly speaking, we measure "how strongly unique", they are. In Section 4 we classify all the bounded sets in a grid $\mathcal{A}$ which are uniquely determined by a set of four directions of uniqueness for $\mathcal{A}$ that contains the coordinate directions. In this important case, we also compute the proportion of non-additive sets of uniqueness with respect to additive sets in $\mathcal{A}$. This ratio depends only on the number of X-ray directions whereas it is independent on the size of $\mathcal{A}$, and hence it is constant in the case study. Finally, we show how to explicitly construct non-additive sets of uniqueness. These results partially answer an open question posed by Fishburn and Shepp in [8].

2 Background

Let $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and $a \geq 0$, with the further assumption that $b = 1$ if $a = 0$. We call (a, b) a *lattice direction*. By *lines with direction* $(a, b) \in \mathbb{Z}^2$ we mean lattice lines defined in the x, y plane by equations of the form $ay = bx + t$, where $t \in \mathbb{Z}$. We refer to a finite subset of $\mathbb{Z}^2$ as a *lattice set*.

Let $\mathcal{A} = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i < m, 0 \leq j < n\}$ be a finite grid. We refer to the lattice sets $E \subseteq \mathcal{A}$ as *bounded sets*. A bounded set $E \subset \mathcal{A}$ can be identified, in a natural way, with its characteristic function $\chi_E : \mathcal{A} \to \{0, 1\}$ defined by $\chi_E(i, j) = 1$ for $(i, j) \in E$, and $\chi_E(i, j) = 0$ otherwise.

For a function $f : \mathcal{A} \to \mathbb{Z}$, we write $|f| = \max_{(i,j)\in\mathcal{A}} \{|f(i, j)|\}$. Further, the *line sum* of f along the lattice line with equation $ay = bx + t$ is defined as $\sum_{aj=bi+t} f(i, j)$.

Given a lattice direction (a, b), the *X-ray* of E in the direction (a, b) is the function giving the number of points in E on each line parallel to (a, b). Two sets $E, F \subseteq \mathbb{Z}^2$ are said to be *tomographically equivalent* with respect to a set S of lattice directions if E and F have the same X-rays in the directions in S. A finite set $E \subseteq \mathbb{Z}^2$ is a *set of uniqueness* with respect to a set S of lattice directions, or simply S-*unique*, if E is uniquely determined by its X-rays taken in the directions belonging to S. In other words, if F is tomographically equivalent to E with respect to S, then $F = E$. Given a finite set S of lattice directions, we say that two functions $f, g : \mathcal{A} \to \{0, 1\}$ are *tomographically equivalent* if they have equal line sums along the lines corresponding to the directions in S. Note that two non trivial functions $f, g : \mathcal{A} \to \{0, 1\}$ which are tomographically equivalent can be interpreted as characteristic functions of two lattice sets which are tomographically equivalent.

Let (a, b) be a lattice direction. Set

$$f_{(a,b)}(x, y) = \begin{cases} x^a y^b - 1, & \text{if } a > 0, b > 0 \\ x^a - y^{-b}, & \text{if } a > 0, b < 0 \\ x - 1, & \text{if } a = 1, b = 0 \\ y - 1, & \text{if } a = 0, b = 1. \end{cases}$$

Given a finite set S of lattice directions, we denote by $F_S(x, y)$ the polynomial associated to S defined by (see [13, p. 19]):

$$F_S(x, y) = \prod_{(a,b)\in S} f_{(a,b)}(x, y) = \sum_{(i,j)\in\mathcal{A}} f(i, j) x^i y^j.$$

For any function $h : \mathcal{A} \to \mathbb{Z}$, its *generating function* is the polynomial defined by

$$G_h(x, y) = \sum_{(i,j)\in\mathcal{A}} h(i, j) x^i y^j.$$

Conversely, we say that the function h is *generated* by a polynomial $P(x,y)$ if $P(x,y) = G_h(x,y)$. Notice that the function f generated by the polynomial $F_S(x,y)$ vanishes outside $\mathcal{A}$ if and only if the set $S = \{(a_k,b_k)\}_{k=1}^d$ of d lattice directions satisfies the conditions

$$\sum_{k=1}^{d} a_k < m, \qquad \sum_{k=1}^{d} |b_k| < n. \tag{1}$$

We then say that a set $S = \{(a_k,b_k)\}_{k=1}^d$ of d lattice directions is *valid* for a finite grid $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \,:\, 0 \le i < m,\ 0 \le j < n\}$, if (1) holds. Moreover, the function f generated by $F_S(x,y)$ has zero line sums along the lines $ay = bx + t$ taken in the directions $(a,b) \in S$. For example, if $b \neq 0$

$$F_S(x, x^{-\frac{a}{b}}) = 0 = \sum_{(i,j)\in\mathcal{A}} f(i,j) x^i x^{-\frac{ja}{b}} = \sum_{\substack{(i,j)\in\mathcal{A}\\ aj=bi+t}} f(i,j) x^{-\frac{t}{b}} = \sum_t x^{-\frac{t}{b}} \sum_{\substack{(i,j)\in\mathcal{A}\\ aj=bi+t}} f(i,j).$$

The other cases can be obtained analogously.

Note that a generating function can be interpreted as follows: a monomial with its sign $h(i,j)x^i y^j \in \mathbb{Z}[x,y]$ is associated to the lattice point $\mathbf{p} = (i,j)$, together with the weight $h(i,j)$ which we call *multiplicity.* If $|h(i,j)| > 1$ we say that $\mathbf{p}$ is a *multiple point.* In order to simplify the notation furthermore we will denote a polynomial by $P(x,y)$, its associated lattice set by P, specifying the set of lattice points with positive (resp. negative) multiplicity by P^+ (resp. P^-). In particular we denote by F_S the set of lattice points associated to $F_S(x,y)$, counted with their multiplicities, namely

$$F_S = \left\{((i,j),\ l(i,j)) \in \mathbb{Z}^2 \times \mathbb{Z} \,:\, l(i,j) = f(i,j) \neq 0\right\}.$$

From the geometric point of view, F_S is a *S-weakly bad configuration*, namely a pair of lattice sets (Z,W) consisting of k lattice points $z_1,\dots,z_k \in Z$, and k points $w_1,\dots,w_k \in W$, not necessarily distinct (counted with multiplicity), such that for each direction $(a,b) \in S$, and for each $z_r \in Z$, the line through z_r in direction (a,b) contains a point $w_r \in W$ (see Figure 1). We then say that a lattice set E has a weakly bad configuration if a weakly bad configuration (Z,W) exists for some $k \ge 2$, such that $Z \subseteq E$, $W \subseteq \mathbb{Z}^2 \setminus E$.

If all the points in the sets Z,W are distinct, then (Z,W) is called *S-bad configuration.* This notion plays a crucial role in investigating uniqueness problems, since any set S of directions is a set of uniqueness if and only if it has no bad configurations [6, Theorem 1].

Consider now the following definition. A finite set $E \subseteq \mathbb{Z}^2$ is *additive with respect to S*, or simply *S-additive* if for each $u \in S$, there is a *ridge function* g_u, that is a function defined in $\mathbb{Z}^2$ which is constant on each lattice line parallel to u, such that

$$E = \left\{p \in \mathbb{Z}^2 : \sum_{u\in S} g_u(p) > 0\right\}. \tag{2}$$

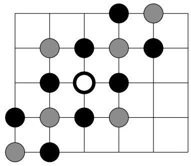

Fig. 1. The weakly bad configuration F_S associated to $F_S(x,y) = (x-1)(y-1)(x^2y-1)(xy^2-1) = x^4y^4 - x^3y^4 - x^4y^3 + x^3y^3 - x^2y^3 + xy^3 - x^3y^2 + 2x^2y^2 - xy^2 + x^3y - x^2y + xy - y - x + 1$ ($S = \{(1,0),(2,1),(1,2),(0,1)\}$). The Z component is formed by the union of the white point (corresponding to $f(2,2) = 2$ and counted twice) with the set of grey points (which correspond to $f(i,j) = 1$), while the set of black points (corresponding to $f(i,j) = -1$) form the W component.

Additivity is also fundamental for treating uniqueness problems, due to the following facts (see [6, Theorem 2]):

1. Every additive set is a set of uniqueness.
2. There exist sets of uniqueness which are not additive.
3. A set is additive if and only if it has no weakly bad configurations.

3 Weakly Bad Configurations

L.Hajdu and R.Tijdman [14, Lemma 3.1] showed that if a function $h : \mathcal{A} \to \mathbb{Z}$ has zero line sums along the lines taken in the directions in S, then $F_S(x,y)$ divides $G_h(x,y)$ over $\mathbb{Z}$ ([14, Lemma 3.1]). In other words, since a weakly bad configuration is algebrically determined by h, we can reformulate the result as follows:

Lemma 1. *Let S be a set of lattice directions. G_h is an S-weakly bad configuration if and only if $G_h(x,y) = H(x,y)F_S(x,y)$, where $H(x,y)$ is a polynomial.*

By Theorem 2.4 in [13], less than four directions are never sufficient to distinguish all the subsets of a given grid $\mathcal{A}$, and consequently $|S| = 4$ represents a minimal choice for our problem. Therefore, throughout the paper, we assume $S = \{u_1, u_2, u_3, u_4\}$, with the further condition that $u_1 + u_2 \pm u_3 = u_4$. Motivation for this position relies on the fact that such cases give the unique situations where F_S represents a weakly bad configuration (see [4]). In particular, F_S has a multiple point, say $p = \frac{1}{2}(u_1+u_2+u_3+u_4)$, and p is positive if $u_4 = u_1+u_2-u_3$, whereas it is negative if $u_4 = u_1 + u_2 + u_3$.

We next show that the weakly bad configuration F_S can be described by means of its multiple point and the given set S of directions. To this end we need some further notations.

We denote $\widehat{S} = \big\{ \pm (v - u_4) : v \in \{u_1, u_2, u_4 - u_1 - u_2\}\big\}$, $\pm S = \{\pm u : u \in S\}$ and $D = \pm S \cup \widehat{S}$. Geometrically, the set D represents all the possible shifts which map the multiple point in F_S to another point of the weakly bad configuration F_S (see Figure 2), as it is shown by the following result.

Proposition 1. *The set F_S is completely determined by the couple (p, D).*

Proof. Assume that $b \geq 0$ for all $(a, b) \in S$, the positive points of F_S are

$$\begin{array}{llll} \mathbf{0}, & u_1+u_2, & u_1+u_3, & u_1+u_4, \\ u_2+u_3, & u_2+u_4, & u_3+u_4, & u_1+u_2+u_3+u_4 \end{array} \tag{3}$$

and the negative points are:

$$\begin{array}{llll} u_1, & u_2, & u_3, & u_4, \\ u_1+u_2+u_3, & u_1+u_2+u_4, & u_1+u_3+u_4, & u_2+u_3+u_4, \end{array} \tag{4}$$

not all necessarily distinct. (We note that if $b < 0$ for some $(a, b) \in S$, then the sets of positive and negative points exchange and are translated by the vector $(0, -h)$, where h is the sum of negative values of b for $(a, b) \in S$. In any case, the properties we are looking for are invariant by integer translations. From the algebraic viewpoint, this corresponds to substituting the polynomial $F_S(x, y)$ by the rational function $y^h F_S(x, y)$.) By direct computation, we can check that all the points of the set F_S can be obtained as the translations of the multiple point p by a vector d in D and, if $p = u_1 + u_2$ (i.e. $f(p) > 0$), the multiplicities are given by:

$$\begin{aligned} l(p) &= f(p) = 2 \\ l(p+d) &= -(f(p) + f(d)) = -1, \text{ if } d \in \pm S \\ l(p+d) &= f(p) - f(d) = 1, \text{ if } d \in D \setminus \pm S. \end{aligned}$$

If $p = u_1 + u_2 + u_3$ (i.e. $f(p) < 0$), exchange the second and third cases and take the opposite signs. □

In general, uniqueness is not a property of the set S of X-ray directions, as for each S there exists a lattice set which is not uniquely determined by S. On the contrary if we restrict to bounded sets in a given rectangular grid $\mathcal{A} = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i < m, 0 \leq j < n\}$, there are whole families of lattice directions which

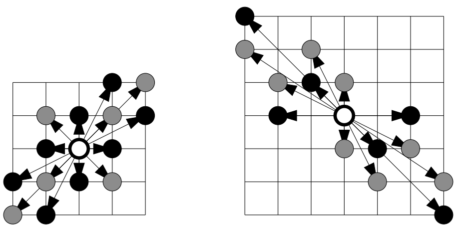

Fig. 2. The set F_S is completely determined by the couple (p, D), for $p = (2, 2)$ and $D = \pm\{(1, 0), (0, 1), (1, 2), (2, 1)\} \cup \pm\{(1, 1), (1, -1), (2, 2)\}$ (left) and for $p = (3, 3)$ and $D = \pm\{(0, 1), (1, -2), (2, -1), (3, -2)\} \cup \pm\{(1, -1), (2, 0), (3, -3)\}$ (right). The arrows show that each point of the configuration F_S can be reached from the multiple point by adding a vector in D.

uniquely determine bounded sets in $\mathcal{A}$. To show this, split D into two disjoint sets A, B, defined as follows.

$$A = \{(a,b) \in D: \ |a| > |b|\}, \tag{5}$$

and

$$B = \{(a,b) \in D: \ |b| > |a|\}. \tag{6}$$

Moreover, if $|a| = |b|$, for some $(a,b) \in D$, and $\sum_{r=1}^{4} a_r = h$, $\sum_{r=1}^{4} |b_r| = k$ (where $(a_r, b_r) = u_r$), we then include (a,b) in A if $\min\{m-h, n-k\} = m-h$, while $(a,b) \in B$ otherwise. Thus we have $D = A \cup B$, where one of the sets A, B may be empty. Then we get the following Theorem 1, proved in [5].

Theorem 1. *Let $S = \{u_1, u_2, u_3, u_4 = u_1 + u_2 \pm u_3\}$ be a valid set for the grid $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2,\ 0 \le i < m,\ 0 \le j < n\}$ and $\sum_{r=1}^{4} a_r = h$, $\sum_{r=1}^{4} |b_r| = k$, being $(a_r, b_r) = u_r$ where $r = 1, ..., 4$. Suppose that $g : \mathcal{A} \to \mathbb{Z}$ has zero line sums along the lines in the directions in S, and $|g| \le 1$. Then g is identically zero if and only if*

$$\min_{|a|} A \ge \min\{m-h, n-k\} \qquad \textit{and} \qquad \min_{|b|} B \ge \min\{m-h, n-k\}, \tag{7}$$

and

$$m-h < n-k, \ \Rightarrow \ \forall (a,b) \in B \ \left(|a| \ge m-h \ \textit{or} \ \ |b| \ge n-k\right), \tag{8}$$

$$n-k < m-h, \ \Rightarrow \ \forall (a,b) \in A \ \left(|a| \ge m-h \ \textit{or} \ \ |b| \ge n-k\right), \tag{9}$$

where, if one of the sets A, B is empty, the corresponding condition in (7) drops.

Notice that Proposition 1, together with Lemma 1, suggests that in order to get a bad configuration one has to translate F_S so that its multiple point overlaps a point of F_S with multiplicity of opposite sign, without producing new multiple points. The above theorem provides the conditions for which such situations cannot occur within the grid $\mathcal{A}$.

4 Non-additive Sets of Uniqueness

By means of Theorem 1 we can check if a set of four directions S uniquely determines bounded sets in a grid $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \le i < m,\ 0 \le j < n\}$. In general, any lattice set is unique (or $\mathcal{G}$-unique) if does not exists a different lattice set with the same X-rays in the tomographic grid $\mathcal{G}$. Unless $\mathcal{G} \subseteq \mathcal{A}$, uniqueness in $\mathcal{A}$ does not imply uniqueness in the tomographic grid $\mathcal{G}$. Therefore it is interesting to try to understand which bounded sets, among those uniquely determined by S in $\mathcal{A}$, are also $\mathcal{G}$-unique and/or additive.

Trivially, the sets S non valid for $\mathcal{A}$ uniquely determine all the subsets of $\mathcal{A}$, since no weakly bad configuration, and hence no bad configuration, can be constructed inside the grid. In general such bounded sets are not $\mathcal{G}$-unique, since $\mathcal{G} \setminus \mathcal{A} \neq \emptyset$. Differently, if $\mathcal{G} \subseteq \mathcal{A}$ (for example if $\{(1,0),(0,1)\} \in S$, and S is non valid for $\mathcal{A}$), then bounded sets are additive and therefore $\mathcal{G}$-unique.

Consider now a set S of four directions valid for $\mathcal{A}$ which ensures uniqueness in the grid $\mathcal{A}$. As before, if $\mathcal{G} \subseteq \mathcal{A}$, then bounded sets are trivially $\mathcal{G}$-unique. Non-additive sets of uniqueness are those E among bounded sets such that ${F_S}^+ \subseteq E$ (resp. ${F_S}^- \subseteq E$) and $E \cap {F_S}^- = \emptyset$ (resp. $E \cap {F_S}^+ = \emptyset$), because they have F_S as weakly bad configuration. More in general,

Theorem 2. *Let $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \leq i < m,\ 0 \leq j < n\}$, and let $S = \{u_1, u_2, u_3, u_4 = u_1 + u_2 \pm u_3\}$ be a set of uniqueness valid for $\mathcal{A}$. A set $E \subset \mathcal{A}$ is non-additive if and only if there exists a polynomial $P(x,y) = H(x,y)F_S(x,y)$, such that $|P| \geq 1$, $P^+ \subseteq E$ and $P^- \cap E = \emptyset$ (or $P^- \subseteq E$ and $P^+ \cap E = \emptyset$).*

Proof. Assume $E \subset \mathcal{A}$ is non-additive with respect to S. Then E has a weakly S-bad configuration (Z, W), where $Z \subseteq E$ and $W \cap E = \emptyset$ (or conversely). By [14, Lemma 3.1], we know that $Z = P^+$ and $W = P^-$ (or conversely), where $P(x,y) = H(x,y)F_S(x,y)$ for some polynomial $H(x,y)$. Therefore $P^+ \subset E$ and $P^- \cap E = \emptyset$ (or $P^- \subset E$ and $P^+ \cap E = \emptyset$).

Assume now that a polynomial $P(x,y) = H(x,y)F_S(x,y)$ exists, such that $|P| \geq 1$, $P^+ \subseteq E$ and $P^- \cap E = \emptyset$ (or $P^- \subseteq E$ and $P^+ \cap E = \emptyset$). Then E has the weakly bad configuration (P^+, P^-), so that it is non-additive. □

Corollary 1. *Assume $\mathcal{A}$ and S are as in Theorem 2. Let $P(x,y) = H(x,y)F_S(x,y)$ be any polynomial satisfying $deg_x P(x,y) < m$ and $deg_y P(x,y) < n$. Then the set $E = P^+$ (or $E = P^-$) is non-additive and it is uniquely determined in $\mathcal{A}$.*

Proof. Since $\deg_x P(x,y) < m$ and $\deg_y P(x,y) < n$, $E \subset \mathcal{A}$, so that E is unique within $\mathcal{A}$. Moreover, E has the bad configuration (P^+, P^-), which is consequently a weakly bad configuration, and E is also non-additive. □

Differently, every nonempty lattice set $E \subseteq F_S$, with $E \neq F_S^-$, $E \neq F_S^+$, is additive. This immediately follows by the observation that the tomographic grid related to E is contained in the tomographic grid related to F_S.

We can use Corollary 1 to explicitly construct non-additive (bounded) sets of uniqueness.

Algorithm 3. 1. Select a set S according to Theorem 1.
2. Compute the weakly bad configuration associated to $F_S(x,y)$.
3. Select any polynomial $H(x,y)$ such that $P(x,y) = H(x,y)F_S(x,y)$ satisfies $\deg_x P(x,y) < m$ and $\deg_y P(x,y) < n$.
4. Take each set $E \subset \mathcal{A}$ such that $P^+ \subset E$ and $P^- \cap E = \emptyset$ (or $P^- \subset E$ and $P^+ \cap E = \emptyset$).

Example 1. Assume $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \leq i < 8, \ 0 \leq j < 7\}$, and let $S = \{(0,1),(1,-2),(2,-1),(3,-2)\}$. Then S is a valid set for $\mathcal{A}$, and we have

$$\begin{aligned}
&S_1 - S_2 = \{\pm(-3,3), \pm(-2,0), \pm(-1,1)\}\\
&D = \{\pm(0,1), \pm(1,-2), \pm(2,-1), \pm(3,-2), \pm(-3,3), \pm(-2,0), \pm(-1,1)\}\\
&A = \{\pm(2,-1), \pm(3,-2), \pm(-2,0)\}\\
&B = \{\pm(0,1), \pm(1,-2)\} \cup \{\pm(-3,3)\}\\
&\min\{m-h, n-k\} = 1.
\end{aligned}$$

Therefore, $\min_{|a|} A = 2 > \min\{m-h, n-k\}$, $\min_{|b|} B = 1 = \min\{m-h, n-k\}$, and $|a| \geq m-h = 2$ for all $(a,b) \in A$, so that conditions of Theorem 5 in [5] are satisfied. Consequently, S is a set of uniqueness for $\mathcal{A}$.

Assume $H(x,y) = x - 1$, and consider the following polynomial

$$\begin{aligned}
&P(x,y) = (x-1)F_S(x,y) = (x-1)(y-1)(x-y^2)(x^2-y)(x^3-y^2) =\\
&= x^7y - x^7 - x^6y^3 + x^6y^2 - x^6y + x^6 + x^5y^3 - \boxed{2x^5y^2} + x^5y + x^4y^4 -\\
&- \boxed{2x^4y^3} + \boxed{2x^4y^2} - x^4y + x^3y^5 - \boxed{2x^3y^4} + \boxed{2x^3y^3} - x^3y^2 - x^2y^5 +\\
&+ \boxed{2x^2y^4} - x^2y^3 - xy^6 + xy^5 - xy^4 + xy^3 + y^6 - y^5.
\end{aligned}$$

Note that several multiple points appear. Let E be the set of lattice points corresponding to the monomials with positive sign. Then E is a non-additive set S-unique in $\mathcal{A}$. The same holds for the set of lattice points corresponding to the monomials with negative sign.

We now consider the special case where $\{(1,0),(0,1)\} \subset S$ (S is a valid set of uniqueness for the given grid). Notice that all the non-additive sets of uniqueness are obtained as described above in Theorem 2 with $H(x,y) = 1$, whereas all the other bounded sets are additive. In fact, in the latter case, the bounded sets cannot have a weakly bad configuration since every weakly bad configuration is obtained by the product $P(x,y) = H(x,y)F_S(x,y)$ for some $H(x,y)$, and hence possibly "enlarging" F_S. But in this case $deg_x P(x,y) \geq m$ or $deg_y P(x,y) \geq n$, since $deg_x H(x,y) \geq 1$ or $deg_y H(x,y) \geq 1$ and $m-h = n-k = 1$ (h,k as in Theorem 1).

Therefore, if $\{(1,0),(0,1)\} \subset S$, we get a complete classification of bounded sets. We can summarize the results as follows:

Theorem 4. *Let $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \ : \ 0 \leq i < m, \ 0 \leq j < n\}$, and let $S = \{u_1, u_2, u_3, u_4 = u_1 + u_2 \pm u_3\}$ be a set of uniqueness for $\mathcal{A}$, where $\{(1,0),(0,1)\} \subset S$.*

1. *If S is not valid for $\mathcal{A}$, then all bounded sets are additive.*
2. *If S is valid for $\mathcal{A}$, then:*
   - *every bounded set E such that ${F_S}^+ \subseteq E \ \wedge \ E \cap {F_S}^- = \emptyset$, or ${F_S}^- \subseteq E \ \wedge \ E \cap {F_S}^+ = \emptyset$ is a non-additive set of uniqueness;*
   - *all the other bounded sets are additive.*

In [8] Fishburn et al. notice that an explicit construction of non-additive sets of uniqueness has proved rather difficult even though it might be true that non-additive uniqueness is the rule rather than exception. In particular they suggest that for some set of X-ray directions of cardinality larger than 2 the proportion of lattice sets E of uniqueness that are not also additive approaches 1 as E gets large. They leave it as an open question in the discussion section.

By means of Theorem 4 we can count the number of bounded additive and bounded non-additive sets of uniqueness as follows.

Theorem 5. *Let $\mathcal{A} = \{(i,j) \in \mathbb{Z}^2 \, : \, 0 \leq i < m, \, 0 \leq j < n\}$, and let $S = \{u_1, u_2, u_3, u_4\}$ be a set of uniqueness, valid for $\mathcal{A}$, where $\{(1,0),(0,1)\} \subset S$. The proportion of bounded non-additive sets of uniqueness w.r.t. those additive is $\frac{2}{2^{|F_S|}-2}$.*

Proof. The non-additive sets of uniqueness contained in F_S are precisely F_S^- and F_S^+, therefore $\frac{2 \cdot 2^{|\mathcal{A} \setminus F_S|}}{2^{|\mathcal{A}|} - 2 \cdot 2^{|\mathcal{A} \setminus F_S|}} = \frac{2}{2^{|F_S|}-2}$. □

The theorem shows that for this family of sets of four directions, the proportion does not depend on the size of the lattice sets into consideration.

As a final remark, we note that a strong tool to treat these questions is provided by linear programming. Indeed the reconstruction problem can be reformulated as the following integer linear program (ILP):

$$Pz = q \text{ and } z \in \{0,1\}^{m \times n}, \tag{10}$$

where the vector z represents the set $\mathcal{A}$, vector q contains the values of the line sums, and P is a $0,1$-matrix whose rows specify which points of $\mathcal{A}$ are on each line l parallel to the directions in S. Each equation corresponds to a line sum on a lattice line l. The NP-hardness of the reconstruction problem for more than two directions is reflected in the integrality constraint for z. Therefore, in order to deal with this problem one can transform (10) by replacing the integrality constraint with an interval constraint, that is, $z \in [0,1]^{m \times n}$. This approach has been introduced by Aharoni et al. in [1] and then studied by several authors (see, for instance [12, 21, 22]). To our aim, we can follow the method described in [7]. Indeed the authors proved that additive (lattice) sets are the unique solutions of the relaxed linear program (LP). Differently, for non-additive (lattice) sets there are many fuzzy sets with the given line sums, even if the lattice set solution is unique.

We refer the reader to the cited paper for a formal description of the method. However we recall that Fishburn and Shepp [8] choose to use interior point methods not only for efficiency "but because they produce solutions that lie in the center of optimality." Moreover the set of all the solutions of LP is convex, that is, if z and z' are solutions, then $\lambda z + (1-\lambda)z'$ is a solution with $0 < \lambda < 1$. There follows that integral solutions are extreme members of the set. This allows to deduce that if the solution z of the relaxation determined by interior point method is such that for some $i \in \{1, \ldots, m \times n\}$ $z_i = 1$ (resp. $z_i = 0$) then all the solutions z' have $z'_i = 1$ (resp. $z'_i = 0$).

Here we just sketch the possible outputs for our concern.

1) if LP does not admit any solution, then no solution exists for (10).
2) if the solution z of LP is such that $z \in \{0,1\}^{m\times n}$, then it follows from the interior point method, convexity, and Theorems 1 and 2 of [6] that the corresponding lattice set is additive;
3) else if the solution z of LP is such that $z \in [0,1]^{m\times n}$, then the situation is ambiguous and if there is exactly one extreme solution, then the corresponding lattice set is unique, even if non-additive.

Recently issues about uniqueness and additivity have been reviewed and settled by a more general treatment in [11]. In particular the concept of J-additivity has been introduced to study invariant sets, i.e. sets each point of which either belongs to all or does not belong to any solution of the reconstruction problem. As a further step in our research we would like to explore possible connections between our approach and the notion of J-additivity. We also plan to conduct some experiments to estimate, for a given grid $\mathcal{A}$ and set S of uniqueness for $\mathcal{A}$, which bounded sets are additive and/or non-additive but unique, in the cases where the tomographic grid $\mathcal{G}$ is not contained in $\mathcal{A}$.

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