Finite Element Discretization in Shape Optimization Problems for the Stationary Navier-Stokes Equation

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Abstract. For shape optimization problems associated to stationary Navier-Stokes equations, we introduce the corresponding finite element approximation and we prove convergence results.

Keywords: shape optimization, full discretization, finite elements, convergence.

1 Introduction

Optimal design and optimal control problems for partial differential equations are extensively studied in the recent mathematical literature. In the case of stationary Navier-Stokes equations, we quote the works Casas, Mateos and Raymond [2007], Rösch and Vexler [2006], Los Reyes and Tröltzsch [2007] devoted to optimal control problems or to approximation procedures. Shape optimization problems related to fluid mechanics have been discussed in Borrvall and Petersson [2003], Mohammadi and Pironneau [2001], Posta and Roubicek [2007], Roubicek and Tröltzsch [2003], Halanay and Tiba [2009]. See as well [7], [8] for related problems and arguments.

This work is concerned with the discretization and the associated convergence analysis, in the spirit of general shape optimization problems for linear elliptic systems, as discussed in Chenais and Zuazua [2006] and in Tiba [2011]. Another approximation procedure for such problems is due to Neittaamäki, Pennanen and Tiba [2009].

In the next section we formulate the problem and review briefly some preliminaries, necessary in the subsequent parts. Section 3 investigates some approximation properties of the stationary Navier-Stokes equation under our discretization approach. The last section introduces the fully discretized optimization problem and studies its convergence.

2 Problem Formulation and Preliminaries

Let $\Omega \subset R^d$ be an (unknown) lipschitzian domain, such that $E \subset \Omega \subset D \subset R^d$ with $E \subset D$ some given bounded domains and d an arbitrary natural number. We recall from Temam [1979] the definition of the following spaces :

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$$\mathcal{V}(\Omega) = \{y \in \mathcal{D}(\Omega)^d; \quad div\ y = 0\}, \tag{1}$$

$$V(\Omega) = \textit{ closure of } \mathcal{V}(\Omega) \textit{ in } H_0^1(\Omega)^d. \tag{2}$$

Then, it is known that $V(\Omega) = \{y \in H_0^1(\Omega)^d, div\ y = 0\}$, as Ω is assumed lipschitzian. For any $y \in V(\Omega)$, if $\widetilde{y}$ is its extension by 0 to D, then $\widetilde{y} \in V(D)$ and conversely, if $\widetilde{z} \in V(D)$ and $\widetilde{z} = 0$ a.e. in $D \backslash \Omega$; then $z = \widetilde{z}|_\Omega \in V(\Omega)$. Such properties may be partially extended to domains with the segment property, Wang and Yang [2008].

The weak formulation of the stationary Navier-Stokes equation with Dirichlet (no-slip) boundary conditions is

$$\int_\Omega (\nu \sum_{i,j=1}^d \frac{\partial y_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} + \sum_{i,j=1}^d y_i \frac{\partial y_j}{\partial x_i} v_j) dx = \int_\Omega \sum_{j=1}^d f_j v_j dx, \forall v \in V(\Omega) \tag{3}$$

where $f = (f_1, \cdots, f_d) \in H^{-1}(D)^d$ and $\nu > 0$ is the viscosity.

By Theorem 1.2 from Temam [1979], the equation (3) has at least one solution $y \in V(\Omega)$. If $d > 4$, the supplementary condition $y \in [L^d(\Omega)]^d$ should be included in the definition (1), (2) of $V(\Omega)$.

We associate to (3) an integral cost functional of the form

$$\int_\Lambda j(x, y(x)) dx \tag{4}$$

where Λ is either $E \subset \Omega$ or Ω and y is one of the weak solutions of (3). The integrand $j : D \times R^d \to R$ satisfies measurability and continuity properties to be precised later.

The shape optimization problem considered in this paper consists in the minimization of the performance index (4) subject to the state system (3) and to the constraints

$$E \subset \Omega \subset D, \tag{5}$$

for any $\Omega \in \mathcal{O}$, where $\mathcal{O}$ is a prescribed family of domains. If the Lipschitz assumption is valid for any $\Omega \in \mathcal{O}$ with a uniform constant, then $\mathcal{O}$ is compact with respect to the Hausdorff-Pompeiu complementary metric. A similar compactness result holds for domains with the uniform segment property according to Theorem A3.9, Neittaanmäki, Sprekels and Tiba [2006]. The following existence result is a simplified version of Theorem 1 in Halanay and Tiba [2009].

Theorem 1. *Assume that $j(x, y_n(x)) \to j(x, y(x))$ weakly in $L^2(E)$ if $y_n \to y$ strongly in $L^2(E)$ and $\mathcal{O}$ is compact. Then, the shape optimization problem (3)-(5), with $\Lambda = E$ has at least one optimal pair $[\Omega^*, y^*] \in \mathcal{O} \times V(\Omega^*)$ if it has an admissible pair.*

Remark. This theorem should be understood in the sense of singular control problems Lions [1983], Neittaanmäki, Sprekels and Tiba [2006, 3.1.3.1]. The state system is ill-posed (nonuniqueness), but the optimization problem (3)-(5) is well defined as minimization over admissible pairs $[\Omega, y]$, $\Omega \in \mathcal{O}$ satisfying (5) and $y \in V(\Omega)$ being one of the weak solutions of (3).

3 Discretization of the State Equation

We assume now that D is a smooth bounded subdomain of R^2 and we consider a family of uniformly regular finite element meshes $\{\mathcal{T}_h\}_{h>0}$ in D with $h = \max_{T_h \in \mathcal{T}_h} diam(T_h)$.

For any admissible $\Omega \in \mathcal{O}$, we define its discrete approximation as follows (Chenais and Zuazua [2006] or Tiba [2011] where other variants are also discussed) :

$$\Omega_h = int \cup \{\overline{T}_h; T_h \in \mathcal{T}_h, T_h \subset \Omega\} \tag{6}$$

According, for instance, to Temam [1979], there are many possibilities to introduce a finite element space V_h in Ω_h approximating (2), that is approximating $H_0^1(\Omega)$ and the divergence free condition. In particular, the piecewise linear finite elements are not possible to be used in this setting. One also has to impose null values on $\partial\Omega_h$ in order to take account the Dirichlet boundary condition and any $y_h \in V_h$ may be extended by 0 to Ω, respectively to D. We shall also write $V_h(\Omega)$ or $V_h(D)$ in order to avoid possible confusions.

One example of space V_h (in dimension 2 as assumed here) is the space of continuous functions, vanishing outside Ω_h, that are polynomials of degree less or equal two on any simplex $T \in \mathcal{T}_h$ and satisfy :

$$\int_T divy_h dx = 0, \forall T \in \mathcal{T}_h, \forall y_h \in V_h \tag{7}$$

On V_h we take the scalar product $(\cdot,\cdot)_h$ induced by $H_0^1(\Omega)$. Note that V_h is an external approximation of V due to (7). The discrete approximation of (3) is

$$\nu(y_h, v_h)_h + b_h(y_h, y_h, v_h) = \int_\Omega f \cdot v_h dx, \forall v_h \in V_h \tag{8}$$

Notice that the last integral in (8) is over Ω_h in fact, as v_h vanishes outside Ω_h. We have denoted by "$\cdot$" the scalar product in R^2 and $b_h(\cdot,\cdot,\cdot)$ is the trilinear form approximating

$$b(y, v, w) = \sum_{i,j=1}^{2} \int_\Omega y_i D_i v_j w_j dx, \forall y, v, w \in H_0^1(\Omega).$$

A detailed construction of $b_h(\cdot,\cdot,\cdot)$ and the proof of

$$b_h(u_h, u_h, r_h v) \to b(u, u, v), \forall v \in \mathcal{V}(\Omega) \tag{9}$$

if $u_h \to u$ weakly in $H_0^1(\Omega)$ can be found in Teman [1979], Ch. II.3.

Here $r_h v \in V_h$ is given by a term that takes the same values as $v \in \mathcal{V}(\Omega)$ in the interior nodes and edge midpoints of Ω_h plus a correction term defined in Temam [1979, p.81]. On $\partial\Omega_h$, $r_h v$ should be zero.

Then, the following convergence property is also valid.

Proposition 3.1. Under the above conditions, there exists at least one $u_h \in V_h$, solution of (8) *, for each $h > 0$.*

The Family. $\{u_h\}$ in $H_0^1(\Omega)$ has strong accumulation points, denoted $\bar{u}$, which are solutions of (3)

Remark. If the uniqueness property is valid for (3), the convergence is valid without taking subsequences. In Casas, Mateos and Raymond [2007] and in Girault and Raviart [1989] Ch. II 4, finite element approximations with uniform convergence properties are indicated, including error estimates.

4 Approximation of the Shape Optimization Problem

We also discretize the cost functional (4) and the constraint (5) :

$$J_h(y_h) = \int_{E_h} j(x, y_h(x))dx \tag{10}$$

where y_h is any of the solutions of (8), associated to Ω_h and E_h is obtained as in (6), starting from E ;

$$E_h \subset \Omega_h \subset D. \tag{11}$$

Notice that for any admissible $\Omega \in \mathcal{O}$, restriction (11) is automatically fulfilled by our discretization construction. The collection of all admissible discretized open sets is denoted by $\mathcal{O}_h$. The discrete shape optimization problem is defined by (8), (10), (11). By (6), the family $\mathcal{O}_h$ is always finite, for any given $h > 0$. Then, the discrete minimization problem has at least one discrete optimal solution denoted by $\Omega_h^* \in \mathcal{O}_h$. Since (8) may have, in principle, an infinity of solutions y_h^n, we remark that in each $T \in \mathcal{T}_h$, $T \subset \Omega_h$, the corresponding coefficients of y_h^n are bounded, by the construction of the finite elements. This is a consequence of $|y_h^n|_{V_h}$ bounded and it is enough to pass to the limit in (8), (10) on a minimizing sequence (with respect to n) of admissible states (h and Ω_h are fixed here). The minimization in (10) should be understood as minimization over pairs $[\Omega_h, y_h] \in \mathcal{O}_h \times V_h(\Omega_h)$, similar to the situation in *Theorem 1.*

We recall first some convergence properties of the admissible pairs $[\Omega_h, y_h] \in \mathcal{O}_h \times V_h(\Omega_h)$, when $h \to 0$.

Proposition 4.1 i) If $\Omega \in \mathcal{O}$, then $\Omega_h \in \mathcal{O}_h$ and $\Omega_h \to \Omega$ in the Hausdorff-Pompeiu complementary topology.

ii) If $\Omega_h \in \mathcal{O}_h$ and $\Omega_h \to \hat{\Omega}$ in the Hausdorff-Pompeiu complementary topology, then $\hat{\Omega} \in \mathcal{O}$.

Remark. At point ii), the discrete sets Ω_h are not necessarily constructed via (6) starting from $\hat{\Omega}$. Point i) also applies to the discretization of E and $E_h \to E$ in the Hausdorff-Pompeiu complementary topology. The proof of this proposition and other related properties may be found in Chenais and Zuazua [2006] and in Tiba [2011].

In the sequel, a crucial role is played by the following result which is an extension of *Proposition 3.1.*

Theorem 2. *If $\Omega_h \in \mathcal{O}_h$ and $y_h \in V_h$ is any solution of (8) and if $\Omega_h \to \hat{\Omega}$ in the Hausdorff-Pompeiu complementary topology, then for any subdomain $\mathcal{K}$, compactly included in $\hat{\Omega}$ there is $h_0 > 0$ such that $\mathcal{K} \subset \Omega_h$, $h < h_0$ and*

$$y_h|_{\mathcal{K}} \to \hat{y}|_{\mathcal{K}} \tag{12}$$

weakly in $H^1(\mathcal{K})$, on a subsequence, where $\hat{y} \in V(\hat{\Omega})$ is a solution of (3) in $\hat{\Omega} \in \mathcal{O}$.

Proof
The fact that $\hat{\Omega} \in \mathcal{O}$ is a consequence of *P4.1.* The inclusion $\mathcal{K} \subset \Omega_h$ for $h < h_0$ is known as the Γ-property of the Hausdorff-Pompeiu complementary convergence, Neittaanmäki, Sprekels and Tiba [2006], p. 63.

Extend y_h by 0 to D and denote it by $\tilde{y}_h \in H_0^1(D)$. By Temam [1979], p. 209, we have

$$b_h(u_h, v_h, v_h) = 0, |b_h(u_h, v_h, w_h) \ \leq \ c|u_h|_{V_h}|v_h|_{V_h}|w_h|_{V_h} \tag{13}$$

for any u_h, v_h, w_h in V_h, where $c > 0$ is an absolute constant.

Fixing $v_h = y_h \in V_h$ in (8) we get that $\{|y_h|_{V_h}\}$ is bounded, due to (13), and $\{\tilde{y}_h\}$ is bounded in $H_0^1(D)$. On a subsequence, we have $\tilde{y_h} \to \tilde{y} \in H_0^1(D)$. A simple distributions argument gives that $\tilde{y}|_{D\setminus\hat{\Omega}} = 0$ almost everywhere. Then $\tilde{y}|_{\hat{\Omega}} \in H_0^1(\hat{\Omega})$ as we have assumed that any admissible domain $\hat{\Omega} \in \mathcal{O}$ is lipschitzian and the trace theorem may be applied.We also get $\widetilde{y} \in V(\hat{\Omega})$ by an adaptation of Proposition 4.3, Temam[1979], p.83. In particular $y_h|_{\mathcal{K}} \to \tilde{y}|_{\mathcal{K}}$ weakly in $H^1(\mathcal{K})$, on a subsequence.

We have to show that $\tilde{y}|_{\hat{\Omega}}$ is a solution of (3). We fix in (8) $v_h = r_h v$ for any $v \in \mathcal{V}(\hat{\Omega})$. In particular *supp* $v \subset \hat{\Omega}$ is a compact subset and the Γ-property gives that *supp* $v \subset \Omega_h$ for $h < h_0$. Consequently $r_h v \in V_h$ for $h < h_0$ and may be used in (8). Moreover, by (9) we have

$$b_h(y_h, y_h, r_h v) \to b(\tilde{y}, \tilde{y}, v), \forall v \in \mathcal{V}(\hat{\Omega}). \tag{14}$$

Relation (14) is obtained by applying (9) in D as $\tilde{y} \in H_0^1(D)$, $v \in \mathcal{V}(D)$ by extending it with 0 outside $\hat{\Omega}$ and since $\tilde{y}_h \to \tilde{y}$ weakly in $H_0^1(D)$. The formulas for $b(\cdot,\cdot,\cdot)$ and $b_h(\cdot,\cdot,\cdot)$ are not affected by these extensions.

One can pass to the limit in (8) by (14) and the strong convergence $\widetilde{r_h v} \to \tilde{v}$ in $H_0^1(D)$ due to the regularity of $v \in \mathcal{V}(\hat{\Omega})$. This ends the proof since $\mathcal{V}(\hat{\Omega})$ is dense in $V(\hat{\Omega})$ and (3) may be obtained.

Remark. In fact, we have shown that the extensions

$$\tilde{y}_h \to \tilde{y}$$

weakly in $H_0^1(D)$, on a subsequence. If the solution of (3) is unique, the convergence is valid on the whole sequence.

Theorem 3. *i) Any accumulation point of any sequence $\{\Omega_h^*\}_{h\to 0}$ of discrete minimizers of (10) is a continuous minimizer Ω^* of (4).*

ii) $J_h(\Omega_h^*) \to J(\Omega^*)$ for $h \to 0$, on the initial sequence.

Proof i) Clearly $\{\Omega_h^*\}$, $h > 0$ is relatively compact in the Hausdorff-Pompeiu complementary metric and we may assume that $\Omega_h^* \to \hat{\Omega}$ on a subsequence; where $\hat{\Omega} \in \mathcal{O}$ by *Proposition 4.1.*

By *Theorem 2*, we get $\tilde{y}_h|_E \to \hat{y}|_E$ strongly in $L^2(E)$, where $\tilde{y}_h$ is the extension by 0 of y_h and $\hat{y}$ is a solution of (3) in $\hat{\Omega}$. The convergence is valid on a subsequence.

We have $J_h(\Omega_h^*) \to J(\hat{\Omega})$. This is a consequence of $j(x,\tilde{y}_h) \to j(x,\hat{y})$ weakly in $L^2(E)$ (see the assumption on $j(\cdot,\cdot)$ in *Theorem 1*) and of

$$J_h(\Omega_h^*) = \int_{E_h} j(x,y_h)dx = \int_E j(x,\tilde{y}_h)dx - \int_{E\backslash E_h} j(x,\tilde{y}_h)dx \tag{15}$$

The last integral in (15) converges to 0 as $meas(E\backslash E_h) \to 0$, Tiba [2011], and $j(x,\tilde{y}_h)$ is bounded in $L^2(E)$, which is argued above.

For any $\Omega \in \mathcal{O}$, we can construct Ω_h as in (6) and again by *Theorem 2* and *Proposition 4.1* we obtain that $J_h(\Omega_h) \to J(\Omega)$. Taking into account that

$$J_h(\Omega_h^*) \le J_h(\Omega_h)$$

we infer that $J(\hat{\Omega}) \le J(\Omega)$ for any $\Omega \in \mathcal{O}$, i.e. $\hat{\Omega}$ is optimal for the problem (3)-(5) and we redenote it by Ω^*.

ii) This is a consequence of i) as the minimal value $J(\Omega^*)$ is uniquely associated to $\mathcal{O}$.

Remark. The results of this section may be extended to the cost functional corresponding to the choice $\Lambda = \Omega$ by using supplementary arguments as in Neittaanmäki, Sprekels and Tiba [2006], p. 472.

Remark. The approach of this paper is based on a fixed grid given in the whole domain D, i.e. it is a fixed domain method. It should be noticed that the finite dimensional optimization problem is nonconvex and it is not easy to find a global minimum Ω_h^* , $h > 0$.

Starting with some initial guess $\tilde{\Omega} \in \mathcal{O}$, one can define $\tilde{\Omega}_h \in \mathcal{O}_h$ by (6) and use it as initial iteration in some descent algorithm for the finite dimensional problem. Denote by $\dot{\Omega}_h$ the obtained finite dimensional "solution" (which is not necessarily a global minimum of J_h). Then, reading (6) in the converse sense, we get at least one $\dot{\Omega} \in \mathcal{O}$, corresponding to $\dot{\Omega}_h$. If the descent property for J_h "dominates" the approximation error between (3) and (8), then $J(\dot{\Omega}) < J(\tilde{\Omega})$, i.e. the method may find a better admissible domain from the point of view of the cost J.

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