Concentrated Curvature for Mean Curvature Estimation in Triangulated Surfaces

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Abstract. We present a mathematical result that allows computing the discrete mean curvature of a polygonal surface from the so-called concentrated curvature generally used for Gaussian curvature estimation. Our result adds important value to concentrated curvature as a geometric and metric tool to study accurately the morphology of a surface.

Keywords: Curvature, Gaussian and mean curvature, Discrete curvature, Triangulated surfaces.

1 Introduction

Curvature is an important geometric tool generally used to study the metric and topological properties of a surface. Indeed, Gauss-Bonnet theorem [10] links the topology of the surface (or of a patch of it) to its total Gaussian curvature. The convexity and concavity of a surface can be studied through mean and Gaussian curvatures and its main morphological features can be deduced from the the critical values of mean curvature. The behavior of geodesic segments (i.e. the shortest segment linking two points on a surface) can be studied through curvature values and their sign over the surface. Curvature has been widely studied in the smooth case and later in the discrete one where several attempts have been made to give adequate definitions for both Gaussian and mean curvatures. Discrete methods either interpolate the discrete values of the surface by a smooth function, or define discrete approaches that guarantee similar properties as the ones available in the smooth case (see [5] for more details). Such methods are based on approximations and, thus, the values they produce suffer from error optimization and control or from the approximation convergence when refining the mesh to tend to a smooth surface.

Concentrated curvature has been defined by Aleksandrov in [3] as the total curvature of spherical caps that approximate a triangulated surface at its vertices. It turns out that concentrated curvature depends only on the total angle around a vertex and does not depend on the radii of the approximating caps. Concentrated curvature produces, thus, an accurate value for each vertex of the surface and does not suffer from computation errors and convergence problems (see Section 3). Moreover, concentrated curvature satisfies a discrete version of

M. Ferri et al. (Eds.): CTIC 2012, LNCS 7309, pp. 79–87, 2012.

the above mentioned Gauss-Bonnet theorem that links the topology of a surface to its metric [10].

In [7], we have introduced *discrete distortion* as a generalization of concentrated curvature to three-combinatorial manifolds, and in [8], we have shown that its restriction to surface boundary of volumetric shape gives a good discrete estimator of mean curvature.

The aim of this paper is to show that concentrated curvature is linked to the restriction of discrete distortion via a simple relation that makes the computation of mean curvature possible from concentrated curvature. As a consequence, principal curvature computation becomes possible as the solution of two simple equations. This result gives to concentrated curvature a crucial role in combinatorial geometry to study the metric properties of a surface.

The reminder of this paper is organized as follows. In Section 2, we present some theoretical background on analytic curvatures. In Section 3, we present concentrated curvature as a Gaussian curvature estimator. In Section 4, we describe how concentrated curvature can be generalized to 3-dimensional manifolds and how its restriction to the boundary surfaces defines a new mean curvature estimator, called discrete distortion. In Section 5, we present the duality between concentrated curvature and discrete distortion. Finally, in Section 6, we present some experiments that highlight such duality, and we draw some conclusions and directions of future development.

2 Background Notions

In this section, we briefly review some fundamental notions on curvature (see [4] for details). Let C be a curve having parametric representation $(c(t))_{t \in R}$. The curvature $k(p)$ of C at a point $p = c(t)$ is given by

$$k(p) = \frac{1}{\rho} = \frac{|c'(t) \wedge c''(t)|}{|c'(t)|^3},$$

where ρ, called the *curvature radius*, corresponds to the radius of the osculatory circle tangent to C at p.

Let S be a smooth surface (at least C^2). Let $\overrightarrow{n_p}$ be the normal vector to the surface at a point p. Let Π be the plane which contains the normal vector $\overrightarrow{n_p}$. Plane Π intersects S at a curve C containing p: the curvature k_p of C at point p is called *normal curvature* at p. When plane Π turns around $\overrightarrow{n_p}$, curve C varies. There are two extremal curvature values $k_1(p) \leq k_2(p)$ which bound the curvature values of all curves C. The corresponding curves C_1 and C_2 are orthogonal at point p [4]. These extremal curvatures are called *principal normal curvatures*. Since the surface is smooth, then *Euler formula* (also called *Dupin indicatrix*) indicates that the curvatures at a point p have an elliptic behavior described by $k(p) = k_1(p)\cos^2(\theta) + k_2(p)\sin^2(\theta)$, where parameter $\theta \in [0; 2\pi]$. The *Gaussian curvature* $K(p)$ and the *mean curvature* $H(p)$ at point p are the quantities

$$K(p) = k_1(p) * k_2(p), \tag{1}$$

and

$$H(p) = \frac{1}{2\pi} \int_0^{2\pi} k(p) d\theta = \frac{k_1(p) + k_2(p)}{2}. \tag{2}$$

Gaussian curvature and the mean curvature strongly depend on the (local) geometrical shape of the surface. *Mean curvature* can identify saddle regions and ridge/ravine lines, and mean curvature combined with *Gaussian curvature* can identify convex, concave and locally flat regions. These are relevant properties of curvature for surface analysis:

- Let p be a point with positive Gaussian curvature (i.e., both principal curvatures have the same sign). If the mean curvature is positive [negative] at p, then the surface is locally convex [concave] at p.
- A negative Gaussian curvature at a point p implies that the principal curves lie in two different half spaces with respect to the tangent plane, and thus p is *a saddle point.*
- If the principal curvatures at a point p are null (i.e., the Gaussian and the mean curvatures are null), then the surface is "infinitesimally" *flat* at p.
- If the Gaussian curvature is null and the mean curvature is different from zero at a point p, then the surface is flat in one principal direction and convex [concave] in the other one (if the mean curvature of p is positive or negative, respectively). *Ridge and ravine* lines correspond to such a situation.

A remarkable property of Gaussian curvature is given by *Gauss-Bonnet Theorem*, which relates the metric property given by the Gaussian curvature to the topology of the surface (given by its Euler characteristic) [4].

Theorem 1 (Gauss-Bonnet Theorem). *For a compact surface S with a possible boundary components ∂S we have*

$$\int\int_S K(p) ds + \int_{\partial S} k_g(p) dl = 2\pi\chi(S), \tag{3}$$

where χ is Euler characteristic of surface S (i.e., $\chi = 2(1-g)$, where g is the genus of the surface), and k_g denotes the geodesic curvature at boundary points (i.e., the geodesic curvature is the norm of the projection of the normal vector of the curve on the tangent plane to the surface).

3 Concentrated Curvature

In [3] a mathematical definition of a discrete Gaussian curvature has been given by means of angle deflection. The author calls it *concentrated curvature* and justifies mathematically this name. Much more recently in [1,2], other authors propose to use concentrated curvature to define a stable alternative to Gaussian curvature.

Let Σ be a (piecewise linear) triangulated surface and let p be a vertex of the triangle mesh. Let $\Delta_1, \cdots, \Delta_n$ be the triangles incident at p such that Δ_i

and Δ_{i+1} are edge-adjacent. If a_i, b_i are the vertices of triangle Δ_i different from p, then the total angle Θ_p at p, also called conical angle, is given by $\Theta_p = \sum_{i=1}^{n} \widehat{a_i p b_i}$.

Around p the surface is isometric to a cone of angle Θ_p at its apex. If $\Theta_p < 2\pi$, then we can approximate the cone by a spherical cap from its interior. Each point on the cap has a constant Gaussian curvature equal to the square of the inverse of the cap radius. The total Gaussian curvature of the cap is then equal to its area normalized by the radius square. By simple computation, this number is equal to $2\pi - \Theta_p$ and is radius independent. This fact implies that approximating the cone by smaller caps, the total Gaussian curvature is always the same. This leads us to the definition of concentrated curvature.

Definition 1. *[10] The* concentrated Gaussian curvature $K_C(p)$*, at a vertex p of the triangulated surface, is the value*

$$K_C(p) = \begin{cases} 2\pi - \Theta_p & \text{if } p \text{ is an interior vertex, and} \\ \pi - \Theta_p & \text{if } p \text{ is a boundary vertex,} \end{cases}$$

where Θ_p is the conical angle at p.

For an internal vertex, the quantity $2\pi - \Theta_p$ is computed by approximating the surface at each vertex by spherical caps. The total curvature of each spherical cap is equal to $2\pi - \Theta_p$ and does not depend on the radius of the cap. The detailed justification can be found in [6].

Thus, concentrated curvature is, simply, the angle defect between the flat Euclidean case (i.e., a plane) and the surface. Concentrated curvature for boundary vertices is the angle defect between the case of boundary points of a half plane and the surface.

A simple computation on the number of triangles, edges and vertices within the surface gives the following discrete version of Gauss-Bonnet theorem [10]:

Theorem 2. *Let Σ be a closed orientable triangulated surface, and $\chi(\Sigma)$ be the Euler characteristic of Σ. Then*

$$\sum_{p \text{ vertex of } \Sigma} K_C(p) = 2\pi\chi(\Sigma).$$

4 Discrete Distortion

The principle underlying concentrated curvature can be extended to combinatorial (triangulated) 3-manifolds, by comparing the total solid angle around a vertex with 4π which is the total solid angle around a point in R^3. Let p be a vertex of a combinatorial 3-manifold Ω. *Vertex distortion* at p is thus defined as

$$D(p) = \begin{cases} 4\pi - S_p & \text{if } p \text{ is an interior vertex, and} \\ 2\pi - S_p & \text{if } p \text{ is a boundary vertex,} \end{cases} \quad (4)$$

where S_p is the solid angle at p within the manifold.

We have proven in [7] that, if Σ is a shape embedded in R^3, then internal vertices have null vertex distortion. This is an important property that we use to define the restriction of distortion on the boundary of the 3-manifold without considering the tetrahedra in its interior.

For triangulated surfaces embedded in R^3, the restriction of discrete distortion to a surface reduces to compare the internal solid angles at vertices with 2π. In this case, distortion at a vertex p can be expressed in a simpler way as

$$D(p) = \sum_{e \in St(p)} (\pi - \Theta_e), \tag{5}$$

where $St(p)$ is the set of edges incident to p, and Θ_e is the dihedral angle around edge e. In [8], we have shown, through the use of Conolly functions, that the restriction of distortion to surfaces provides a good discrete approximation of mean curvature.

Mean curvature of a polyhedral surface is usually defined in literature (see, e.g., [9]) by

$$|H| = \tfrac{1}{4|A|} \textstyle\sum_{i=1}^{n} \|\overrightarrow{e_i}\| |\pi - \Theta_i|, \tag{6}$$

where $|A|$ is the the area of the Voronoi or barycentric region around a vertex p, e_i is one of the n edges incident in p with a dihedral angle Θ_i. Formula (6) produces only positive values. A positive or negative sign is given depending on the angle formed by the surface normal at p with the vector obtained by summing all edges, weighted with $|\pi - \Theta_i|$. However, there is another issue when using Formula (6) for mean curvature estimation: curvature values depend on the length of the edges incident at vertex p, and, thus, are area-dependent.

5 Concentrated Curvature versus Discrete Distortion

We show here that there is a natural duality between discrete distortion and concentrated curvature. Let p be a vertex on a triangulated surface Σ embedded in the Euclidean space. Let $(\Delta_i = u_i p u_{i+1})_{i=1\cdots n}$ be the set of all triangles incident at p on Σ and let $(\overrightarrow{N_i})_{i=1\cdots n}$ be their unit normal vectors. Vectors $\overrightarrow{N_i}$ generate a polyhedral cone $\mathcal{C}(p)$ of summit p where each face F_i $(i = 1 \cdots n)$ is defined by two consecutive vectors $\overrightarrow{N_i}$ and $\overrightarrow{N_{i+1}}$ $(i = 1 \cdots n \ mod(n))$, see Figure 1. Vertex p belongs thus to two surfaces Σ and $\mathcal{C}(p)$.

The following theorem implies that concentrated curvature can be used in different ways to estimate both Gaussian and mean curvatures through simple geometric constructions.

Theorem 3. *Concentrated curvature and distortion of surfaces Σ and $\mathcal{C}(p)$ at vertex p are linked by the following formulas, where indexes refer to the corresponding surface:*

$$D_{\mathcal{C}}(p) + K_{\Sigma}(p) = 2\pi, \quad \text{and} \quad D_{\Sigma}(p) + K_{\mathcal{C}}(p) = 2\pi. \tag{7}$$

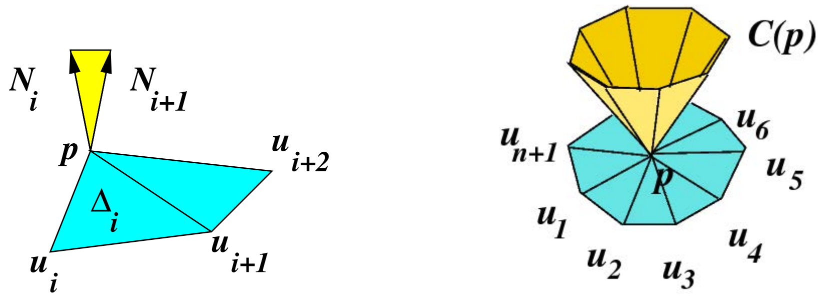

Fig. 1. Duality between distortion and concentrated curvature. Unit normal vectors to triangles incident to p generate a cone $\mathcal{C}(p)$.

Proof. Let $\widehat{u_i}$ be the dihedral angle at edge pu_i shared by triangles Δ_{i-1} and Δ_i. Similarly, let $\widehat{N_i}$ be the dihedral angle at edge $\overrightarrow{N_i}$ within the cone $\mathcal{C}(p)$. Simple geometric considerations, imply that the angle between $\overrightarrow{N_i}$ and $\overrightarrow{N_{i+1}}$ is given by

$$\sphericalangle(\overrightarrow{N_i}, \overrightarrow{N_{i+1}}) = \pi - \widehat{u_{i+1}}. \tag{8}$$

Conversely, vectors $\overrightarrow{pu_i}$ are perpendicular to triangles generated by $(p, \overrightarrow{N_{i-1}}, \overrightarrow{N_i})$ of cone $\mathcal{C}(p)$. The above relation implies that

$$\widehat{u_{i-1}pu_i} = \sphericalangle(\overrightarrow{pu_{i-1}}, \overrightarrow{pu_i}) = \pi - \widehat{N_i}. \tag{9}$$

Hence, there is a duality between angles at p of its incident triangles on surface Σ and dihedral ones on cone $\mathcal{C}(p)$, and vice versa. The above results, together with (5), imply that the distortion at p on surface Σ is equal to the total angle at p of all triangles on $\mathcal{C}(p)$, and vice versa. Hence we have:

$$D_\Sigma(p) = \sum_{i=1}^{n} \sphericalangle(\overrightarrow{N_i}, \overrightarrow{N_{i+1}}), \qquad D_\mathcal{C}(p) = \sum_{i=1}^{n} (\widehat{u_ipu_{i+1}}). \tag{10}$$

On the other hand, we know that concentrated curvature is the angle deficit on the sum of all triangles incident to a vertex on a surface. Then we have

$$D_\Sigma(p) + 2\pi - \sum_{i=1}^{n} \sphericalangle(\overrightarrow{N_i}, \overrightarrow{N_{i+1}}) = 2\pi, \tag{11}$$

and

$$D_\mathcal{C}(p) + 2\pi - \sum_{i=1}^{n} (\widehat{u_ipu_{i+1}}) = 2\pi, \tag{12}$$

which leads to relations (7), and therefore proves the theorem.

Principal curvatures k_1 and k_2 can be obtained as a common solution of both equation $k_1 + k_2 = 2D(p)$ and $k_1 \times k_2 = K(p)$. The result expressed by Theorem 3 provides a new interesting use for concentrated curvature and allows, with the corresponding principal curvatures, a local control of geometry via dual cones, in addition to its topological role described by the discrete Gauss-Bonnet theorem [10].

6 Concluding Remarks

We have implemented the methods defined in (4) and (5) to compute discrete distortion. We have experimentally compared the results and evaluated the efficiency of the computation. In Table 1, we report the order of magnitude of the difference between the values obtained with each method. We can see that the difference in distortion values computed with the two methods is negligible (less than $1/10^9$ of the values range). Moreover, the version with cone angles is slightly faster.

Figures 2 and 3 show the values of distortion and of mean curvature estimated with equation (6), with computation of sign, in a color scale. Color corresponds to negative and to positive values, respectively, in the two figures, and white corresponds to the remaining values. The two methods give the same image.

Table 1. Range of values of distortion, order of magnitude of the maximum difference between the two methods, and execution times (averaged over 100 executions), in seconds

Mesh	Vertices	Distortion range	Difference	Execution time (1)	(2)
Bunny	34k	$[-4.8, 5.2]$	e^{-09}	.398	.378
Bumpy Torus	17k	$[-7.4, 6.4]$	e^{-11}	.192	.185
Octopus	17k	$[-5.6, 6.1]$	e^{-11}	.199	.187
Kitten	11k	$[-6.1, 6.3]$	e^{-11}	.128	.122
Happy Buddha	544k	$[-8.9, 2.6]$	e^{-14}	6.43	6.31

We have shown that Gaussian and mean curvatures can be described through a pair of concentrated curvature values at each vertex. This gives concentrated curvature an additional geometric role besides its topological role described by the discrete Gauss-Bonnet theorem.

Surfaces where mean curvature is null everywhere, called minimal surfaces, play a great role in many scientific fields (DNA structures, architecture, etc.). In mathematics, generating and tracking such surfaces is a hard problem due to the complexity of their defining PDE equations. Our work can help studying minimal surfaces through the duality between concentrated curvature and distortion.

Troyanov has shown in [10] that, given a set of points with a corresponding set of weights, then, under some conditions, there exists a polyhedral surface whose vertices are the given set of points and whose concentrated curvatures are the corresponding weights. We project to exploit such result to construct and study the *dual* surface whose vertices are the same as the original surface and whose concentrated curvatures are those data (i.e., concentrated curvature values) coming from the dual cones that we have constructed here. This may reveal other interesting properties linking concentrated curvature to distortion or reveal geometrical and topological properties relating the two surfaces.

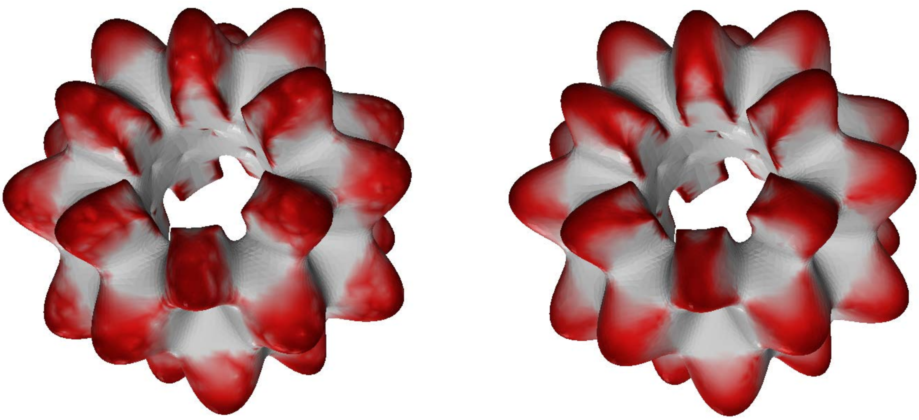

Fig. 2. Positive values of distortion (left) and mean curvature (right) in false colors: red (dark grey for black-and-white version) represent positive values, white represents negative or null values

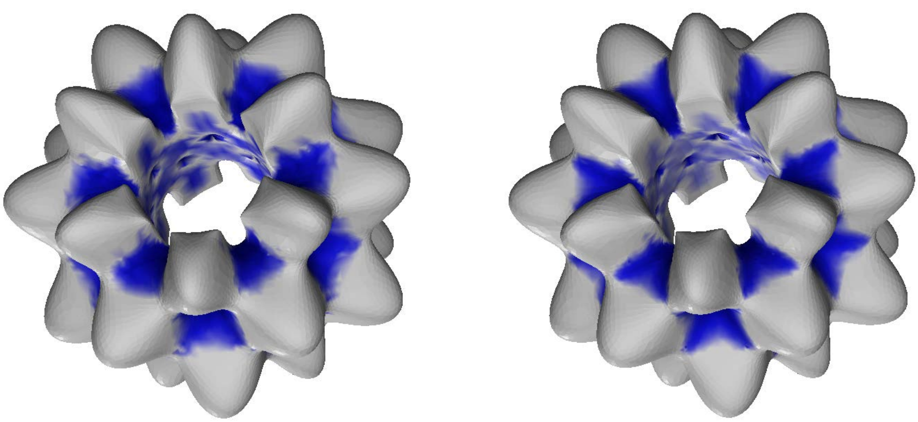

Fig. 3. Negative values of distortion (left) and mean curvature (right) in false colors: blue (dark grey for black-and-white version) represent negative values, white represents positive or null values

Acknowledgements. This work has been partially supported by the National Science Foundation under grant number IIS-1116747, and by the Italian Ministry of Education and Research under the PRIN 2009 program.

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