Chapter 8
Some Properties of the Eigenvalues of $Q^{w}_{(\alpha, \beta)}(x, D)$

In this chapter we shall show some properties of the eigenvalues of a NCHO $Q_{(\alpha, \beta)}$ (when $\alpha \beta > 1$ and $\alpha, \beta > 0$), and in particular give some upper and lower bounds to the lowest eigenvalue.

In the first place, we establish the following simple consequence of Theorem 3.1.12, which is useful when one is willing to study the behavior of the eigenvalues of the NCHO $Q_{(\alpha, \beta)}$ with respect to $\alpha, \beta \in \mathbb{R}$.

**Lemma 8.0.1.** We have that, for any given $\alpha, \beta \in \mathbb{R}$, the symbols $Q_{(\alpha, \beta)}(x, \xi)$ and $Q_{(\beta, \alpha)}(x, \xi)$ have the same eigenvalues, for all $(x, \xi) \in \mathbb{R} \times \mathbb{R}$, and, at the operator level, $Q^{w}_{(\alpha, \beta)}(x, D)$ is unitarily equivalent to $Q^{w}_{(\beta, \alpha)}(x, D)$, that is, there exists a unitary transformation $U: L^{2}(\mathbb{R}; \mathbb{C}^{2}) \rightarrow L^{2}(\mathbb{R}; \mathbb{C}^{2})$ such that

$$U^{*}Q^{w}_{(\alpha, \beta)}(x, D)U = Q^{w}_{(\beta, \alpha)}(x, D).$$

Moreover, $U$ is an automorphism of $\mathcal{S}'(\mathbb{R}; \mathbb{C}^{2})$ and $\mathcal{S}(\mathbb{R}; \mathbb{C}^{2})$, so that the operators $Q^{w}_{(\alpha, \beta)}(x, D)$ and $Q^{w}_{(\beta, \alpha)}(x, D)$ are also equivalent in $\mathcal{S}'$ and $\mathcal{S}$.

**Proof.** Let $A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$, $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Consider the system

$$KQ_{(\alpha, \beta)}(x, \xi)K = KAKp_{0}(x, \xi) - iJx\xi.$$

As $KAK = \begin{bmatrix} \beta & 0 \\ 0 & \alpha \end{bmatrix}$, by considering the symplectic transformation $\kappa: \mathbb{R} \times \mathbb{R} \ni (x, \xi) \longmapsto (\xi, -x) \in \mathbb{R} \times \mathbb{R}$, we have that

$$Q_{(\beta, \alpha)}(x, \xi) = (KQ_{(\alpha, \beta)}K \circ \kappa)(x, \xi).$$

It thus follows, since

$$\text{Tr}Q_{(\alpha, \beta)}(x, \xi) = \text{Tr}Q_{(\alpha, \beta)}(\xi, -x),$$

and

\[ \det Q_{(\alpha, \beta)}(x, \xi) = \det Q_{(\alpha, \beta)}(\xi, -x), \]

that \( Q_{(\alpha, \beta)}(x, \xi) \) and \( Q_{(\beta, \alpha)}(x, \xi) \) have exactly the same eigenvalues, and that by Theorem 3.1.12, with \( U_\kappa \) the metaplectic operator (the normalized Fourier transform) associated with \( \kappa \),

\[ Q^w_{(\beta, \alpha)}(x, D) = U_\kappa^{-1} K Q^w_{(\alpha, \beta)}(x, D) K U_\kappa = (K Q_{(\alpha, \beta)}(\kappa \circ \kappa))^w(x, D) \]

are unitarily equivalent, for \( U := K U_\kappa : L^2(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2) \) is unitary. Since clearly \( U \) is also an automorphism of \( \mathcal{S}'(\mathbb{R}; \mathbb{C}^2) \) and \( \mathcal{S}(\mathbb{R}; \mathbb{C}^2) \), this concludes the proof of the lemma.

From now on, unless otherwise specified, we shall assume \( \alpha, \beta > 0 \) and \( \alpha \beta > 1 \), so that \( Q = Q^w_{(\alpha, \beta)} \) is elliptic and positive (as a differential operator), and hence has a discrete spectrum contained in \( \mathbb{R}_+ \).

### 8.1 The Ichinose and Wakayama Bounds

In [33] Ichinose and Wakayama proved the following theorem about estimates on upper and lower bounds for the eigenvalues of \( Q \).

**Theorem 8.1.1.** Let \( \lambda_{2j-1}, \lambda_{2j}, j = 1, 2, \ldots \), be the \((2j-1)\)-st and \(2j\)-th eigenvalues of \( Q \). Then

\[
(j - \frac{1}{2}) \min \{\alpha, \beta\} \sqrt{\frac{\alpha \beta - 1}{\alpha \beta}} \leq \lambda_{2j-1} \leq \lambda_{2j} \leq (j - \frac{1}{2}) \max \{\alpha, \beta\} \sqrt{\frac{\alpha \beta - 1}{\alpha \beta}}. \tag{8.1}
\]

**Proof.** Put \( K(t) = K(t, x, y) = e^{-tQ}(x, y), t > 0 \). Then

\[ (\partial_t + Q)K(t) = 0, \quad K|_{t=0} = \delta(x-y). \]

Now, following Parmeggiani-Wakayama [58, 59], define

\[
Q' := A^{-1/2} Q A^{-1/2} = I \left( -\frac{\partial_x^2 + x^2}{2} \right) + \gamma J(x \partial_x + \frac{1}{2}) \\
= \frac{1}{2} (\partial_x - i \gamma J x)^2 + \frac{1 - \gamma^2}{2} x^2,
\]

where \( \gamma := 1/\sqrt{\alpha \beta} \). One then has that \( K'(t) = e^{-tQ'}(x, y) \) solves

\[ (\partial_t + Q')K'(t) = 0, \quad K'|_{t=0} = \delta(x-y), \]
and
\[
K'(t,x,y) = (1 - \gamma^2)^{1/4} e^{\gamma(x^2 - y^2)/2} p_H(\sqrt{1 - \gamma^2} t, \sqrt{1 - \gamma^2} x, \sqrt{1 - \gamma^2} y),
\]
where
\[
p_H(t,x,y) = e^{-tH}(x,y)
\]
is the heat-kernel of the harmonic oscillator \( H = p_0^w \). This follows from [58] (see also [59]), for \( Q' \) is unitarily equivalent to
\[
Q_0 := \frac{1}{2} \gamma I - \frac{1}{2},
\]
so that
\[
\text{Spec}(Q') = \left\{ \sqrt{1 - \gamma^2 (j - \frac{1}{2})}; \ j \geq 1 \right\},
\]
with each eigenvalue of multiplicity 2. Therefore, it also follows that
\[
\zeta_{Q'}(s) = \text{Tr} Q'^{-s} = (1 - \gamma^2)^{-s/2} \zeta_{Q_0}(s) = 2 \frac{2^s - 1}{(1 - \gamma^2)^{s/2}} \zeta(s).
\]
Now, since each eigenvalue \( \lambda'_j \) of \( Q' \) has multiplicity 2, we have that the \((2j - 1)\)-st eigenvalue \( \lambda'_{2j-1} \) and the \(2j\)-th eigenvalue \( \lambda'_{2j} \) coincide and
\[
\lambda'_{2j-1} = \lambda'_{2j} = (j - \frac{1}{2}) \sqrt{1 - \gamma^2} = (j - \frac{1}{2}) \sqrt{\frac{\alpha \beta - 1}{\alpha \beta}}.
\]
We now use the fact that
\[
Q = A^{1/2} Q' A^{1/2},
\]
and the Minimax Principle (4.1). For \( n = 2j - 1 \) or \( n = 2j \), with
\[
[u_1, \ldots, u_{n-1}]^\perp := \text{Span}(u_1, \ldots, u_{n-1})^\perp \cap D(Q),
\]
\( D(Q) \) being the domain of \( Q \), we thus have that
\[
\lambda_n = \sup_{u_1, \ldots, u_{n-1} \text{ lin. ind.}} \left( \inf_{0 \neq u \in [u_1, \ldots, u_{n-1}]^\perp} \frac{(Qu, u)}{\|u\|_0^2} \right) = \sup_{u_1, \ldots, u_{n-1} \text{ lin. ind.}} \left( \inf_{0 \neq u \in [u_1, \ldots, u_{n-1}]^\perp} \frac{(Q'A^{1/2}u, A^{1/2}u)}{\|u\|_0^2} \right)
\]
(note that the vectors \( u_1, \ldots, u_{n-1} \) are linearly independent if and only if the vectors \( A^{-1/2} u_1, \ldots, A^{-1/2} u_{n-1} \) are linearly independent, for \( A^{1/2} : L^2(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2) \) is an isomorphism, i.e. bounded with bounded inverse, and that \( u \in D(Q) \) or \( u \in D(Q') \) if and only if \( A^{-1/2} u \) does)
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We thus have

\[
\sup_{u_1, \ldots, u_{n-1} \text{ lin. ind.}} \left( \inf_{0 \neq A^{1/2}v \in [u_1, \ldots, u_{n-1}]^\perp} \frac{(Q' A^{1/2} u, A^{1/2} u)}{\|u\|^2_0} \right) = \sup_{u_1, \ldots, u_{n-1} \text{ lin. ind.}} \left( \inf_{0 \neq A^{1/2}v \in [u_1, \ldots, u_{n-1}]^\perp} \frac{(Q' A^{1/2} u, A^{1/2} u)}{\|u\|^2_0} \right)
\]

(putting $v = A^{1/2} u$)

\[
= \sup_{u_1, \ldots, u_{n-1} \text{ lin. ind.}} \left( \inf_{0 \neq v \in [u_1, \ldots, u_{n-1}]^\perp} \frac{(Q' v, v)}{\|A^{-1/2} v\|^2_0} \right)
\]

\[
= \sup_{u_1, \ldots, u_{n-1} \text{ lin. ind.}} \left( \inf_{0 \neq v \in [u_1, \ldots, u_{n-1}]^\perp} \frac{(Q' v, v)}{\|v\|^2_0} \right)
\]

Put $m = \min\{\alpha, \beta\}$ and $M = \max\{\alpha, \beta\}$. Then

\[
M^{-1} \|v\|^2_0 \leq (A^{-1} v, v) = \|A^{-1/2} v\|^2_0 \leq m^{-1} \|v\|^2_0, \quad \forall v \in L^2(\mathbb{R}; \mathbb{C}^2),
\]

and

\[
m \leq \frac{\|v\|^2_0}{\|A^{-1/2} v\|^2_0} \leq M, \quad \forall v \neq 0.
\]

We thus have

\[
\lambda_{2j-1} \leq \lambda_{2j} \leq M \sup_{u_1, \ldots, u_{2j-1} \text{ lin. ind.}} \left( \inf_{0 \neq v \in [u_1, \ldots, u_{2j-1}]^\perp} \frac{(Q' v, v)}{\|v\|^2_0} \right) = M \lambda_{2j} = M \lambda_{2j-1},
\]

and

\[
\lambda_{2j} \geq \lambda_{2j-1} \geq m \sup_{u_1, \ldots, u_{2j-2} \text{ lin. ind.}} \left( \inf_{0 \neq v \in [u_1, \ldots, u_{2j-2}]^\perp} \frac{(Q' v, v)}{\|v\|^2_0} \right) = m \lambda_{2j-1},
\]

which concludes the proof. \(\square\)

The bounds in (8.1) make up an interval

\[
I_j := \left( (j - \frac{1}{2}) \min\{\alpha, \beta\} \sqrt{\frac{\alpha \beta - 1}{\alpha \beta}} , (j - \frac{1}{2}) \max\{\alpha, \beta\} \sqrt{\frac{\alpha \beta - 1}{\alpha \beta}} \right).
\]

When $j < k$ we have

\[
I_j \cap I_k = \emptyset \iff \frac{2k - 1}{2j - 1} > \frac{\max\{\alpha, \beta\}}{\min\{\alpha, \beta\}}.
\]
It follows that the eigenvalue $\lambda_{2j-1}$ or $\lambda_{2j}$ has a multiplicity less than or equal to 2 if

$$j < \frac{\max\{\alpha, \beta\} + \min\{\alpha, \beta\}}{2(\max\{\alpha, \beta\} - \min\{\alpha, \beta\})} = \frac{|\alpha + \beta|}{2|\alpha - \beta|}.$$

Otherwise, the eigenvalue will possibly happen to have a multiplicity greater than 2.

However, we will see in Chapter 12, Section 12.4, that, as a consequence of a more general argument (see Section 12.3), under certain assumptions on the periods of the Hamilton-trajectories associated with the eigenvalues of the symbol, the spectrum of $Q^w_{(\alpha, \beta)}(x, D)$ “clusters” and is simple (see also Parmeggiani [52, 55]).

As regards the lowest eigenvalue, it was shown by Nakao, Nagatou and Wakayama in [45], and by Parmeggiani in [51], that it is always simple for $\alpha \beta$ large. Moreover, in [51], using perturbation theory in the limit $\alpha \beta \to +\infty$ and $\alpha/\beta$ a fixed constant $\neq 1$, it is seen that the lowest eigenvalue is always smaller, for $\alpha \beta$ sufficiently large ($\alpha/\beta = \text{constant} \neq 1$), than the lowest eigenvalue of the operator

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \left( -\frac{\partial_x^2}{2} + \frac{x^2}{2} \right).$$

Perturbation theory may be used, for one writes

$$Q^w_{(\alpha, \beta)}(x, D) = \sqrt{\alpha \beta} \left[ \sqrt{\frac{\alpha}{\beta}} \begin{bmatrix} 0 \\ \sqrt{\frac{\beta}{\alpha}} \end{bmatrix} \left( -\frac{\partial_x^2}{2} + \frac{x^2}{2} \right) + \frac{1}{\sqrt{\alpha \beta} J(x \partial_x + \frac{1}{2})} \right].$$

Put then $\varepsilon = 1/\sqrt{\alpha \beta} \to 0^+$ and $\omega_0 = \sqrt{\alpha/\beta}$ with $\omega_0$ fixed with $\omega_0 \neq 1$. Since

$$Q^w_{\omega_0}(x, D) = \begin{bmatrix} \omega_0 & 0 \\ 0 & \omega_0^{-1} \end{bmatrix} \left( -\frac{\partial_x^2}{2} + \frac{x^2}{2} \right),$$

is an elliptic system of GPDOs, it therefore possesses a parametrix. It is then easy to see that $E^w(x, D) = J \left( x \partial_x + \frac{1}{2} \right)$ is bounded with respect to $Q^w_{\omega_0}(x, D)$ (see Kato’s book [35]). Hence Rellich’s theory can be applied as $\varepsilon \to 0^+$ to study the spectrum of

$$\varepsilon Q^w_{(\alpha, \beta)}(x, D) = Q^w_{\omega_0}(x, D) + \varepsilon E^w(x, D),$$

in terms of the spectrum of $Q^w_{\omega_0}(x, D)$. See Parmeggiani [51] for more on this.

### 8.2 A Better Upper-Bound for the Lowest Eigenvalue

We now show, by using some elementary symplectic linear algebra, how to obtain an upper bound for the lowest eigenvalue of $Q$ which is more accurate than that of Theorem 8.1.1.
Theorem 8.2.1. For the lowest eigenvalue $\lambda_1$ of $Q$ we have

$$\frac{1}{2} \min\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}} \leq \lambda_1 \leq \frac{\sqrt{\alpha\beta} \sqrt{\alpha\beta - 1}}{\alpha + \beta + |\alpha - \beta| (\alpha\beta - 1)^{1/4} \text{Re } \omega},$$

where $\omega \in \mathbb{C}$ is the solution of $\omega^2 = \sqrt{\alpha\beta - 1} - i$ with $\text{Re } \omega > 0$.

Remark 8.2.2. Since, as is readily seen,

$$\frac{\sqrt{\alpha\beta} \sqrt{\alpha\beta - 1}}{\alpha + \beta + |\alpha - \beta| (\alpha\beta - 1)^{1/4} \text{Re } \omega} \leq \frac{1}{2} \max\{\alpha, \beta\} \sqrt{\frac{\alpha\beta - 1}{\alpha\beta}},$$

the upper bound in (8.2) is better than the one by Ichinose and Wakayama given in (8.1).

When $\alpha = \beta$ all the bounds reduce to the actual value of $\lambda_1 = \sqrt{\alpha^2 - 1/2}$. △

Proof (of Theorem 8.2.1). The lower bound in (8.2) has already been proved in Theorem 8.1.1.

Let $e(x, \xi) := x\xi$, and $\delta := \sqrt{\alpha\beta}$. We may therefore write

$$Q = \frac{1}{\delta} A^{1/2} \left( \delta p_0^w(x, D) I + iJ e^w(x, D) \right) A^{1/2}.$$

Let next $v_{\pm} = \frac{1}{\sqrt{2}} \left[ 1 \mp i \right]$ be the normalized eigenvectors of $J$, so that $Jv_{\pm} = \pm iv_{\pm}$.

Notice that $KJv_{\pm} = v_{\mp}$.

Set

$$L_\delta(x, \xi) := \frac{1}{2}(\xi^2 + (\delta^2 - 1)x^2).$$

Consider the following symplectic transformations of $\mathbb{R} \times \mathbb{R}$, and the corresponding metaplectic operators,

$$\kappa_\delta(x, \xi) = (\delta^{1/2}x, \frac{1}{\delta^{1/2}} \xi), \quad (U_\delta f)(x) = \frac{1}{\delta^{1/4}} f\left(\frac{x}{\delta^{1/2}}\right),$$

$$\kappa_{\pm}(x, \xi) = (x, \xi \pm x), \quad (U_{\pm} f)(x) = e^{\pm ix^2/2} f(x).$$

Define also

$$a_\pm^w(x, D) := \delta p_0^w(x, D) \mp e^w(x, D).$$

With $\mu_{\pm} := (\sqrt{\alpha} \pm \sqrt{\beta})/2$, we may write

$$A^{-1/2} = \frac{1}{\delta} \left( \mu_+ I - \mu_- KJ \right).$$
We also write
\[ f = f_+ v_+ + f_- v_-, \quad \forall f \in \mathcal{H}(\mathbb{R}; \mathbb{C}^2). \]

Let hence
\[ \frac{\lambda_1 = \inf_{0 \neq f \in \mathcal{H}(\mathbb{R}; \mathbb{C}^2)} \frac{(Qf, f)}{\|f\|_0^2}}{\inf_{0 \neq f \neq f_+ v_+ + f_- v_- \in \mathcal{H}(\mathbb{R}; \mathbb{C}^2)} \frac{\left( (\delta p_0^w(x, D)I + iJ e^w(x, D))(f_+ v_+ + f_- v_-) \right)}{\delta \|A^{-1/2}(f_+ v_+ + f_- v_-)\|_0^2}}. \]

In the basis \( \{v_+, v_-\} \) of \( \mathbb{C}^2 \),
\[ (\delta p_0^w(x, D)I + iJ e^w(x, D))(f_+ v_+ + f_- v_-) \]
is represented by
\[ \begin{bmatrix} a_+^w(x, D) & 0 \\ 0 & a_-^w(x, D) \end{bmatrix} \begin{bmatrix} f_+ \\ f_- \end{bmatrix}, \]
for we have that
\[ (\delta p_0^w(x, D)I + iJ e^w(x, D))(f_+ v_+ + f_- v_-) = \left( (\delta p_0^w(x, D) - e^w(x, D))f_+ \right) v_+ + \left( (\delta p_0^w(x, D) + e^w(x, D))f_- \right) v_- = (a_+^w(x, D)f_+)v_+ + (a_-^w(x, D)f_-)v_- \]
Hence, \( \{v_+, v_-\} \) being a unitary basis of \( \mathbb{C}^2 \),
\[ \lambda_1 = \frac{1}{\delta} \inf_{0 \neq f \neq f_+ v_+ + f_- v_- \in \mathcal{H}(\mathbb{R}; \mathbb{C}^2)} \frac{(a_+^w(x, D)f_+, f_+) + (a_-^w(x, D)f_-, f_-)}{\|A^{-1/2}(f_+ v_+ + f_- v_-)\|_0^2}. \]
Now, it is readily seen that
\[ \delta p_0(x, \xi) \equiv x\xi = (L_\delta \circ \kappa_+ \circ \kappa_0^{-1})(x, \xi). \]
Hence, by Theorem \( 3.1.12 \), we have
\[ a_\pm^w(x, D) = (L_\delta \circ \kappa_+ \circ \kappa_0^{-1})^w(x, D) = U_\delta (L_\delta \circ \kappa_+)^w(x, D)U_\delta^{-1} = U_\delta U_\pm^{-1} L_\delta^w(x, D)U_\pm U_\delta^{-1}. \]
One next computes
\[ A^{-1/2}(f_+ v_+ + f_- v_-) = \frac{1}{\delta} \left( (\mu_+ f_+ - \mu_- f_-)v_+ + (\mu_+ f_+ - \mu_- f_-)v_- \right), \]
whence

\[ \| A^{-1/2}(f_v + f_{-v}) \|_0^2 = \frac{1}{\delta^2} \left( \| \mu_+ f_+ - \mu_- f_- \|_0^2 + \| \mu_+ f_+ - \mu_- f_- \|_0^2 \right). \]

We thus have

\[ \lambda_1 = \delta \inf_{(0,0) \neq (f_+, f_-) \in \mathcal{X}(\mathbb{R}, \mathbb{C})} \left[ \frac{(L_w^\delta(x,D)U_-U_\delta^{-1}f_+ + U_-U_\delta^{-1}f_-)}{\| \mu_+ f_+ - \mu_- f_- \|_0^2 + \| \mu_+ f_+ - \mu_- f_- \|_0^2} \right]. \]

Let \( \varphi_0 = \varphi_0(x) = ce^{-(\alpha\beta-1)/2}x^2/2 \) be the ground state of \( L_w^\delta(x,D) \), with \( c \) so chosen that \( \| \varphi_0 \|_0 = 1 \). Thus

\[ L_w^\delta(x,D)\varphi_0 = \frac{\sqrt{\alpha\beta - 1}}{2} \varphi_0, \quad \text{and} \quad c = \left( \frac{(\alpha\beta - 1)^{1/4}}{\sqrt{\pi}} \right)^{1/2}. \]

We now choose \( f_\pm \) to be

\[ f_\pm = U_\delta U_{\pm}^{-1} \varphi_0, \]

that is

\[ f_\pm(x) = \frac{1}{\delta^{1/4}} e^{\pm ix^2/2\delta} \varphi_0(\frac{x}{\delta^{1/2}}). \]

It follows that

\[ (a_w^\delta(x,D)f_+, f_+) = \frac{1}{2} \sqrt{\alpha\beta - 1}. \]

Now, for \( r, s \in \mathbb{R} \),

\[ \| rf_+ + sf_- \|_0^2 = \| (re^{ix^2/2} + se^{-ix^2/2}) \varphi_0 \|_0^2 = (r^2 + s^2) \| \varphi_0 \|_0^2 + 2rs \int \cos(x^2) \| \varphi_0(x) \|^2 dx, \]

so that

\[ \| A^{-1/2}f \|_0^2 = \frac{1}{\delta^2} \left( \mu_+^2 + \mu_-^2 - 2\mu_+ \mu_- \int \cos(x^2) \| \varphi_0(x) \|^2 dx \right) \]

\[ = \frac{1}{\delta^2} \left( \alpha + \beta - (\alpha - \beta) \int \cos(x^2) \| \varphi_0(x) \|^2 dx \right). \]
We next compute

\[ \int \cos(x^2) |\phi_0(x)|^2 dx = \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\pi}} \int \cos(x^2) e^{-\sqrt{\alpha \beta - 1} x^2} dx \]

\[ = \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\pi}} \Re \int e^{-(\sqrt{\alpha \beta - 1} - i)x^2} dx. \]

Let \(\omega \in \mathbb{C}\) be the unique solution to \(\omega^2 = \sqrt{\alpha \beta - 1} - i\) with \(\Re \omega > 0\). Then, as is well-known,

\[ \int e^{-(\sqrt{\alpha \beta - 1} - i)x^2} dx = \frac{\sqrt{\pi}}{\omega}, \]

whence

\[ \int \cos(x^2) |\phi_0(x)|^2 dx = \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\pi}} \sqrt{\pi} \Re \frac{1}{\omega} = \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\alpha \beta}} \Re \omega, \]

and

\[ \|A^{-1/2} f\|_0^2 = \frac{1}{\alpha \beta} (\alpha + \beta - (\alpha - \beta) \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\alpha \beta}} \Re \omega) > 0. \]

We therefore have that

\[ \lambda_1 \leq \frac{1}{\sqrt{\alpha \beta}} \frac{\sqrt{\alpha \beta - 1}}{\sqrt{\alpha \beta - 1}} + \frac{1}{\sqrt{\alpha \beta}} \frac{\sqrt{\alpha \beta - 1}}{\sqrt{\alpha \beta - 1}} \frac{1}{\alpha + \beta - (\alpha - \beta) \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\alpha \beta}} \Re \omega} \]

\[ = \frac{\sqrt{\alpha \beta} \sqrt{\alpha \beta - 1}}{\alpha + \beta - (\alpha - \beta) \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\alpha \beta}} \Re \omega}. \]

Since, by Lemma 8.0.1, \(Q^w_{(\alpha, \beta)}(x, D)\) and \(Q^w_{(\beta, \alpha)}(x, D)\) are unitarily equivalent, they have the same lowest eigenvalue. Thus (with the same \(\omega\))

\[ \lambda_1 \leq \min \left\{ \frac{\sqrt{\alpha \beta} \sqrt{\alpha \beta - 1}}{\alpha + \beta - (\alpha - \beta) \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\alpha \beta}} \Re \omega}, \frac{\sqrt{\alpha \beta} \sqrt{\alpha \beta - 1}}{\alpha + \beta - (\beta - \alpha) \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\alpha \beta}} \Re \omega} \right\}, \]

that is

\[ \lambda_1 \leq \frac{\sqrt{\alpha \beta} \sqrt{\alpha \beta - 1}}{\alpha + \beta + |\alpha - \beta| \frac{(\alpha \beta - 1)^{1/4}}{\sqrt{\alpha \beta}} \Re \omega}. \]

This shows that inequality (8.2) holds and concludes the proof of the theorem. \(\square\)
8.3 Notes

The problem of determining the lowest eigenvalue, and its multiplicity, of a generic elliptic positive NCHO is an open, and important, problem, which should be explored in depth.