Chapter 5
The Heat-Semigroup, Functional Calculus and Kernels

We shall review in this chapter some elementary properties of the heat-semigroup associated with a globally elliptic operator, along with its functional calculus and some properties of the Schwartz kernels involved that give rise to the definition of the “trace” of the operator.

5.1 Elementary Properties of the Heat-Semigroup

Let \( a \in S_{\text{cl}}(m^\mu, g; M_N) \) be globally elliptic with \( \mu \in \mathbb{N} \) and principal symbol \( a_\mu = a_\mu^* > 0 \) (as an Hermitian matrix), and suppose that \( A^* = A = a^w(x, D) \) be positive on \( \mathcal{S}^0(\mathbb{R}^n; \mathbb{C}^N) \), that is there exists \( c > 0 \) such that

\[
(Au, u) \geq c\|u\|^2_0, \quad \forall u \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N).
\]

Let \( s \in \mathbb{R} \) and let also

\[
A_s : D(A_s) = B^{s+\mu}(\mathbb{R}^n; \mathbb{C}^N) \subset B^s(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow B^s(\mathbb{R}^n; \mathbb{C}^N), \quad A_s := A|_{B^s},
\]

be a realization of \( A \) as an unbounded operator in \( B^s(\mathbb{R}^n; \mathbb{C}^N) \). Then we know that \( A_s = A^*_s \), \( \text{Spec}(A_s) = \text{Spec}(A_0) = : \text{Spec}(A) \), that the eigenfunctions of \( A_s \) are the eigenfunctions of \( A \), and that they belong to \( \mathcal{S}^0 \), by the elliptic regularity (i.e., the existence of the parametrix). We have that the resolvent

\[
R_s(\lambda) = (A_s - \lambda)^{-1} : B^s(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow B^s(\mathbb{R}^n; \mathbb{C}^N)
\]

is defined, bounded and holomorphic for all \( \lambda \in \mathbb{C} \setminus \text{Spec}(A) \). It is important to note that

\[
s' \leq s \implies R_{s'}(\lambda)\big|_{B^s} = R_s(\lambda), \quad \forall \lambda \in \mathbb{C} \setminus \text{Spec}(A), \tag{5.1}
\]

for if \( \lambda \not\in \text{Spec}(A) \), then \((A_{s'} - \lambda)u = f \in B^s\) is solvable in \( B^{s'+\mu} \) and, by the elliptic regularity, for the solution \( u \) we have \( u \in B^{s+\mu} \subset B^{s'+\mu} \).
By virtue of (5.1), since \( \bigcap_{s \in \mathbb{R}} B^s = \mathcal{S} \) and \( \bigcup_{s \in \mathbb{R}} B^s = \mathcal{S}' \), we may hence define, as a continuous map,

\[
R(\lambda): \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N), \quad R(\lambda)|_{B^s} := R_s(\lambda), \quad \forall \lambda \in \mathbb{C} \setminus \text{Spec}(A). \tag{5.2}
\]

Note that \( R(\lambda): \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \) is also continuous. We may then simply write \( R(\lambda) \) for the resolvent of \( A \), regardless the domain used to realize \( A \) as an unbounded operator. It is hence well-known that for any given \( s \in \mathbb{R} \) there is \( C_s > 0 \) such that

\[
\| R(\lambda) \|_{B^s \rightarrow B^s} \leq \frac{C_s}{\text{dist}(\lambda, \text{Spec}(A))}.
\]

Let \( \delta_A := \min \text{Spec}(A) \). We may therefore choose \( \delta' \in (0, \delta_A) \) such that

\[
D_{\delta'} := \{ \lambda \in \mathbb{C}; \text{Re} \lambda \leq \delta' \} \cup \{ \lambda = \rho e^{i\phi}; \rho \geq \frac{\delta'}{2}, \phi \in [\frac{\pi}{4}, \frac{7\pi}{4}] \} \subset \mathbb{C} \setminus \text{Spec}(A).
\]

Hence there exists \( C'_s > 0 \) such that

\[
\lambda \in D_{\delta'} \implies \| R(\lambda) \|_{B^s \rightarrow B^s} \leq \frac{C'_s}{1 + |\lambda|}.
\]

Let \( \partial D_{\delta'} \) be oriented in such a way that \( D_{\delta'} \) is kept to the right-hand side. Then (see Kato’s book [35]) we may define

\[
\begin{cases}
e^{-tA} = \frac{1}{2\pi i} \int_{\partial D_{\delta'}} e^{-\lambda t} R(\lambda) d\lambda, & t > 0, \\
e^{-tA}|_{t=0} = I.
\end{cases}
\]

Hence, for any fixed \( s \in \mathbb{R}, \{e^{-tA}\}_{t \geq 0} \) is a strongly continuous semigroup in \( B^s \) with generator \(-A_s\). Furthermore, for every \( \delta \in (0, \delta_A) \), we have

\[
\sup_{t \geq 1} e^{\delta t} \| e^{-tA} \|_{B^s \rightarrow B^s} < +\infty.
\]

Using the same arguments as in Chazarain-Piriou [6], one can prove the following lemma. (Recall that \( \mathbb{R}_+ = [0, +\infty) \).)

**Lemma 5.1.1.** Let \( f \in B^s(\mathbb{R}^n; \mathbb{C}^N) \). Put \( F(t) = e^{-tA} f \). Then for every \( p, j \in \mathbb{Z}_+ \),

\[
t^p (\frac{d}{dt})^j F \in C^0(\mathbb{R}_+; B^{s+(p-j)\mu}(\mathbb{R}^n; \mathbb{C}^N)),
\]

and

\[
\sup_{t \geq 0} \| t^p (\frac{d}{dt})^j F \|_{B^{s+(p-j)\mu}} \leq C_{p,j} \| f \|_{B^s}.
\]
From the lemma we may therefore think of $e^{-tA}$ as a map

$$e^{-tA} : \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \mathcal{S}(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)),$$

and as a map

$$e^{-tA} : \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N) \longrightarrow \mathcal{S}(\mathbb{R}^n; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N)).$$

We also have the following fact about the heat-equation associated with $-A$.

**Lemma 5.1.2.** For any given $f \in \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N)$ and $g \in \mathcal{S}(\mathbb{R}^n; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N))$, there exists a unique $u(t) = u \in \mathcal{S}(\mathbb{R}_+; \mathcal{S}'(\mathbb{R}^n; \mathbb{C}^N))$ such that

$$\begin{cases}
\frac{du}{dt} + Au(t) = g(t), \quad t > 0, \\
\quad u(0) = f,
\end{cases}$$

with

$$u(t) = e^{-tA}f + \int_0^t e^{-(t-t')A}g(t')dt'.$$

Moreover,

$$f \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N), \quad g \in \mathcal{S}(\mathbb{R}_+; \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)) \implies u \in \mathcal{S}(\mathbb{R}_+; \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)).$$

## 5.2 Direct Definition of $\text{Tr} e^{-tA}$

In this section we give a direct proof that the kernel $e^{-tA}(x, y), \quad t > 0$, of $e^{-tA}$ (the heat-kernel of $A$) is a rapidly decreasing function of $(x, y)$, and that

$$\text{Tr} e^{-tA} = \sum_{j \geq 1} e^{-t\lambda_j}, \quad t > 0.$$

Let $\{\varphi_j\} \subset \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ be an ON systems of $L^2(\mathbb{R}^n; \mathbb{C}^N)$ made of eigenfunctions of $A$.

To start with, note that if $u = \sum_{j=1}^{\infty} u_j \varphi_j \in L^2$, i.e. $\sum_{j=1}^{\infty} |u_j|^2 < +\infty$, then

$$e^{-tA}u = \sum_{j=1}^{\infty} e^{-t\lambda_j} (u, \varphi_j) \varphi_j,$$

so that, $e^{-tA} : \mathcal{S} \longrightarrow \mathcal{S} \hookrightarrow \mathcal{S}'$ being continuous (and recalling that $(v^* \otimes w)\zeta = v^*(\zeta)w$), we have by the Schwartz-kernel Theorem that

$$e^{-tA}(x, y) = \sum_{j \geq 1} e^{-t\lambda_j} \varphi_j(y)^* \otimes \varphi_j(x) \in \mathcal{S}', \quad t > 0.$$
If we proceed formally, we have
\[
\text{Tr} e^{-tA} = \int_{\mathbb{R}^n} \text{Tr} e^{-tA}(x,x) dx = \sum_{j \geq 1} e^{-t\lambda_j} \int_{\mathbb{R}^n} |\varphi_j(x)|^2 dy = \sum_{j \geq 1} e^{-t\lambda_j},
\]
where Tr denotes the matrix-trace.

We next show that the formal procedure is actually correct. Since \( A\varphi_j = \lambda_j \varphi_j \), we get
\[
A_r \varphi_j = \lambda_r \varphi_j, \quad \text{for all } r \in \mathbb{N},
\]
and
\[
||\varphi_j||_{p,q} \lesssim ||\varphi_j||_0^2 + ||A_r \varphi_j||_0^2 = (1 + \lambda_j^{2r}) ||\varphi_j||_0^2. \tag{5.3}
\]
(The reader may prove (5.3) as an exercise by using a parametrix of \( A \), or else look at Lemma 5.3.1 below.) From (5.3) and (3.31) we thus get that given any \( \mathcal{S} \)-seminorm \( | \cdot |_{p,q}, p,q \in \mathbb{Z}_+ \), there exists \( r \) so large that
\[
||\varphi_j||_{p,q} \leq C_{pq}(1 + \lambda_j^{2r}) ||\varphi_j||_0^2 = C_{pq}(1 + \lambda_j^{2r}). \tag{5.4}
\]
We now notice that
\[
(t \lambda_j)^k e^{-t\lambda_j} \leq C e^{-t\lambda_j/2}, \quad t > 0, \tag{5.5}
\]
where \( C = (2k/e)^k \). Indeed, it is easy to see that
\[
\tau^a e^{-\tau} \leq \left( \frac{a}{e} \right)^a, \quad \forall \tau, a > 0, \tag{5.6}
\]
so that
\[
\tau^a e^{-\tau} = \tau^a e^{-\tau/2} e^{-\tau/2} = \left( \frac{\tau}{2} \right)^a e^{-\tau/2} 2^a e^{-\tau/2} \leq \left( \frac{a}{e} \right)^a 2^a e^{-\tau/2} = \left( \frac{2a}{e} \right)^a e^{-\tau/2}.
\]
Hence we obtain from (5.4) and (5.5), recalling that \( \lambda_j \to +\infty \) as \( j \to +\infty \),
\[
|e^{-tA}(\cdot,\cdot)|_{p,q} \leq C_{pq} t^{-k} \sum_{j \geq 1} e^{-t\lambda_j/2} < +\infty, \quad \forall t > 0,
\]
for we already know that \( \lambda_j \approx j^{\mu/2n} \) as \( j \to +\infty \). Hence,
\[
t \mapsto e^{-tA}(\cdot,\cdot) \in C^\infty(\mathbb{R}_+; \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{M}_N)),
\]
and
\[
\text{Tr} e^{-tA} = \sum_{j \geq 1} e^{-t\lambda_j}, \quad t > 0,
\]
as we wanted.
The next step, to be carried out in Chapter 6, will be to study the singularity as $t \to 0+$ of the trace of the heat-kernel. We shall accomplish this by constructing a parametrix approximation of $e^{-tA}$ (the parametrix we are referring to is a parametrix for $d/dt + A$), that will give the sought information. After that, we shall relate the singularity of $\text{Tr} e^{-tA}$ as $t \to 0+$ to the counting function $N(\lambda)$ through the Karamata theorem.

But before doing that, we continue with a section about the abstract functional calculus of an elliptic global system, a section about kernels, and finally close the chapter with a section about $f(A)$ as a global pseudodifferential operator.

### 5.3 Abstract Functional Calculus

We recall in this section how to obtain an abstract functional calculus for an elliptic $0 < A = A^* \in \text{OPS}_{\text{cl}}(m^\mu, g; M_N)$, $\mu > 0$, with a discrete spectrum, $\text{Spec}(A)$, made of a sequence $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to +\infty$ of eigenvalues repeated according to multiplicity, and with eigenfunctions $\{\varphi_j\}_{j \in \mathbb{N}} \subset \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ that form an orthonormal basis of $L^2(\mathbb{R}^n; \mathbb{C}^N)$. Let $f: \text{Spec}(A) \to \mathbb{C}$. Define the unbounded operator $f(A): D(f(A)) \subset L^2(\mathbb{R}^n; \mathbb{C}^N) \to L^2(\mathbb{R}^n; \mathbb{C}^N)$ by

$$D(f(A)) := \left\{ u = \sum_{j \geq 1} u_j \varphi_j \in L^2(\mathbb{R}^n; \mathbb{C}^N); \sum_{j \geq 1} |f(\lambda_j)|^2 |u_j|^2 < +\infty \right\};$$

$$D(f(A)) \ni u = \sum_{j \geq 1} u_j \varphi_j \mapsto f(A)u := \sum_{j \geq 1} f(\lambda_j) u_j \varphi_j.$$

Then $f(A)$ is a closed operator with dense domain, since the $\varphi_j \in D(f(A))$. If $f$ is real-valued, then $f(A) = f(A)^*$.

Now we want to have conditions on $f$ which ensure that $f(A): \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \to \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N)$ be continuous. The first step is to understand $D(A^p)$, for $p \in \mathbb{N}$.

**Lemma 5.3.1.** Define, for $p \in \mathbb{N}$, the unbounded operator $A^p: D(A^p) \subset L^2 \to L^2$, by $D(A^p) = \{u \in D(A^{p-1}); A^{p-1}u \in D(A)\}$, and $A^p u = A(A^{p-1}u)$. Remark that the domain is dense in $L^2$ since $\mathcal{S} \subset D(A^p)$. Using the eigenfunctions $\{\varphi_j\}_{j \in \mathbb{N}}$, we then have

$$D(A^p) = \left\{ u = \sum_{j \geq 1} u_j \varphi_j \in L^2; \sum_{j \geq 1} \lambda_j^{2p} |u_j|^2 < +\infty \right\} = B^{p\mu}(\mathbb{R}^n; \mathbb{C}^N),$$

and

$$Au = \sum_{j \geq 1} \lambda_j u_j \varphi_j.$$

**Proof.** Note in the first place that $\varphi_j \in \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \subset B^s(\mathbb{R}^n; \mathbb{C}^N)$, for all $s \in \mathbb{R}$. If $u \in D(A^p)$ we want then to prove that $A^{p\mu} u \in L^2$, where, recall, $A^{p\mu} u$ is taken in the sense of distributions. Using a parametrix $Q$ of $A^p$ such that $QA^p = I + R$ gives
\[ u \in D(A^p) \implies \Lambda^{p\mu} u = \Lambda^{p\mu}(Q A^p - R) u = (\Lambda^{p\mu} Q) A^p u - \Lambda^{p\mu} R u \in L^2. \]

And conversely, supposing \( u \in B^{p\mu} \), and using a parametrix \( E \) of \( \Lambda^{p\mu} \) such that \( E \Lambda^{p\mu} = I + \tilde{R} \), yields

\[ S' \ni A^p u = A^p(E \Lambda^{p\mu} - \tilde{R}) u = (A^p E) \Lambda^{p\mu} u - A^p \tilde{R} u \in L^2, \]

and this concludes the proof. \( \square \)

Remark 5.3.2. Note hence that for \( p \in \mathbb{N} \),

\[ \| u \|_{B^{p\mu}} \approx \| u \|_{D(A^p)}^2 = \| u \|_0^2 + \| A^p u \|_0^2 \approx \| A^p u \|_0^2, \quad (5.7) \]

for, by our assumption, \( \text{Spec}(A) \subset (0, +\infty) \). \( \triangle \)

When \( f \) is slowly increasing on the spectrum of \( A \) one has that \( f(A) \) is continuous from \( \mathcal{S} \) into itself. One has in fact the following lemma.

Lemma 5.3.3. Suppose \( f \) is slowly increasing on \( \text{Spec}(A) \), i.e. there exists \( C > 0 \) and \( p \in \mathbb{Z}_+ \) such that

\[ |f(\lambda_j)| \leq C(1 + \lambda_j)^p, \quad \forall j \in \mathbb{N}. \quad (5.8) \]

Then \( f(A) : \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \to \mathcal{S}(\mathbb{R}^n; \mathbb{C}^N) \) is continuous.

Proof. For all \( r \in \mathbb{Z}_+ \) we have

\[ |(1 + \lambda_j)^r f(\lambda_j) u_j| \leq C(1 + \lambda_j)^{r+p} |u_j|, \quad \forall j \in \mathbb{N}. \quad (5.9) \]

By Lemma 5.3.1 we have \( D(A^p) = B^{p\mu} \), whence (5.9) implies the continuity of

\[ f(A)|_{B^{(r+p)\mu}} : D(A^{r+p}) = B^{(r+p)\mu} \to D(A^r) = B^{r\mu}, \quad \forall r \in \mathbb{Z}_+, \]

the spaces being endowed with the respective Hilbert-space structures (and making use of (5.7)). This proves the lemma. \( \square \)

Remark 5.3.4. One has the following immediate consequences.

1. If \( f \) is bounded on \( \text{Spec}(A) \) (e.g. \( p = 0 \) in (5.8)) then \( f(A) : L^2 \to L^2 \) is continuous.
2. Since \( A > 0 \), we may consider \( f(A) \) when \( f(\lambda) = \lambda^s, \lambda > 0, s \in \mathbb{R} \).
3. If \( f \in C^0_0(\mathbb{R}) \) then \( f(A) \) is smoothing and compact (and actually of finite rank).
4. If \( f(\lambda) = e^{it\lambda}, t \in \mathbb{R} \), then

\[ e^{itA} : L^2 \xrightarrow{\text{unitary}} L^2, \quad (e^{itA})^* = e^{-itA}. \]
5. If $f \in \mathcal{S}([0, +\infty))$, then $f(A)$ is smoothing. This is in particular the case when $f(\lambda) = e^{-i\lambda}$, for $t, \lambda > 0$.

It is useful to have also the following proposition about the Schrödinger group.

**Proposition 5.3.5.** The function

$$t \mapsto e^{-itA} \in C^\infty(\mathbb{R}_t; \mathcal{L}(\mathcal{S}, \mathcal{S})).$$

Moreover, with $D_t = -i\partial_t$, it solves the Cauchy problem

$$\begin{cases}
(D_t + A)e^{-itA} = 0, \text{ in } C^\infty(\mathbb{R}_t; \mathcal{L}(\mathcal{S}, \mathcal{S})), \\
e^{-itA}|_{t=0} = I.
\end{cases}$$

**Proof.** One directly sees, from the very definition, that for all $m, p \in \mathbb{N}$

$$t \mapsto e^{-itA} \in C^{m-1}(\mathbb{R}_t; \mathcal{L}(D(A^{m+p}), D(A^p))),$$

and that the equation is satisfied. \(\square\)

## 5.4 Kernels

We now come to the study of the Schwartz kernel of $f(A)$. Let hence $f$ be slowly increasing on $\text{Spec}(A)$. Then, by Lemma 5.3.3 $f(A): \mathcal{S} \rightarrow \mathcal{S}$ is continuous so that, using the embedding $\mathcal{S} \hookrightarrow \mathcal{S}'$, $f(A): \mathcal{S} \rightarrow \mathcal{S}'$ is also continuous. Hence, by the Schwartz-kernel theorem, $f(A)$ has a distribution kernel: for any given $u, v \in \mathcal{S}$ we have

$$(f(A)u, v) = \langle K \hat{v} \otimes u, \hat{\phi}_j \rangle_{\mathcal{S}', \mathcal{S}}.$$ 

We have the following proposition.

**Proposition 5.4.1.** If $f$ is slowly increasing on $\text{Spec}(A)$, then the distribution kernel $K$ of $f(A)$ belongs to $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n; \mathcal{M}_N)$, and one has

$$K = \mathcal{S}' \lim_{m \rightarrow +\infty} K_m,$$

where $K_m(x, y) = \sum_{j=1}^{m} f(\lambda_j) \phi_j(y)^* \otimes \phi_j(x)$,

where $\phi_j(y)^* \otimes \phi_j(x) v = \langle v, \phi_j(y) \rangle_{\mathcal{C}^N} \phi_j(x)$, for all $v \in \mathcal{C}^N$ (that is, $\phi_j(y)^* \otimes \phi_j(x) = \phi_j(x)^t \phi_j(y)$, column-times-row).
Proof. Let $k_0 \in \mathbb{N}$ be such that
\[
\frac{f(\lambda_j)}{(1 + \lambda_j)^{2k_0}} \longrightarrow 0, \quad j \to +\infty. \tag{5.10}
\]
Let $P_m$ be the $L^2$-orthogonal projection onto $\text{Span}\{\varphi_j\}_{1 \leq j \leq m}$. Then $R_m := f(A)P_m$ is the operator whose kernel is $K_m$. In addition, $R_m$ can be extended as an operator belonging to $\mathcal{L}(D(A^{k_0}), D(A^{k_0})^*)$, where, recall, $D(A^{k_0})^*$ is the dual space of $D(A^{k_0})$. By (5.10), $\{R_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(D(A^{k_0}), D(A^{k_0})^*)$. Hence there exists
\[
\lim_{m \to +\infty} R_m = R \in \mathcal{L}(D(A^{k_0}), D(A^{k_0})^*).
\]
But for all $j \geq 1$
\[
R \varphi_j = \lim_{m \to +\infty} R_m \varphi_j = \lim_{m \to +\infty} f(A) \varphi_j = f(A) \varphi_j.
\]
Since $\mathcal{S} \hookrightarrow D(A^{k_0})$, and $D(A^{k_0})^* \hookrightarrow \mathcal{S}'$, we have
\[
R \in \mathcal{L}(\mathcal{S}, \mathcal{S}'),
\]
with
\[
R \varphi_j = f(A) \varphi_j, \quad \forall j \geq 1.
\]
Hence $R = f(A)$ in $\mathcal{L}(\mathcal{S}, \mathcal{S}')$, so that $f(A) = \lim_{m \to +\infty} R_m$. \qed

Proposition 5.4.2. If $f$ is rapidly decreasing on $\text{Spec}(A)$ then
\[
K_m \overset{\mathcal{S}}{\longrightarrow} K, \quad m \to +\infty.
\]

Proof. From Proposition 5.4.1 with $k_0 = 0$ we obtain that $K_m \overset{L^2}{\longrightarrow} K$ as $m \to +\infty$.
Let $A_x$ be the operator $A$ acting on $x$-functions (or distributions). Then, with
\[
\left( A^*_y \otimes A^*_x \right) (v^* \otimes u) = (A_y v)^* \otimes (A_x u),
\]
we have, for $r \in \mathbb{N},$
\[
\left( \left( A^*_y \right)^r \otimes A^*_x \right) K_m (x, y) = \sum_{j=1}^{m} \lambda_j^{2r} f(\lambda_j) \varphi_j(y)^* \otimes \varphi_j(x).
\]
Hence, on the one hand, by Proposition 5.4.1,
\[
\left( \left( A^*_y \right)^r \otimes A^*_x \right) K_m \longrightarrow \left( \left( A^*_y \right)^r \otimes A^*_x \right) K, \quad m \to +\infty,
\]
in the topology of $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n; M_N)$, and on the other hand $\lambda_j^{2r} f(\lambda_j)$ is rapidly decreasing for $j \to +\infty$, whence it also follows that
\[
\left( \left( A^*_y \right)^r \otimes A^*_x \right) K_m \longrightarrow \left( \left( A^*_y \right)^r \otimes A^*_x \right) K, \quad m \to +\infty,
\]
in the topology of \( L^2(\mathbb{R}^n \times \mathbb{R}^n; M_N) \), for all \( r \in \mathbb{Z}_+ \). Since \( A_y^* \) and \( A_x \) are elliptic we obtain, by a natural vector-valued regularity theorem for \( A_x \) or \( A_y^* \) starting from \( L^2_{x,y} \), that
\[
K_m \longrightarrow K, \quad \text{as} \quad m \rightarrow +\infty,
\]
in the topology of \( B'_x \otimes B'_y \), for all \( r \in \mathbb{Z}_+ \). Since \( B'_x \otimes B'_y \subset B'_{x,y} \) and \( \bigcap_{r \in \mathbb{Z}_+} B'_{x,y} = \mathcal{S}_{x,y} \), the claim follows.

**Corollary 5.4.3.** If \( f \) is rapidly decreasing on \( \text{Spec}(A) \) then
\[
\text{Tr} f(A) := \sum_{j \geq 1} f(\lambda_j) = \int_{\mathbb{R}^n} \text{Tr}(K(x,x)) \, dx,
\]
where, recall, \( \text{Tr} \) denotes the matrix-trace.

To make this section self-contained, we prove the following “rough” information on the behavior of the eigenvalues of \( A \).

**Proposition 5.4.4.** There exists \( k_0 \in \mathbb{N} \) such that
\[
\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{2k_0}} < +\infty.
\]

**Proof.** For simplicity we consider the scalar case. We know that the operator \( A^{-k} \in \text{OPS}_c(m^{-k\mu}, g) \), \( k \in \mathbb{N} \). Using Theorem 3.1.16 (or Theorem 3.2.17), write \( A^{-k} = b(x,D) + R \), where \( R \) is smoothing. Then for the Schwartz-kernel \( K \) of \( A^{-k} \) we have
\[
K(x,y) = K_0(x,y) + K_1(x,y),
\]
where \( K_1 \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \) is the Schwartz-kernel of \( R \), and \( K_0 \) is the Schwartz-kernel of \( b(x,D) \), given by
\[
K_0(x,y) = (2\pi)^{-n} \int e^{i(x-y,\xi)} b(x,\xi) d\xi \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)
\]
(in the sense of oscillatory integrals, see Helffer [17] or Shubin [67]). Let us choose in the first place the integer \( k \) so that \( k\mu > n \). Then \( K_0 \in C(\mathbb{R}^n \times \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n) \). If \( k \) is picked so that \( k\mu > n + 1 \), we have in addition that
\[
y_j K_0(x,y) = (2\pi)^{-n} \int y_j e^{i(x-y,\xi)} b(x,\xi) d\xi
\]
\[
= -(2\pi)^{-n} \int D_{\xi_j} (e^{i(x-y,\xi)}) b(x,\xi) d\xi + x_j (2\pi)^{-n} \int e^{i(x-y,\xi)} b(x,\xi) d\xi
\]
\[
= (2\pi)^{-n} \int e^{i(x-y,\xi)} \left( (D_{\xi_j} b)(x,\xi) + x_j b(x,\xi) \right) d\xi,
\]
from which we deduce that $y_jK_0 \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$. By the same token, we may finally conclude that there is $k_0 = k_0(n)$ such that picking $k = k_0$ gives

$$x^\alpha y^\beta K_0 \in C(\mathbb{R}^n \times \mathbb{R}^n) \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n), \forall \alpha, \beta \in \mathbb{Z}^n, |\alpha| + |\beta| \leq 2(n + 1),$$

whence $K_0 \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, and therefore also $K \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Since $\{ \phi_j \}_{j \geq 1}$ and $\{ \tilde{\phi}_j \}_{j \geq 1}$ are orthonormal systems in $L^2(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively, we get that $\phi_j(y) \otimes \phi_{j'}(x) = \phi_{j'}(x)\tilde{\phi}_j(y)$ in the scalar case is an orthonormal system in $L^2_{x,y}$. The Fourier coefficients of $K$ with respect to this basis are given by

$$a_{jj'} := \iint K(x,y)\phi_j(y)\overline{\phi_{j'}(x)}dxdy = (A^{-k_0}\phi_j, \phi_{j'}) = \lambda_j^{-k_0}(\phi_j, \phi_{j'}) = \lambda_j^{-k_0}\delta_{jj'}.$$

Hence

$$\iint |K(x,y)|^2dxdy = \sum_{j,j'=1}^\infty |a_{jj'}|^2 = \sum_{j=1}^\infty \frac{1}{\lambda_j^{2k_0}} < +\infty,$$ and this concludes the proof. \hfill \Box

It is useful to define also the trace of $t \mapsto e^{-\mu A}$, that is the trace of the Schrödinger group (which is clearly important in its own right, and basic when studying Poisson-like relations). We have the following proposition.

**Proposition 5.4.5.** The map

$$\mathcal{S}(\mathbb{R}) \ni \phi \longmapsto \text{Tr} \hat{\phi}(A), \hat{\phi}(t) = \int e^{-itA} \phi(x)dx,$$

defines a distribution in $\mathcal{S}'(\mathbb{R})$, denoted by Tr $e^{-\mu A}$.

**Proof.** The map $\phi \longmapsto \hat{\phi}$ is continuous in $\mathcal{S}$. It suffices therefore to prove that for some $C > 0$ we have

$$|\text{Tr} \hat{\phi}(A)| \leq C|\hat{\phi}|_{p,q}, \quad (5.11)$$

where $|\hat{\phi}|_{p,q}$ is some suitable $\mathcal{S}$-seminorm of $\hat{\phi}$. By Proposition 5.4.4 (alternatively, in case $\mu = 2\nu$ by Theorem 4.3.4, Remark 4.3.5, Theorem 4.4.1 and Remark 4.4.2, we may also use the behavior $\lambda_j \approx j^{\nu/n}$ as $j \to +\infty$), there exists $r \in \mathbb{N}$ sufficiently large such that $\sum_{j \geq 1} (1 + \lambda_j^2)^{-r} < +\infty$. Hence

$$\left| \sum_{j \geq 1} \hat{\phi}(\lambda_j) \right| = \left| \sum_{j \geq 1} (1 + \lambda_j^2)^{-r}(1 + \lambda_j^2)^r \hat{\phi}(\lambda_j) \right| \leq \left( \sum_{j \geq 1} (1 + \lambda_j^2)^{-r} \right) \sup_{t \in \mathbb{R}} |(1 + t^2)^r \hat{\phi}(t)|,$$

which proves (5.11) and the proposition. \hfill \Box
5.5 \( f(A) \) as a Pseudodifferential Operator

The following theorem, which concludes the chapter, is also useful (see Helffer \[17\] and Robert \[63–65\]). It is obtained through the calculus with parameters.

**Theorem 5.5.1.** Let \( 0 < A = A^* \in \text{OP}_{\text{cl}}(m^\mu, g), \mu \in \mathbb{N} \), be a scalar elliptic GPDO of order \( \mu \). Let \( f : (-c, +\infty) \rightarrow \mathbb{C} \) satisfy, for some \( r \in \mathbb{R} \), the inequalities

\[
\forall k \in \mathbb{Z}_+, \exists C_k > 0, \text{ such that } |f^{(k)}(\lambda)| \leq C_k(1 + |\lambda|)^{r-k}.
\]

Then the operator-function \( f(A) = F^w(x, D) \in \text{OP}(m^\mu, g), \) with

\[
F \sim \sum_{j=0}^\mu F_{r\mu-j}, \quad F_{r\mu-j} \in S(m^\mu-r-j, g),
\]

and

- \( F_{r\mu} = f(a_\mu) \);
- \( F_{r\mu-1} = a_{\mu-1}f'(a_\mu) \) (and hence it is equal to 0, for in our case \( a_{\mu-1} = 0 \));
- \( F_{r\mu-j} = \sum_{k=1}^j \frac{d_{jk}(a)}{k!} f^{(k)}(a_\mu), \quad j \geq 2, \) where the \( d_{jk} \in S(m^\mu-r-j, g) \) and depend only on the symbol \( a \).

Furthermore, if \( f \) is classic, that is \( f(\lambda) \sim \sum_{j \geq 0} c_j \lambda^{r-j}, \lambda > 0 \), then \( f(A) \in \text{OP}_{\text{cl}}(m^\mu, g) \).

**Remark 5.5.2.** In the matrix-valued case we shall need the result of Theorem 5.5.1 only for the complex power \( A^z \), \( z \in \mathbb{C} \), of an elliptic system of GPDOs \( 0 < A = A^* \in \text{OP}_{\text{cl}}(m^\mu, g; M_N) \). By a result due to Robert \[63\], one has that \( A^z \) belongs to the class \( \text{OP}_{\text{cl}}(m^\mu \text{Re}z, g; M_N) \).

**Corollary 5.5.3.** In particular, for \( Q_{(\alpha, \beta)} \in S_{\text{cl}}(m^2, g; M_2) \), which is globally positive elliptic for \( \alpha, \beta > 0 \) and \( \alpha\beta > 1 \), Remark 5.5.2 holds true.

5.6 Notes

The reader is addressed to Helffer’s book \[17\] for the functional calculus of global pseudodifferential operators (see also the paper by Helffer and Robert \[19\]).