4-Labelings and Grid Embeddings of Plane Quadrangulations

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Finding aesthetic drawings of planar graphs is a main issue in graph drawing. Of special interest are *rectangle of influence drawings*.The graphs considered here are quadrangulations, that is, planar graphs all whose faces have degree four. We show that each quadrangulation on n vertices has a closed rectangle of influence drawing on the $(n-2) \times (n-2)$ grid. Biedl, Bretscher and Meijer [2] proved that every planar graph on n vertices without separating triangle has a closed rectangle of influence drawing on the $(n-1) \times (n-1)$ grid.Our method, which is completely different from that of [2], is in analogy to Schnyder's algorithm for embedding triangulations on an integer grid [9] and gives a simple algorithm.

Schnyder [9] showed that labeling the angles of a triangulation T with 3 colors, with special rules, gives a 3-coloring and 2-orientation of the edges of T such that the edges of each color form a directed tree. For each interior vertex of T, the three colored paths to the sinks of the respective trees divide T into three regions. Counting the number of faces in each region gives the coordinates of the interior vertex in the grid drawing. Felsner [3] extended this result to the class of 3-connected plane graphs. In [8] it was studied to adapt this method to quadrangulations. In this case, the angles of a quadrangulation Q can be colored with 2 colors, which gives an analogous 2-coloring and 2-orientation of the edges of Q such that the edges of each color form a directed tree, and for each interior vertex the two colored paths to the respective sinks divide Q into two regions. In [5] it is shown that counting the number of faces in a region of an interior vertex v of Q gives the coordinate of v in a book embedding of Q with two pages. Each page in this book embedding for Q contains one of the two trees. Book embeddings of quadrangulations were also found in [6]. Whether this approach also gives a grid embedding for quadrangulations remained open.

We show here that labeling the angles of Q with 4 colors instead of 2 (which gives a 4-coloring and 2-orientation of the edges) allows to obtain a pair of book embeddings of Q such that the coordinates of a vertex v in the two book embeddings are the coordinates of v in the grid drawing of Q. It turns out that this embedding is a closed rectangle of influence drawing. It has the further property that edges of different colors are oriented in different directions (north-east, south-east, south-west, north-west). As a by-product of the rectangle of influence drawing, we also obtain a grid drawing of a quadrangulation on an

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$\lceil \frac{n}{2} \rceil \times \lceil \frac{3n}{4} \rceil$ grid by simple scaling. This is not optimal, because quadrangulations on n vertices have a straight-line embedding on an $(\lceil \frac{n}{2} \rceil - 1) \times \lfloor \frac{n}{2} \rfloor$ grid. However, the known algorithms by Biedl and Brandenburg [1] and Fusy [7], both require to add edges to make the quadrangulation 4-connected. An advantage of our simple algorithm is that it does not need to add edges and also works for quadrangulations with connectivity 2.

Quadrangulations Q are known to admit a touching segment representation: de Fraysseix, de Mendez and Pach [6] showed that one can assign vertical segments and horizontal segments to the vertices of Q such that two segments touch if and only if the two corresponding vertices of Q are adjacent. A different proof of this result, based on book embeddings of Q, is by Felsner et al. [4], who provided a bijection between the two trees of book embeddings of quadrangulations and rectangulations of a diagonal point set. The 4-labeling of a quadrangulation Q gives two book embeddings and therefore two rectangulations by [4]. This pair of rectangulations has the further nice property that in each rectangulation the boxes correspond isomorphically to the faces of Q (that is, the dual graphs are isomorphic), both rectangulations have the same fixed outer face, and each segment intersects the line with slope 1 in one rectangulation and intersects the line with slope -1 in the other one.

This work builds upon previous results on binary labelings of quadrangulations from [5,8]. The novelty is the use of four colors instead of two and its application to the grid drawing. 4-labelings are in bijection with binary labelings from [5]. Using four colors allows us to get more insight into the combinatorial structure of quadrangulations.

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