3D Visibility Representations by Regular Polygons

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Abstract. We study 3D visibility representations of complete graphs where vertices are represented by equal regular polygons lying in planes parallel to the xy-plane. Edges correspond to the z-parallel visibility among these polygons.

We improve the upper bound on the maximum size of a complete graph with a 3D visibility representation by regular n-gons from $2^{O(n)}$ to $O(n^4)$.

1 Introduction

In this paper we study 3D visibility drawings that represent vertices by two-dimensional sets placed in planes parallel to the xy-plane. Two vertices are connected by an edge if and only if they can see each other in the direction that is orthogonal to their planes, i.e., parallel to the z-axis.

This type of representation was introduced as a generalization of the 2D visibility drawing. The 2D rectangle visibility drawing received a wide attention because of its connection to VLSI routing and circuit board layout [7,8].

The representation of vertices by rectangles remains popular also in the 3D visibility drawing. A lot of papers are focused on the maximum size of a complete graph with a 3D visibility representation by rectangles. Rote and Zelle provide a representation of K_{22} (see [6]). On the other hand, Bose et al. [4] showed that no complete graph with more than 102 vertices has such a representation. This result was then improved to 55 by Fekete et al. [3] and recently by Štola [5] to 50.

If the vertices are represented by unit squares then the largest complete graph with this type of representation is K_7 according to [3]. This is the only exact result known about representations by equal regular n-gons. Only estimates are known for $n \neq 4$. Babilon et al. [2] show that K_{14} can be represented by equal triangles. They also present a lower bound $\lfloor \frac{n+1}{2} \rfloor + 2$ on the maximum size of a complete graph with a 3D visibility representation by equal regular n-gons. Štola [1] then moved this bound to $n+1$. The first upper bound 2^{2^n} was given by Babilon et al. [2]. This doubly-exponential estimate was improved by Štola [1] to an exponential $\binom{6n-3}{3n-1} - 3 \approx 2^{6n}$. The main result of this paper is another significant improvement of this bound. We present a polynomial upper bound $O(n^4)$.

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2 Preliminaries

Let P be a regular n-gon inscribed in a unit circle (with the center c). Let $v_0, v_1, \ldots, v_n = v_0$ be the vertices of P, $s_0 = \overline{v_0 v_1}, \ldots, s_{n-1} = \overline{v_{n-1} v_n}, s_n = s_0$ the sides of P, m_i the center of s_i and p_i the half-line $\overrightarrow{cm_i}$. If P_i is a copy of P (shifted by a vector $\boldsymbol{w_i}$) then we denote its vertices by v^i_j and the sides by s^i_j.

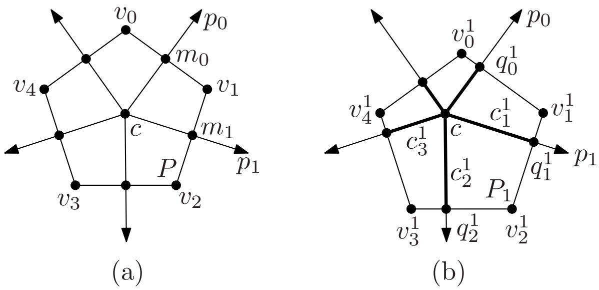

Fig. 1.

The distance of v_j and p_j is $\sin(\pi/n)$, similarly $\text{dist}(v_j, p_{j-1}) = \sin(\pi/n)$ and $\text{dist}(s_j, c) = \cos(\pi/n)$. Hence, if $|\boldsymbol{w_i}| < \sin(\pi/n)$ then v^i_j (the shifted copy of v_j) remains in the angle $\widehat{m_{j-1} c m_j}$. If in addition $|\boldsymbol{w_i}| < \cos(\pi/n)$ then s^i_j intersects p_j.

Definition 1. *Let $\{P_i, P_i = P + \boldsymbol{w_i}\}$ be the set of shifted copies of a regular n-gon P (inscribed in a unit circle). We say that this set is* a short-distance set *if $\forall i : |\boldsymbol{w_i}| < \min(\sin(\pi/n), \cos(\pi/n))$.*

The definition of a short-distance set requires a reference polygon P that is close to every polygon from the set. If the polygons $P_i = P + \boldsymbol{w_i}$ are far from P but close to each other, i.e., $\forall i, j : |\boldsymbol{w_i} - \boldsymbol{w_j}| < \min(\sin(\pi/n), \cos(\pi/n))$ then they also form a short-distance set because we can take any P_i as a reference polygon in this case.

For a polygon P_i from a short-distance set we can define $q^i_j = p_j \cap s^i_j$ and $c^i_j = \text{dist}(c, q^i_j)$ (see Figure 1b). We call the n-tuple $(c^i_j)^n_{j=1}$ *the coordinates* of P_i.

Every polygon can be reconstructed from its coordinates (see Figure 2). If H^i_j is the half-plane with its boundary line h^i_j such that $c \in H^i_j$, $h^i_j \perp p_j$ and $\text{dist}(h^i_j, c) = c^i_j$ then $P_i = \bigcap^n_{j=1} H^i_j$. Therefore the intersection $P_i \cap P_k = \bigcap^n_{j=1} (H^i_j \cap H^k_j)$ can be described by coordinates $(\min(c^i_j, c^k_j))^n_{j=1}$.

We assume in the sequel that P is a regular n-gon inscribed in a unit circle and $\{P_i = P + \boldsymbol{w_i}, i = 1, \ldots, m\}$ is a 3D visibility representation of a complete graph K_m. We assume that the z-coordinate of P_i is i but we use it to identify polygons that can block visibility between other polygons only. Otherwise, we ignore the z-coordinate and work with the polygons as if they were in the same

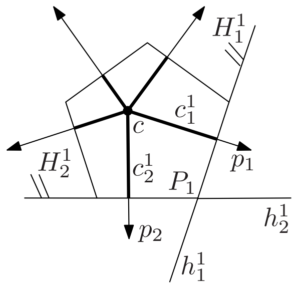

Fig. 2. Reconstruction of the polygon P_1 from its coordinates

xy-parallel plane. Formally, these operations represent operations over orthogonal projections of the relevant objects (points, lines, polygons) into a common xy-parallel plane and the projection of the results (for example, intersection points) into individual planes of the polygons.

Lemma 1. *Polygons P_i and P_k can see each other if and only if there exists l such that $\forall j, i < j < k : (c_l^j < \min(c_l^i, c_l^k)$ or $c_{l+1}^j < \min(c_{l+1}^i, c_{l+1}^k))$.*

Proof. $Q = P_i \cap P_k$ is a polygon given by coordinates $(\min(c_j^i, c_j^k))_{j=1}^n$. Let Q_l be the intersection of Q with the angle $\widehat{m_l c m_{l+1}}$ and q_l be the (only) vertex of Q in $\widehat{m_l c m_{l+1}}$.

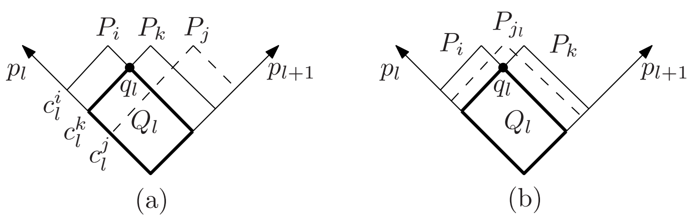

Fig. 3.

If $c_l^j < \min(c_l^i, c_l^k)$ or $c_{l+1}^j < \min(c_{l+1}^i, c_{l+1}^k)$ then P_j doesn't block the visibility of P_i and P_k in the neighborhood of q_l, see Figure 3a. Hence, if for a fixed l this condition holds for all polygons P_j between P_i and P_k then P_i and P_k can see each other in the neighborhood of q_l.

On the other hand, if $\forall l\ \exists j_l : i < j_l < k$, $c_l^{j_l} \geq \min(c_l^i, c_l^k)$ and $c_{l+1}^{j_l} \geq \min(c_{l+1}^i, c_{l+1}^k)$ then P_{j_l} blocks the visibility of P_i and P_k in the angle $\widehat{m_l c m_{l+1}}$, see Figure 3b. Therefore P_i cannot see P_k. □

Lemma 1 describes a sufficient and necessary condition for the visibility between two polygons from a short-distance set. If we shift the polygon P_i by a sufficiently small vector then we don't break any of the strict inequalities in Lemma 1. In other words, the shifted polygon can see all polygons that the original polygon can see. Therefore we can replace the original polygon P_i by the shifted one without breaking the completeness of the represented graph. This observation allows us to assume in the sequel that j-th coordinates of polygons are distinct, i.e., $\forall i, j, k, i \neq k : c_j^i \neq c_j^k$.

Lemma 2. *Let P_i be a regular n-gon with coordinates $(c_j^i)_{j=1}^n$ and $P_k = P_i + \boldsymbol{w}$ a shifted copy of P_i with coordinates $(c_j^k)_{j=1}^n$. If n is even then there are exactly $n/2$ adjacent coordinates with $\operatorname{sgn}(c_j^k - c_j^i) = 1$ and $n/2$ adjacent coordinates with the opposite signum. If n is odd then there are $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ adjacent coordinates with $\operatorname{sgn}(c_j^k - c_j^i) = 1$ and the rest with the opposite signum.*

Proof. The length of the orthogonal projection of $\boldsymbol{w}$ into a line containing p_j is $|c_j^k - c_j^i|$. The difference $c_j^k - c_j^i$ is positive (resp. negative) if this projection of $\boldsymbol{w}$ has the same (resp. the opposite) orientation as p_j.

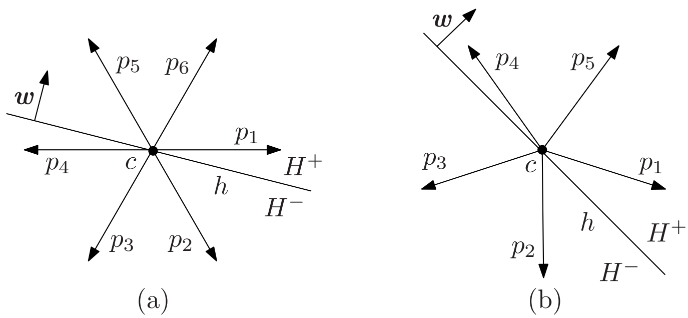

Fig. 4.

Let h be a line such that $h \perp \boldsymbol{w}$ and $c \in h$. h divides the plane into half-planes H^+ and H^-. Let H^+ be the half-plane in the direction of the vector $\boldsymbol{w}$. p_j lies in H^+ resp. H^- if $c_j^k > c_j^i$ resp. $c_j^i > c_j^k$.

If n is even then exactly $n/2$ adjacent half-lines from $(p_j)_{j=1}^n$ lie in H^+ and $n/2$ adjacent half-lines lie in H^-, see Figure 4a. If n is odd then $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ adjacent half-lines lie in H^+ and the rest of them lie in H^-, see Figure 4b. □

The next lemma shows that every 3D visibility representation of a complete graph contains a large short-distance subset. The following sections focus on these subsets.

Lemma 3. *Let $\{P_i = P + \boldsymbol{w_i}, i = 1, \ldots, m\}$ be a set of regular n-gons. If $\{P_i\}$ is a 3D visibility representation of a complete graph K_m then $\{P_i\}$ contains a short-distance subset with at least $\lceil m/16n^2 \rceil$ polygons.*

Proof. Every two polygons P_j, P_k from the representation have to intersect (to see each other). Polygons $\{P_i\}$ are shifted copies of P (a polygon inscribed into a unit circle). Hence, P_j can intersect P_k only if the distance of their centers is at most 2. Therefore the set C of centers of polygons from $\{P_i\}$ has the diameter at most 2.

Let S be a square that contains all points from C and whose side-length is 2. We can divide this square into $4n \times 4n = 16n^2$ sub-squares with the side-length $1/2n$. At least one of these sub-squares must contain at least $\lceil m/16n^2 \rceil$ points of C. We claim that the polygons with the center in this sub-square form a short-distance set.

It is sufficient to show that two points in one sub-square have the distance lower than $\min(\sin(\pi/n), \cos(\pi/n))$. For $x \in (0, \pi/3\rangle$ we have $\frac{x}{\sqrt{2}\pi} < \min(\sin x, \cos x)$. Hence, for $n \geq 3$ we have $\frac{1}{\sqrt{2}n} < \min(\sin(\pi/n), \cos(\pi/n))$ and $\frac{1}{\sqrt{2}n}$ is the maximum distance of two points in one sub-square. □

3 Regular $2k$-gons

The goal of this section is a polynomial upper bound on the maximum size of a complete graph with a 3D visibility representation by regular $2k$-gons. We start with a lemma that points out an important forbidden configuration of three polygons.

Lemma 4. *Let $\{P_1, P_2, P_3\}$ be a short-distance set of regular $2k$-gons. If $\{P_1, P_2, P_3\}$ is a 3D visibility representation of a complete graph K_3 then it cannot happen that $c_1^1 < c_1^2 < c_1^3$ and $c_2^1 > c_2^2 > c_2^3$ (where $(c_j^i)_{j=1}^n$ are coordinates of P_i).*

Proof. If $c_1^1 < c_1^2 < c_1^3$ and $c_2^1 > c_2^2 > c_2^3$ then $c_l^1 > c_l^2 > c_l^3$ for $l \in \{2, \ldots, k+1\}$ and $c_l^1 < c_l^2 < c_l^3$ for $l \in \{k+2, \ldots, 2k\} \cup \{1\}$ by Lemma 2. Therefore, $c_l^2 > \min(c_l^1, c_l^3)$ for $l \in \{1, \ldots, 2k\}$ and P_1 cannot see P_3 according to Lemma 1 but this is a contradiction. □

The following lemma shows that if the sequence $(c_1^i)_i$ of the first coordinates is monotone then the size of the representation is small.

Lemma 5. *Let $\{P_i, i = 1, \ldots, m\}$ be a short-distance set of regular $2k$-gons. If $\{P_i\}$ is a 3D visibility representation of a complete graph K_m and $(c_1^i)_{i=1}^m$ is a monotone sequence (where $(c_j^i)_{j=1}^n$ are coordinates of P_i) then $m \leq k+1$.*

Proof. We assume that the sequence $(c_1^i)_{i=1}^m$ is increasing. The proof for a decreasing sequence is similar. Let $I = \{\{i,j\} : i < j, c_2^i > c_2^j\}$, i.e., the pairs of polygons whose boundaries intersect in $\widehat{m_1 c m_2}$. We claim that $I = \emptyset$ or $\bigcap I \neq \emptyset$.

We proceed by contradiction. Let's assume that $I \neq \emptyset$ and $\bigcap I = \emptyset$. At first we show that there must be (at least) two disjoint pairs in I. Let's assume that there aren't two disjoint pairs in I. If $\{a, \overline{a} : a < \overline{a}\} \in I$ then there exist $B = \{b, \overline{b} : b < \overline{b}\}$ and $C = \{c, \overline{c} : c < \overline{c}\}$ in I such that $a \notin B$ and $\overline{a} \notin C$ (because $a, \overline{a} \notin \bigcap I$). Moreover $\overline{a} \in B$ and $a \in C$ because the pairs $\{a, \overline{a}\}$ and B (resp. C) are not disjoint. If $\overline{a} = b$ then $c_1^a < c_1^{\overline{a}} = c_1^b < c_1^{\overline{b}}$ and $c_2^a > c_2^{\overline{a}} = c_2^b > c_2^{\overline{b}}$ which is in contradiction with Lemma 4. Therefore $\overline{a} = \overline{b}$ and $B = \{b, \overline{a}\}$.

An analogous argument shows that $a = c$ and $C = \{a, \overline{c}\}$. The pairs B and C are not disjoint according to our assumption. This can happen only if $\overline{c} = b$ but then $c_1^a < c_1^{\overline{c}} = c_1^b < c_1^{\overline{a}}$ and $c_2^a > c_2^{\overline{c}} = c_2^b > c_2^{\overline{a}}$ which is in contradiction with Lemma 4 again. This means that there must be two disjoint pairs in I.

Let $\{a, \overline{a} : a < \overline{a}\}$ and $\{b, \overline{b} : b < \overline{b}\}$ be disjoint pairs in I. We can assume without loss of generality that $a < b$.

Let's assume that $\overline{a} < \overline{b}$ (see Figure 5):

$$a < \overline{a} < \overline{b}, a < b < \overline{b}, (c_1^i)_i \text{ increasing} \Rightarrow c_1^a < c_1^{\overline{a}} < c_1^{\overline{b}}, c_1^a < c_1^b < c_1^{\overline{b}}$$
$$\{a, \overline{a} : a < \overline{a}\}, \{b, \overline{b} : b < \overline{b}\} \in I \Rightarrow c_2^a > c_2^{\overline{a}}, c_2^b > c_2^{\overline{b}}$$
$$c_1^b < c_1^{\overline{b}}, c_2^b > c_2^{\overline{b}} \Rightarrow c_l^b > c_l^{\overline{b}}, l \in \{2, \ldots, k+1\} \text{ by Lemma 2}$$
$$c_1^a < c_1^{\overline{a}}, c_2^a > c_2^{\overline{a}} \Rightarrow c_l^a < c_l^{\overline{a}}, l \in \{k+2, \ldots, 2k\} \cup \{1\} \text{ by Lemma 2}$$
$$c_1^{\overline{a}} < c_1^{\overline{b}} \Rightarrow c_{k+1}^{\overline{b}} < c_{k+1}^{\overline{a}} \text{ by Lemma 2}$$

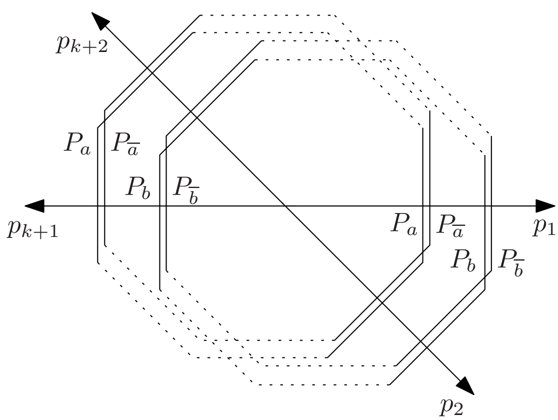

Fig. 5.

We can see that $c_1^a < c_1^b$ and $c_l^b > c_l^{\overline{b}}, l \in \{2, \ldots, k+1\}$. Therefore $c_l^b > \min(c_l^a, c_l^{\overline{b}})$, $l \in \{1, \ldots, k+1\}$. Similarly, $c_{k+1}^{\overline{b}} < c_{k+1}^{\overline{a}}$ and $c_l^a < c_l^{\overline{a}}$, $l \in \{k+2, \ldots, 2k\} \cup \{1\}$, i.e., $c_l^{\overline{a}} > \min(c_l^a, c_l^{\overline{b}})$, $l \in \{k+1, \ldots, 2k\} \cup \{1\}$. Hence, P_a cannot see $P_{\overline{b}}$ according to Lemma 1 but this cannot happen because $\{P_i\}$ is a representation of a complete graph. Therefore, it cannot be $\overline{a} < \overline{b}$.

If $\overline{b} < \overline{a}$ then $a < b < \overline{b} < \overline{a}$ and $c_1^a < c_1^b < c_1^{\overline{b}} < c_1^{\overline{a}}$ because $(c_1^i)_i$ is increasing. $c_2^{\overline{a}} < c_2^a$ and $c_2^{\overline{b}} < c_2^b$ because $\{a, \overline{a} : a < \overline{a}\}$, $\{b, \overline{b} : b < \overline{b}\} \in I$. If $c_2^{\overline{a}} < c_2^{\overline{b}}$ then $P_b, P_{\overline{b}}$ and $P_{\overline{a}}$ are in contradiction with Lemma 4. Similarly, if $c_2^b < c_2^a$ then P_a, P_b and $P_{\overline{b}}$ are in contradiction with Lemma 4. Therefore it must be $c_2^{\overline{b}} < c_2^{\overline{a}} < c_2^a < c_2^b$ but this means that the disjoint pairs $\{a, \overline{b} : a < \overline{b}\}$, $\{b, \overline{a} : b < \overline{a}\}$ satisfy assumptions of the previous paragraph and we again have a contradiction with the completeness of the represented graph.

We know that $\overline{a} \neq \overline{b}$ because $\{a, \overline{a}\}$ and $\{b, \overline{b}\}$ are disjoint. On the other hand, both possibilities $\overline{a} < \overline{b}$ and $\overline{b} < \overline{a}$ lead to a contradiction. Hence, the original assumption that $I \neq \emptyset$ and $\bigcap I = \emptyset$ cannot be satisfied. It must be either $I = \emptyset$ or $\bigcap I \neq \emptyset$.

If $I = \emptyset$ then $(c_2^i)_i$ is increasing. If $I \neq \emptyset$ then there exists $a \in \bigcap I$. This means that if $i < j$ and $c_i^2 > c_j^2$ then $i = a$ or $j = a$. In other words, the sequence $(c_2^i)_{i \in \{1,\dots,m\} \setminus \{a\}}$ is increasing.

We can repeat this proof with $c_2, c_3, \dots, c_k$ subsequently and show that there is a set A such that $|A| \leq k$ and $(c_{k+1}^i)_{i \in \{1,\dots,m\} \setminus A}$ is increasing. On the other hand, this sequence is also decreasing by Lemma 2 because $(c_1^i)_{i \in \{1,\dots,m\} \setminus A}$ is increasing. Therefore the sequence $(c_{k+1}^i)_{i \in \{1,\dots,m\} \setminus A}$ has length at most 1 and $1 \geq |\{1, \dots, m\} \setminus A| \geq m - k$. □

Now we are ready to prove the main theorem of this section.

Theorem 1. *If $\{P_i, i = 1, \dots, m\}$ is a 3D visibility representation of a complete graph K_m by regular n-gons (where $n = 2k$) then $m \leq 4n^2(n+2)^2$.*

Proof. The set $\{P_i\}$ contains a short-distance subset $\{P_i'\}$ with at least $\lceil m/16n^2 \rceil$ polygons according to Lemma 3. Let $(c_j^i)_{j=1}^n$ be coordinates of P_i'. If $\lceil m/16n^2 \rceil \geq (k+1)^2 + 1$ then due to Erdős-Szekeres theorem [9] the sequence $(c_1^i)_{i=1}^{\lceil m/16n^2 \rceil}$ contains a monotone subsequence of length $k+2$ which is in contradiction with Lemma 5. Therefore $m/16n^2 \leq \lceil m/16n^2 \rceil \leq (k+1)^2$. □

4 Regular $(2k+1)$-gons

We focus on regular $(2k+1)$-gons in this section. We prove a theorem analogous to Theorem 1. Unfortunately, Lemma 4 doesn't hold for $(2k+1)$-gons. We have to use a more complicated version.

Lemma 6. *Let $\{P_1, P_2, P_3, P_4\}$ be a short-distance set of regular $(2k+1)$-gons. If $\{P_i\}$ is a 3D visibility representation of a complete graph K_4 then it cannot happen that $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_2^1 > c_2^2 > c_2^3 > c_2^4$ (where $(c_j^i)_{j=1}^n$ are coordinates of P_i).*

Proof. If $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_2^1 > c_2^2 > c_2^3 > c_2^4$ then $c_l^1 > c_l^2 > c_l^3 > c_l^4$ for $l \in \{2, \dots, k+1\}$ and $c_l^1 < c_l^2 < c_l^3 < c_l^4$ for $l \in \{k+3, \dots, 2k+1\} \cup \{1\}$ by Lemma 2. In other words, $c_l^2 > \min(c_l^1, c_l^3)$ and $c_l^3 > \min(c_l^2, c_l^4)$ for $l \in \{1, \dots, 2k+1\} \setminus \{k+2\}$.

P_1 and P_3 can see each other. Therefore, $c_{k+2}^2 < \min(c_{k+2}^1, c_{k+2}^3)$ according to Lemma 1. Similarly, $c_{k+2}^3 < \min(c_{k+2}^2, c_{k+2}^4)$ because P_2 and P_4 can see each other. But this is a contradiction because the first inequality gives us $c_{k+2}^2 < c_{k+2}^3$ while $c_{k+2}^3 < c_{k+2}^2$ by the second inequality. □

We need the following consequence of Lemma 6 several times in the sequel.

Corollary 1. *Let $\{P_1, P_2, P_3, P_4\}$ be a short-distance set of regular $(2k+1)$-gons. If $\{P_i\}$ is a 3D visibility representation of a complete graph K_4 then it cannot happen that $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_{k+1}^1 < c_{k+1}^2 < c_{k+1}^3 < c_{k+1}^4$ (or $c_{k+2}^1 < c_{k+2}^2 < c_{k+2}^3 < c_{k+2}^4$).*

Proof. If $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_{k+1}^1 < c_{k+1}^2 < c_{k+1}^3 < c_{k+1}^4$ then $c_{k+2}^1 > c_{k+2}^2 > c_{k+2}^3 > c_{k+2}^4$ by Lemma 1 but this is in contradiction with Lemma 6 for coordinates $k+1$ and $k+2$ (Lemma 6 holds for any pair of adjacent coordinates).

Similarly, if $c_1^1 < c_1^2 < c_1^3 < c_1^4$ and $c_{k+2}^1 < c_{k+2}^2 < c_{k+2}^3 < c_{k+2}^4$ then $c_{k+1}^1 > c_{k+1}^2 > c_{k+1}^3 > c_{k+1}^4$ by Lemma 1 and we have a contradiction again. □

The next lemma is an analogy of Lemma 5. The proof of this lemma is more complicated because the representations by $(2k+1)$-gons are more complicated but the main ideas of both proofs (of Lemma 5 and Lemma 7) are the same.

Lemma 7. *Let $\{P_i, i = 1, \dots, m\}$ be a short-distance set of regular $(2k+1)$-gons. There exists $c > 0$ independent of k such that if $\{P_i\}$ is a 3D visibility representation of a complete graph K_m and $(c_1^i)_{i=1}^m$ is a monotone sequence (where $(c_j^i)_{j=1}^n$ are coordinates of P_i) then $m \leq ck$.*

Proof. We assume that the sequence $(c_1^i)_{i=1}^m$ is increasing. The proof for a decreasing sequence is similar. Let $I = \{\{i, j\} : i < j, c_2^i > c_2^j\}$. We claim that there exists $n_0 \in \mathbb{N}$ (independent of k) such that I doesn't contain n_0 pairwise disjoint pairs.

Let's assume that $J \subseteq I : \forall A, B \in J, A \neq B \Rightarrow A \cap B = \emptyset$. Consider a complete graph on the vertex set J. We color the edge $\{\{a, \overline{a} : a < \overline{a}\}, \{b, \overline{b} : b < \overline{b}\} : a < b\}$ by

- color 1 when $\overline{a} < \overline{b}$ and $c_{k+2}^{\overline{a}} < \min(c_{k+2}^a, c_{k+2}^{\overline{b}})$
- color 2 when $\overline{a} < \overline{b}$ and $c_{k+2}^{\overline{a}} > \min(c_{k+2}^a, c_{k+2}^{\overline{b}})$
- color 3 when $\overline{b} < \overline{a}, c_2^a < c_2^b$, $c_2^{\overline{b}} < c_2^{\overline{a}}$ and $c_{k+2}^{\overline{b}} < \min(c_{k+2}^a, c_{k+2}^{\overline{a}})$
- color 4 when $\overline{b} < \overline{a}, c_2^a < c_2^b$, $c_2^{\overline{b}} < c_2^{\overline{a}}$ and $c_{k+2}^{\overline{b}} > \min(c_{k+2}^a, c_{k+2}^{\overline{a}})$
- color 5 when $\overline{b} < \overline{a}, c_2^a < c_2^b$ and $c_2^{\overline{a}} < c_2^{\overline{b}}$
- color 6 when $\overline{b} < \overline{a}, c_2^b < c_2^a$ and $c_2^{\overline{b}} < c_2^{\overline{a}}$
- color 7 when $\overline{b} < \overline{a}, c_2^b < c_2^a$ and $c_2^{\overline{a}} < c_2^{\overline{b}}$

If $\{\{a, \overline{a} : a < \overline{a}\}, \{b, \overline{b} : b < \overline{b}\} : a < b\}$ has the 7th color then $c_1^a < c_1^b < c_1^{\overline{b}} < c_1^{\overline{a}}$ because $a < b < \overline{b} < \overline{a}$ and $(c_1^i)_i$ is increasing. $c_2^{\overline{b}} < c_2^b$ because $\{b, \overline{b} : b < \overline{b}\} \in I$. Therefore $c_2^{\overline{a}} < c_2^{\overline{b}} < c_2^b < c_2^a$ and $P_a, P_b, P_{\overline{b}}, P_{\overline{a}}$ are in contradiction with Lemma 6. Hence, the 7th the color is not used and every edge of K_J has one of the first six colors.

According to Ramsey's theorem [10,11] there exists n_0 such that if $|J| \geq n_0$ then K_J contains a monochromatic subgraph K_S, $S = \{\{a, \overline{a} : a < \overline{a}\}, \{b, \overline{b} : b < \overline{b}\}, \{c, \overline{c} : c < \overline{c}\}, \{d, \overline{d} : d < \overline{d}\} : a < b < c < d\}$.

If K_S has color 1 then $c_{k+2}^{\overline{a}} < c_{k+2}^{\overline{b}} < c_{k+2}^{\overline{c}} < c_{k+2}^{\overline{d}}$, $\overline{a} < \overline{b} < \overline{c} < \overline{d}$ and $c_1^{\overline{a}} < c_1^{\overline{b}} < c_1^{\overline{c}} < c_1^{\overline{d}}$ (because $(c_1^i)_i$ is increasing). This is in contradiction with Corollary 1.

If K_S has color 2 then we have:

$$a < b < \overline{b}, a < \overline{a} < \overline{b}, (c_1^i)_i \text{ increasing} \Rightarrow c_1^a < c_1^b < c_1^{\overline{b}}, c_1^a < c_1^{\overline{a}} < c_1^{\overline{b}}$$
$$\{a, \overline{a} : a < \overline{a}\}, \{b, \overline{b} : b < \overline{b}\} \in I \Rightarrow c_2^{\overline{a}} < c_2^a, c_2^{\overline{b}} < c_2^b$$
$$c_1^b < c_1^{\overline{b}}, c_2^{\overline{b}} < c_2^b \Rightarrow c_l^{\overline{b}} < c_l^b, l \in \{2, \ldots, k+1\} \text{ by Lemma 2}$$
$$c_1^a < c_1^{\overline{a}}, c_2^{\overline{a}} < c_2^a \Rightarrow c_l^a < c_l^{\overline{a}}, l \in \{k+3, \ldots, 2k+1\} \cup \{1\} \text{ by Lemma 2}$$

We can see that $c_1^a < c_1^b$ and $c_l^{\overline{b}} < c_l^b, l \in \{2, \ldots, k+1\}$. Hence, $c_l^b > \min(c_l^a, c_l^{\overline{b}})$ for $l \in \{1, \ldots, k+1\}$. Similarly, $c_l^a < c_l^{\overline{a}}, l \in \{k+3, \ldots, 2k+1\} \cup \{1\}$ and $c_{k+2}^{\overline{a}} > \min(c_{k+2}^a, c_{k+2}^{\overline{b}})$. Therefore $c_l^{\overline{a}} > \min(c_l^a, c_l^{\overline{b}})$ for $l \in \{k+2, \ldots, 2k+1\} \cup \{1\}$. If $c_{k+1}^{\overline{a}} > \min(c_{k+1}^a, c_{k+1}^{\overline{b}})$ then P_a cannot see $P_{\overline{b}}$ according to Lemma 1. It must be $c_{k+1}^{\overline{a}} < \min(c_{k+1}^a, c_{k+1}^{\overline{b}})$, namely $c_{k+1}^{\overline{a}} < c_{k+1}^{\overline{b}}$. The same argument shows that also $c_{k+1}^{\overline{b}} < c_{k+1}^{\overline{c}} < c_{k+1}^{\overline{d}}$. On the other hand, $c_1^{\overline{a}} < c_1^{\overline{b}} < c_1^{\overline{c}} < c_1^{\overline{d}}$ (because $\overline{a} < \overline{b} < \overline{c} < \overline{d}$) which is in contradiction with Corollary 1.

If K_S has color 3 then $c_{k+2}^{\overline{d}} < c_{k+2}^{\overline{c}} < c_{k+2}^{\overline{b}} < c_{k+2}^{\overline{a}}$, $\overline{d} < \overline{c} < \overline{b} < \overline{a}$ and $c_1^{\overline{d}} < c_1^{\overline{c}} < c_1^{\overline{b}} < c_1^{\overline{a}}$ (because $(c_1^i)_i$ is increasing) and we have a contradiction again.

If K_S has color 4 then we proceed in a similar way as with the second color. We have:

$$c_2^a < c_2^b, c_2^{\overline{b}} < c_2^{\overline{a}}$$
$$a < b < \overline{b} < \overline{a}, (c_1^i)_i \text{ increasing} \Rightarrow c_1^a < c_1^b < c_1^{\overline{b}} < c_1^{\overline{a}}$$
$$\{a, \overline{a} : a < \overline{a}\} \in I \Rightarrow c_2^{\overline{a}} < c_2^a$$
$$c_1^b < c_1^{\overline{a}}, c_2^{\overline{a}} < c_2^a < c_2^b \Rightarrow c_l^{\overline{a}} < c_l^b, l \in \{2, \ldots, k+1\} \text{ by Lemma 2}$$
$$c_1^a < c_1^{\overline{b}}, c_2^{\overline{b}} < c_2^{\overline{a}} < c_2^a \Rightarrow c_l^a < c_l^{\overline{b}}, l \in \{k+3, \ldots, 2k+1\} \cup \{1\} \text{ by Lemma 2}$$

We can see that $c_1^a < c_1^b$ and $c_l^{\overline{a}} < c_l^b, l \in \{2, \ldots, k+1\}$. Hence, $c_l^b > \min(c_l^a, c_l^{\overline{a}})$ for $l \in \{1, \ldots, k+1\}$. Similarly, $c_l^a < c_l^{\overline{b}}, l \in \{k+3, \ldots, 2k+1\} \cup \{1\}$ and $c_{k+2}^{\overline{b}} > \min(c_{k+2}^a, c_{k+2}^{\overline{a}})$. Therefore, $c_l^{\overline{b}} > \min(c_l^a, c_l^{\overline{a}})$ for $l \in \{k+2, \ldots, 2k+1\} \cup \{1\}$. If $c_{k+1}^{\overline{b}} > \min(c_{k+1}^a, c_{k+1}^{\overline{a}})$ then P_a cannot see $P_{\overline{a}}$ according to Lemma 1. It must be $c_{k+1}^{\overline{b}} < \min(c_{k+1}^a, c_{k+1}^{\overline{a}})$, namely $c_{k+1}^{\overline{b}} < c_{k+1}^{\overline{a}}$. The same argument shows that also $c_{k+1}^{\overline{d}} < c_{k+1}^{\overline{c}} < c_{k+1}^{\overline{b}}$. On the other hand, $c_1^{\overline{d}} < c_1^{\overline{c}} < c_1^{\overline{b}} < c_1^{\overline{a}}$ (because $\overline{d} < \overline{c} < \overline{b} < \overline{a}$) which is in contradiction with Corollary 1.

If K_S has color 5 then $c_2^{\overline{a}} < c_2^{\overline{b}} < c_2^{\overline{c}} < c_2^{\overline{d}}$, $\overline{d} < \overline{c} < \overline{b} < \overline{a}$ and $c_1^{\overline{d}} < c_1^{\overline{c}} < c_1^{\overline{b}} < c_1^{\overline{a}}$ (because $(c_1^i)_i$ is increasing). This is in contradiction with Lemma 6.

If K_S has color 6 then $c_2^d < c_2^c < c_2^b < c_2^a$, $a < b < c < d$ and $c_1^a < c_1^b < c_1^c < c_1^d$ (because $(c_1^i)_i$ is increasing) and we have a contradiction with Lemma 6 again.

We can see that K_J cannot contain a monochromatic subgraph K_S. Therefore $|J| \leq n_0 - 1$, i.e., I doesn't contain n_0 pairwise disjoint pairs.

Let $J_{max} \subseteq I$ be a maximal subset of pairwise disjoint pairs. We know that $|\bigcup J_{max}| = 2|J_{max}| \leq 2(n_0 - 1)$. For any $A \in I$ there exists $B \in J_{max}$ such that $A \cap B \neq \emptyset$. Hence, the sequence $(c_2^i)_{i \in \{1, \ldots, m\} \setminus \bigcup J_{max}}$ is increasing.

We can repeat this proof with $c_2, c_3, \ldots, c_k$ subsequently and show that there is a set J' such that $|J'| \leq 2(n_0 - 1)k$ and $(c^i_{k+1})_{i \in \{1,\ldots,m\} \setminus J'}$ is increasing. The sequence $(c^i_1)_{i \in \{1,\ldots,m\} \setminus J'}$ is also increasing. Therefore, its length is less than 4 by Corollary 1, i.e., $4 > |\{1, \ldots, m\} \setminus J'| \geq m - 2(n_0 - 1)k$. □

Lemma 7 allows us to prove an analogy of Theorem 1 for regular $(2k + 1)$-gons.

Theorem 2. *There exists $c > 0$ such that if $\{P_i, i = 1, \ldots, m\}$ is a 3D visibility representation of a complete graph K_m by regular n-gons (where $n = 2k + 1$) then $m \leq cn^4$.*

Proof. The proof is the same as the proof of Theorem 1 (using Lemma 7 instead of Lemma 5). □

If we combine Theorem 1 and Theorem 2 then we obtain the following result.

Theorem 3. *If $s(n)$ is the maximum size of a complete graph with a 3D visibility representation by equal regular n-gons then $s(n) = O(n^4)$.*

Proof. Theorem 1 if n is even and Theorem 2 if n is odd. □

5 Conclusion

We show that the maximum size of a complete graph with a 3D visibility representation by regular n-gons is $O(n^4)$. This result is a significant improvement of the previously known exponential bound $\binom{6n-3}{3n-1} - 3 \approx 2^{6n}$ from [1]. We don't attempt to minimize constants in this estimate because there still remains a big gap between the lower bound $\Omega(n)$ and our upper bound $O(n^4)$.

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