Chapter 3
Phase Transitions in Coverage and Connectivity in Two-Dimensional Deployment Fields

This chapter addresses the problems of almost sure integrated coverage and connectivity in two-dimensional wireless sensor networks from the perspective of percolation theory. Specifically, it focuses on finding the critical sensor density above which the network is almost surely connected and the deployment field is almost surely covered. It proposes our solution to this problem using a probabilistic approach. Precisely, each of the above problems is discussed separately. Then, both are investigated in an integrated manner using a suitable integration model.

3.1 Introduction

In wireless sensor networks, sensing coverage reflects the surveillance quality provided by active sensors in a field, whereas network connectivity enables active sensors to communicate with each other in data forwarding to a central gathering node, called the sink. For the correct operation of the network, it is necessary that both sensing coverage and network connectivity be maintained. Assuming perfectly reliable wireless links, both sensing coverage and network connectivity are affected by the sensor spatial density. In this chapter, we compute the value of this sensor density to provide sensing coverage and network connectivity.

Given a field that is initially uncovered, as more and more sensors are continuously added to the network, the size of the partially covered areas increases. At some point, the situation abruptly changes from small fragmented covered areas to a single large covered area in the field. We call this abrupt change as the sensing-coverage phase transition (SCPT) [16]. The SCPT problem can be stated as follows:

Given a field that is initially uncovered, what is the sensor spatial density corresponding to the first appearance of a single large covered component that spans the entire network?

Likewise, given a network that is originally disconnected, the number of connected components changes with the addition of sensors such that the network suddenly becomes connected at some point. We call this sudden change in the network topology as the network-connectivity phase transition (NCPT) [16]. The NCPT problem can be expressed as follows:

Given a network that is initially disconnected, what is the sensor spatial density corresponding to the first appearance of a single large connected component that spans the entire network?
The nature of such phase transitions is a central topic in percolation theory of Boolean models. The process of the ground getting wet during a period of rain [155] gives us a better analogy with the SCPT and NCPT problems. A circular wet patch forms whenever a point of the ground is hit by a raindrop. At the start of the rain, one can see a small wet area within a large dry area. After some time and as many raindrops continue to hit the ground, the situation suddenly changes and one can see a small dry area within a large wet area. This phase transition phenomenon occurs at a given density of the raindrops. This example helps us approach the SCPT and NCPT problems from a perspective of continuum percolation. In this chapter, we propose a probabilistic approach to compute the covered area fraction at critical percolation for both the SCPT and NCPT problems. Then, we derive the corresponding critical sensor density for each of these problems taken separately. In addition, we compute the critical sensor density when both these problems are considered together in an integrated manner. In [12], we propose a different percolation theory-based approach to study coverage and connectivity for three-dimensional wireless sensor networks. This study will be discussed in Chap. 4.

Fig. 3.1 Schematic of overlapping disks (three covered components of size 1, two of size 2, one of size 3, and one of size 4)

As will be discussed later, the specific connection function used in the NCPT problem has not been studied before and hence no bound on the critical covered area fraction is known. Furthermore, given that sensing coverage and network connectivity are not totally orthogonal [30, 205], we propose a new model for percolation in wireless sensor networks, called correlated disk model, which
allows network connectivity and sensing coverage to be studied together in an integrated fashion. We show that the SCPT and NCPT problems have the same solution (i.e., same critical covered area fraction). Precisely, we solve the SCPT and NCPT problems together, where the radii of the sensing disks \( r \) of the sensors and the radii of their transmission disks \( R \) are related by \( R = \alpha r \) with \( \alpha \geq 1 \). We show that network–connectivity phase transition occurs provided that sensing–coverage phase transition arises and the ratio \( R/r \) has certain value.

This study is of practical use for WSN designers to build up more reliable sensing applications in terms of their sensing coverage and network connectivity. For several real-word sensing applications, and in particular, intruder detection and tracking, it is required that each location in a target field be covered (or sensed) by at least one sensor. This would definitely imply better information gathering about the intruder, thus leading to accurate analysis and processing of the situation. On one hand, sensing coverage is associated with all locations in a target field, and hence would guarantee that any event about the intruder is sensed by sensors. On the other hand, network connectivity would enable gathered data about an intruder to reach a central control unit for further analysis and processing. Thus, both sensing coverage and network connectivity should be maintained for high intruder detection and tracking accuracy. Network connectivity, however, depends on sensing coverage. Thus, it is necessary to compute the critical sensor spatial density above which the target field is almost surely guaranteed to be covered and the network is almost surely guaranteed to be connected.
It is worth noting that the exact value of the critical density at which an infinite (or single large) cluster of overlapping disks first appears is still an open problem, and its approximation is either predicted by simulations [166, 173, 188] or computed analytically [86]. From now on, “infinite” means “single large”.

The remainder of this chapter is organized as follows. Section 3.2 solves the SCPT problem. Section 3.3 solves the NCPT problem. Section 3.4 discusses our results. Section 3.5 reviews related work. Section 3.6 summarizes the chapter.

3.2 Phase Transition in Sensing Coverage

This section discusses the SCPT problem and solves it using a percolation-theoretic approach.

Let \( X_\lambda = \{ \xi_i : i \geq 1 \} \) be a two-dimensional homogeneous Poisson point process of density \( \lambda \), where \( \xi_i \) represents the location of a sensor \( s_i \).

Given an initially uncovered field, the SCPT problem is to compute the probability of the first appearance of an infinite (or single large) covered component that spans the entire network. In particular, we are interested in the limiting case of an infinite field, where there exists no single large covered component for sufficiently small density \( \lambda \) and it suddenly appears at a critical percolation density \( \lambda_c \).

3.2.1 Estimation of the Shape of Covered Components

Each covered \( k \)-component \( CC_k \) (Chap. 2, Definition 2.6) is characterized by a reference point, called centre and denoted by \( \xi(k) \). Figure 3.1 shows various covered components of different sizes. Using the Poissonness argument stated in [100, pp. 200–202], as the centres \( \{ \xi_i : i \geq 1 \} \) form a Poisson process with density \( \lambda \), the centres of all covered \( k \)-components also form a Poisson process with density \( \lambda(k) \). In other words, the covered components are randomly and independently distributed according to a Poisson process with a density of \( \lambda(k) \) centres per unit area. We want to determine the smallest shape enclosing a covered \( k \)-component. In fact, the shape of the covered components varies depending on the number of its overlapping sensing disks. For tractability of the problem, we assume that the geometric form that encloses a covered \( k \)-component is a circle (Fig. 3.2), which tends to minimize the area of uncovered region around the covered component. Indeed, the circle is the most compressed shape. Let \( R_k \) be the radius of a circle, denoted by \( C(R_k, k) \), which encloses a covered \( k \)-component. Thus, there is no other sensing disk that could overlap with the boundary of the circle. In other words, the concentric circular band of width \( r \), denoted by \( CCB(r) \) and which surrounds the circle, should not include any other sensing disk. Hence, the annulus between radii \( R_k \) and \( R_k + r \) around the centre \( \xi(k) \) must be empty.
Let $P(k)$ be the conditional probability that the circle encloses only one covered $k$-component. This probability is given by

$$P(k) = \text{Prob}[C(R_k, k) \mid \text{CCB}(r) \text{ is empty}]$$

By definition, this conditional probability is computed as

$$P(k) = \frac{\text{Prob}[C(R_k, k) \land \text{CCB}(r) \text{ empty}]}{\text{Prob}[\text{CCB}(r) \text{ empty}]} \quad (3.1)$$

where $\text{Prob}[C(R_k, k) \land \text{CCB}(r) \text{ empty}]$ can be interpreted as the probability that the circle of radius $R_k + r$ encloses only one covered $k$-component. Thus,

$$\text{Prob}[C(R_k, k) \land \text{CCB}(r) \text{ empty}] = \text{Prob}[C(R_k + r, k)]$$

Using Eq. 2.3 (see Chap. 2), we obtain the following results:

$$\text{Prob}[C(R_k + r, k)] = \frac{\lambda \pi (R_k + r)^2}{k!} e^{-\lambda \pi (R_k + r)^2}$$

$$\text{Prob}[\text{CCB}(r) \text{ empty}] = e^{-\lambda \pi ((R_k + r)^2 - r^2)}$$

Therefore, Eq. 3.1 becomes

$$P(k) = \frac{(\lambda \pi (R_k + r)^2)^k}{k!} e^{-\lambda \pi R_k^2} \quad (3.2)$$

It is worth mentioning that the analysis of SCPT and NCPT problems will be based on the form of conditional probability given in (3.2).

### 3.2.2 Critical Density of Covered Components

Although there exists a few definitions of the average distance between clusters (i.e., covered components), one of them is more appropriate. It is defined as the average of the minimum distance between all pairs of sensing disks, each from one covered component. Indeed, two covered components could be merged together into a single one if there is at least a pair of sensing disks, one from each covered component, such that the distance between their centres is at most equal to $2r$. Lemma 3.1 computes the mean distance between neighbouring covered $k$-components at critical percolation.

**Lemma 3.1:** Let $\{CC_k\}$ be a set of covered $k$-components with density $\lambda(k)$ and $Y$ a random variable representing distances between them. The mean distance $d_{avg}^1$ between two neighbouring covered components at critical percolation is computed as follows:
\[ d_{av}^1 = \frac{1}{2 \sqrt{\lambda_k}} \]  

where \( \lambda_k \) is the density of \( \{CC_k\} \) at critical percolation.

**Proof:** Let \( \omega_k \) be the mean number of covered \( k \)-components in a circular field of radius \( \mathfrak{R} \). Denote by \( p(\sigma) \) the probability that there is a covered component whose centre is located at a distance upper bounded by \( \sigma \) from the centre, say \( \xi(k) \), of a given covered component. We denote by \( P(\sigma)d\sigma \) the probability that a nearest centre of a covered component to a given centre \( \xi(k) \) is located at a distance between \( \sigma \) and \( \sigma + d\sigma \). Hence, \( P(\sigma)d\sigma \) can be viewed as the probability that there exists one of the \( \omega_k - 1 \) covered components at a distance between \( \sigma \) and \( \sigma + d\sigma \) from the centre \( \xi(k) \) and the other \( \omega_k - 2 \) covered components are at a distance larger than \( \sigma \) from \( \xi(k) \). Thus,

\[
P(\sigma)d\sigma = \left( \omega_k - 1 \right) \frac{\partial p(\sigma)}{\partial \sigma} d\sigma (1 - p(\sigma))^{(\omega_k - 2)}
\]

where \( \frac{\partial p(\sigma)}{\partial \sigma} d\sigma \) stands for the probability that there is a covered component whose centre lies within a circular band located at a distance \( \sigma \) from the centre \( \xi(k) \) and whose width is \( d\sigma \). Notice that \( p(\sigma) \) can be computed as the ratio of the number of covered components within the circle of radius \( \sigma \) to the total number of covered components within the field. Thus, we obtain:

\[
p(\sigma) = \frac{\lambda_k \pi \sigma^2}{\lambda_k \pi \mathfrak{R}^2} = \frac{\sigma^2}{\mathfrak{R}^2}
\]

and

\[
\frac{\partial p(\sigma)}{\partial \sigma} = \frac{2 \sigma}{\mathfrak{R}^2}
\]

Substituting Eq. 3.3 in Eq. 3.4 gives

\[
P(\sigma)d\sigma = (\omega_k - 1) \frac{2 \sigma}{\mathfrak{R}^2} d\sigma \left( \frac{\sigma^2}{\mathfrak{R}^2} \right)^{\omega_k - 2})
\]

where \( \omega_k = \lambda_k \pi \mathfrak{R}^2 \). We assume that the circular field contains all covered components. Now, the mean distance between two covered \( k \)-components can be computed as
\[ E[Y] = \int_{0}^{\Re} \sigma \, P(\sigma) \, d\sigma \]

\[ = \frac{2(\omega_k - 1)}{\Re^2} \int_{0}^{\Re} \sigma^2 \left( 1 - \frac{\sigma^2}{\Re^2} \right)^{(\omega_k - 2)} \, d\sigma \]

Using the variable change \( T = \frac{\sigma^2}{\Re^2} \), we obtain

\[ E[Y] = (\omega_k - 1) \Re \int_{0}^{1} \sqrt{T} \, (1 - T)^{(\omega_k - 2)} \, dT \]

Recall that the beta function \([231]\) is defined by

\[ B(m, n) = \int_{0}^{1} u^{m-1} (1 - u)^{n-1} \, du \]

Thus,

\[ E[Y] = (\omega_k - 1) \Re \, B(3/2, \omega_k - 1) \quad (3.6) \]

where

\[ B(m, n) = \frac{\Gamma(m) \, \Gamma(n)}{\Gamma(m + n)} \]

\[ \Gamma(m) = (m - 1)\Gamma(m - 1) = (m - 1)! \]

Hence, Eq. 3.6 becomes

\[ E[Y] = (\omega_k - 1) \Re \, \frac{\Gamma(3/2) \, \Gamma(\omega_k - 1)}{\Gamma(\omega_k + 1/2)} = \Re \, \frac{\Gamma(3/2) \, \Gamma(\omega_k)}{\Gamma(\omega_k + 1/2)} \]

However, Graham et al. \([94]\) proved that

\[ \frac{\Gamma(x + 1/2)}{\Gamma(x)} = \sqrt{x} \left( 1 - \frac{1}{8x} + \frac{1}{128x^2} + \frac{5}{1024x^3} - \frac{21}{32768x^4} + \ldots \right) \]

Thus,

\[ \lim_{x \to \infty} \frac{\Gamma(x + 1/2)}{\Gamma(x)} = \sqrt{x} \]

Notice that at critical percolation, the value of \( \omega_k \) should be large enough \((\omega_k \to \infty)\) so an infinite covered component spanning the network could form.

Since \( \Gamma(3/2) = \frac{\sqrt{\pi}}{2} \) and \( \omega_k = \lambda_k \pi \Re^2 \), the mean distance \( d_{\text{avg}}^1 \) between two neighbouring covered \( k \)-components at critical percolation is given by
\[ d_{\text{avg}}^1 = \lim_{\omega_k \to \infty} \mathbb{E}[Y] = \Re \Gamma(3/2) \frac{1}{\sqrt{\omega_k}} = \frac{1}{2\sqrt{\lambda_c(k)}} \]

where \( \lambda_c(k) \) is the critical density of covered \( k \)-components.

Lemma 3.2 computes the average distance between neighbouring covered \( k \)-components at critical percolation using another approach. As can be seen later, Lemma 3.2 will help us compute the density of covered \( k \)-components at critical percolation.

**Lemma 3.2:** Let \( \{CC_k\} \) be a set of covered \( k \)-components with density \( \lambda(k) \), and \( Y \) a random variable associated with the distances between them. The mean distance \( d_{\text{avg}}^2 \) between two neighbouring covered components at critical percolation is computed as

\[ d_{\text{avg}}^2 = \frac{\text{erf}(2\sqrt{\lambda_c \pi} r) - 4\sqrt{\lambda_c} r e^{-4\lambda_c \pi r^2}}{2\sqrt{\lambda_c}} \]  

(3.7)

where \( \lambda_c \) is the density of a set of sensing disks \( \{D_i(r) : i \geq 1\} \) at critical percolation.

**Proof:** For a homogeneous Poisson point process, the probability that there is no neighbour within distance \( \sigma \) of an arbitrary point is given by \( e^{-\lambda \pi \sigma^2} \) [68]. Therefore, the probability that the distance between a point and its neighbour is less than or equal to \( \sigma \) is given by

\[ \mathbb{P}[Y \leq \sigma] = 1 - e^{-\lambda \pi \sigma^2} \]

Hence, the corresponding probability density function is given by

\[ f(Y | Y \leq \sigma) = 2\lambda \pi \sigma e^{-\lambda \pi \sigma^2} \]

The mean distance \( d_{\text{avg}}^2 \) between two neighbouring covered \( k \)-components of \( \{CC_k\} \) at critical percolation is obtained when the distance \( \sigma \) between two sensing disks, say \( D_i(r) \) and \( D_j(r) \), each from one covered component, belongs to the interval \( [0, 2r] \). Therefore,

\[ d_{\text{avg}}^2 = \mathbb{E}[Y | Y \leq 2r] = \int_0^{2r} \sigma \times f(Y | Y \leq \sigma) d\sigma = \frac{\text{erf}(2\sqrt{\lambda_c \pi} r) - 4\sqrt{\lambda_c} r e^{-4\lambda_c \pi r^2}}{2\sqrt{\lambda_c}} \]

where \( \text{erf}(x) \) is the error function [232].

**Lemma 3.3:** which follows from Lemmas 4.1 and 4.2, computes the density of covered \( k \)-components at critical percolation.
3.2 Phase Transition in Sensing Coverage

**Lemma 3.3:** The critical density of a set of covered $k$-components $\{CC_k\}$ is computed as follows:

$$\lambda_c(k) = \frac{\lambda_c}{\left( \text{erf}(2\sqrt{\lambda_c \pi r}) - 4\sqrt{\lambda_c} \, r \, e^{-\lambda_c \pi r^2} \right)^2}$$

(3.8)

where $\lambda_c$ is the density of sensing disks at critical percolation and $\text{erf}(x)$ is the error function [232].

**Proof:** From Lemma 3.1 (Eq. 3.3) and Lemma 3.2 (Eq. 3.7), the mean distance between two covered $k$-components at critical percolation should verify the following equality $d_{avg}^1 = d_{avg}^2$, which implies that the density of covered $k$-components at critical percolation $\lambda_c(k)$ is given by

$$\lambda_c(k) = \frac{\lambda_c}{\left( \text{erf}(2\sqrt{\lambda_c \pi r}) - 4\sqrt{\lambda_c} \, r \, e^{-\lambda_c \pi r^2} \right)^2}$$

3.2.3 Critical Radius of Covered Components

There is a particular value of the radius $R_k$ of the circular shape enclosing a covered component that almost surely guarantees the formation of special class of covered $k$-components, called critical covered $k$-components. Any non-empty circle of radius $2r$ should enclose a covered $k$-component. In other words, regardless of the number of sensing disks of radius $r$ located in a circle of radius $2r$, these sensing disks should definitely form a covered $k$-component. Moreover, this covered $k$-component is a complete graph in that each pair of sensors, say $s_i$ and $s_j$, whose sensing disks are included in this circle of radius $2r$ are collaborating given that $|\xi_i - \xi_j|_{\text{max}} \leq 2r$. Lemma 3.4 computes the density of critical covered $k$-components at critical percolation.

**Lemma 3.4:** At critical percolation, the density of covered $k$-components, which are enclosed in circles whose radii is equal to $2r$, is given by

$$\lambda_c(k) = \lambda_c \frac{(9 \lambda_c \pi r^2)^k}{k!} e^{-\lambda_c \pi r^2}$$

(3.9)

where $\lambda_c$ and $\lambda_c(k)$ are the densities of sensing disks and covered $k$-components at critical percolation, respectively.

**Proof:** Let $N$ be the total number of sensing disks that are randomly deployed in a circular field of radius $R$ according to a spatial Poisson process with density equal to
\[ \lambda = \frac{N}{\pi R^2} \]  

Using \( \omega_k = \lambda(k) \pi R^2 \), which represents the mean number of covered \( k \)-components in the circular field, and Eq. 3.10, we obtain

\[ \lambda(k) = \lambda \frac{\omega_k}{N} \]  

We can approximate \( \frac{\omega_k}{N} \) by the probability \( P[rad(CC_k) = 2r] \) of finding a covered \( k \)-component whose radius is equal to \( 2r \). Hence, we have

\[ P[rad(CC_k) = 2r] = \frac{\omega_k}{N} \]  

Substituting Eq. 3.12 in Eq. 3.11 gives

\[ \lambda(k) = \lambda P[rad(CC_k) = 2r] \]  

Following the same reasoning as in Sect. 3.2.1, \( P[rad(CC_k) = 2r] \) is the conditional probability of finding \( k \) sensing disks enclosed in a circle with radius \( 2r \) and centred at \( \xi(k) \) such that the annulus between circles of radii \( 2r \) and \( 2r + r \) around the centre \( \xi(k) \) is empty. Substituting \( R_k = 2r \) into Eq. 3.2 gives

\[ P(k) = \frac{(9\lambda \pi r^2)^k}{k!} e^{-4\lambda \pi r^2} \]

and hence Eq. 3.13 becomes

\[ \lambda_c(k) = \lambda_c \frac{(9\lambda_c \pi r^2)^k}{k!} e^{-4\lambda_c \pi r^2} \]

where \( \lambda_c \) and \( \lambda_c(k) \) are the critical densities of sensing disks and covered \( k \)-components, respectively.

### 3.2.4 Characterization of Critical Percolation

Now, we generate an equation that characterizes a set of covered \( k \)-components at critical percolation. By equating Eqs. 3.8 and 3.9, we obtain a new equation \( g_1(\lambda_c, r, k) = 0 \), where

\[ g_1(\lambda_c, r, k) = (erf(2\sqrt{\lambda_c \pi r}) - 4\sqrt{\lambda_c r} e^{-4\lambda_c \pi r^2})^2 \times \frac{(9\lambda_c \pi r^2)^k}{k!} e^{-4\lambda_c \pi r^2} - 1 \]  

(3.14)
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Fig. 3.3 No critical percolation at $k = 2$

Fig. 3.4 No critical percolation at $k = 3$
Instead of focusing on finding the critical value of the density \( \lambda_c \) of sensing disks at which an infinite covered component first appears, we consider a dimensionless metric, i.e., the covered area fraction at critical percolation given by

\[
A_c(r) = 1 - e^{-\lambda_c \pi r^2}
\]

The benefits of using \( A_c(r) \) instead of \( \lambda_c \) are two fold: first, the number of unknown parameters is reduced to two, namely \( A_c(r) \) and \( k \), thus removing any direct dependency of \( g_1(\lambda_c, r, k) \) on \( r \). Hence, the parameter \( r \) will not have any direct impact on the critical percolation density. Second, we know the exact domain of \( A_c(r) \) is \([0,1]\), which helps us study exactly the entire behaviour of the function \( g_1(\lambda_c, r, k) = 0 \) for all values of \( A_c(r) \). Substituting \( A_c(r) \) into Eq. 3.14 and let \( \mu = -\log(1 - A_c(r)) \) gives a new function \( g_1(A_c(r), k) \) given by

\[
g_1(A_c(r), k) = \left( \text{erf}(2\sqrt{\mu}) - \frac{4\sqrt{\mu} e^{-4\mu}}{\sqrt{\pi}} \right)^2 \\
\times \frac{(9\mu)^k}{k!} e^{-4\mu} - 1
\]

(3.15)

Fig. 3.5 Critical percolation at \( k = 4 \) and \( A_c(r) = 0.575 \)
3.3 Phase Transition in Network Connectivity

3.2.5 Numerical Results

Figures 3.3–3.5 plot the function \( g_1(A_c(r), k) \) given in Eq. 3.15 with respect to different values of \( k \) and \( A_c(r) \). Notice that for \( k < 4 \), the function \( g_1(A_c(r), k) \) cannot be equal to zero (Figs. 3.3 and 3.4). Thus, percolation first occurs at \( k = 4 \) and \( A_c(r) = 0.575 \) (Fig. 3.5), which is a bit smaller than the values 0.688 of Vicsek and Kertesz [188], 0.68 of Pike and Seager [166], and 0.62 of Roberts [173] (all predicted by Monte Carlo experiments), and the value 0.67 as calculated by Fremlin [86] for studying the percolation of overlapping homogeneous disks. Thus, when the number of collaborating sensors of a sensor is larger than four \( (k \geq 5) \), it is almost surely that an infinite covered component that spans the entire network will appear for the first time.

3.3 Phase Transition in Network Connectivity

Let \( X = \{\xi_i : i \geq 1\} \) be a two-dimensional homogeneous Poisson point process of density \( \lambda \), where \( \xi_i \) represents the location of sensor \( s_i \).

Given a network that is originally disconnected, the network-connectivity phase transition (NCPT) problem is to compute the sensor spatial density corresponding to the first appearance of an infinite (or single large) connected component that spans the network.

Notice that both of the SCPT and NCPT problems have similar structure although the difference of the concepts of collaboration (SCPT) and communication (NCPT) between the sensors in the SCPT and SCPT problems, respectively, as stated earlier in Sect. 2.2 (see Chap. 2). In the SCPT problem, two sensing disks belong to the same covered component if the distance between them is at most equal to one diameter \((2r)\). However, the NCPT problem requires that two communication disks be at a distance of at most half their diameter (i.e., \( R \)) from each other so they belong to the same connected component, where \( r \) and \( R \) stand for the radii of the sensing and communication disks of the sensors, respectively. To our knowledge, the connection function of the NCPT problem has not been studied previously in the literature.

Some sensing applications require that every location in the field be covered by at least one sensor and that the active sensors be also able to communicate with each other so the sensed data could reach the sink. Indeed, sensed data would be meaningless if connectivity between the sensors is not maintained. Thus, we are mainly interested in the formation of an infinite (or single large) connected covered component that spans the entire network. Next, we study the SCPT and NCPT problems together using percolation theory.

3.3.1 Integrated Sensing Coverage and Network Connectivity

We propose a new model for percolation in wireless sensor networks, called correlated disk model. Each sensor is associated with two concentric disks of radii \( r \)
and $R$ representing the radii of its sensing and communication disks, respectively. This kind of structure reveals a double behaviour of the sensors that can be described by their collaboration and communication. The collaboration between sensors depends on the relationship between the radii of their sensing disks, whereas communication is related to the relationship between the radii of their communication disks. Previous studies by Wang et al. [197] and Ammari and Das [30] showed the existence of certain dependency between the concepts of sensing coverage and network connectivity. Our proposed correlated disk model allows us to study these two concepts together from a percolation-theoretic viewpoint to account for their correlation. This problem can be viewed as a correlated continuum percolation problem. Next, we study the simultaneous percolation of the sensing and communication disks of the sensors based on the ratio $R/r$.

### 3.3.1.1 Simultaneous Phase Transitions When $R \geq 2r$

As it will be discussed later in this chapter, Wang et al. [197] proved that if a wireless sensor network is configured to be covered and the radius $R$ of the communication disk of the sensors is at least double the radius $r$ of their sensing disk, then the network is guaranteed to be connected. Ammari and Das [30] provided a tighter relationship between $R$ and $r$ while achieving network connectivity provided that sensing coverage is guaranteed. In fact, the “worst-case” behaviour is when the sensing disks of the sensors are tangential, i.e., the distance between their corresponding centres is equal to $2r$. Hence, when $R \geq 2r$, there is a dependency between sensing coverage and network connectivity in that the former implies the latter. In other words, collaboration between the sensors will lead to their communication. In this case, the SCPT and NCPT problems are equivalent, and thus have the same critical covered area fraction. Thus, a set of communication disks percolates at $k = 4$ with a covered area fraction $A_c(R) = 0.575$ at critical percolation. Therefore, when the number of communicating sensors of a given sensor is larger than four ($k = 4$), an infinite connected component spanning the network will almost surely form.

### 3.3.1.2 Simultaneous Phase Transitions When $r \leq R < 2r$

The interesting case is when the radii of the sensing and communication disks of the sensors are related by $R = \alpha r$, where $1 \leq \alpha < 2$. Precisely, we focus on the study of the percolation of the sensing disks of the sensors, where two sensors collaborate if and only if the distance between the centres of their sensing disks is equal to $\alpha r$, where $1 \leq \alpha < 2$. The communication disks of the sensors will also percolate given that $R = \alpha r$. Thus, our goal is to compute the critical covered area fraction above which both the sensing and communication disks of the sensors percolate when $r \leq R < 2r$. It is a valid assumption that the radius of the communication disks of the sensors cannot be less than the radius of their sensing disks as shown in Tables 2 and 3 for a wide spectrum of sensor devices [218].
3.3 Phase Transition in Network Connectivity

We consider the previous analysis in Sect. 3.2, where we replace $2r$ by $\alpha r$, with $1 \leq \alpha < 2$. Without repeating those details, we obtain a new equation that characterizes a set of covered $k$-components at critical percolation, which is given by $g_2(\lambda_c, r, \alpha, k) = 0$, where

$$g_2(\lambda_c, r, \alpha, k) = (\text{erf}(\sqrt{\lambda_c \pi \alpha r} - \sqrt{\lambda_c \alpha^2 r e^{-\lambda_c \pi \alpha^2 r^2}})^2 \times \frac{(9 \lambda_c \pi \alpha^2 r^2 / 4)^k}{k!} e^{-\lambda_c \pi \alpha^2 r^2} - 1$$

Let $\mu = -\log(1 - A_c(r))$. We substitute $A_c(r)$ in $g_2(\lambda_c, r, \alpha, k)$ to obtain a new function $g_2(A_c(r), \alpha, k)$ given by

$$g_2(A_c(r), \alpha, k) = \left(\text{erf}(\alpha \sqrt{\mu}) - \frac{\alpha^2 \sqrt{\mu e^{-\alpha^2 \mu}}}{\sqrt{\pi}}\right)^2 \times \frac{(9 \alpha^2 \mu / 4)^k}{k!} e^{-\alpha^2 \mu} - 1$$

(3.16)

Figures 3.6–3.9 show the plots of the function $g_2(A_c(r), \alpha, k)$ given in Eq. 3.16 for different values of $k$ and $\alpha$, where $2 \leq k \leq 5$ and $1 \leq \alpha < 2$. As can be seen from Figures 3.6 and 3.7, the function $g_2(A_c(r), \alpha, k)$ cannot be

![Fig. 3.6 Plot of the function $g_2(A_c(r), \alpha, k)$ for different values of $\alpha$ ($1 \leq \alpha < 2$). No critical percolation occurs at $k = 2$](image-url)
equal to zero for $k < 4$, regardless of the value of $\alpha$. Furthermore, a set of sensing disks percolates (which occurs when $g_2(A_c(r), \alpha, k) = 0$) faster for large values of $\alpha$. For instance, when $\alpha = 1$ (which corresponds to $R = r$), critical percolation occurs at $k = 5$ and $A_c(r) = 0.925$ (Fig. 3.9). Thus, when $\alpha = 1$, it is *almost surely* that an infinite covered component spanning the entire network will appear when the number of collaborating sensors of a sensor is larger than five ($k \geq 6$). However, when $\alpha = 1.5$ (i.e., $R = 1.5\, r$), critical percolation occurs at $k = 4$ and $A_c(r) = 0.580$ (Fig. 3.8).

![Plot of the function $g_2(A_c(r), \alpha, k)$](image)

**Fig. 3.7** Plot of the function $g_2(A_c(r), \alpha, k)$ for different values of $\alpha$ ($1 \leq \alpha < 2$). No critical percolation occurs at $k = 3$

Finally, for $\alpha = 1.25$ (i.e., $R = 1.25\, r$), critical percolation occurs at $k = 4$ and $A_c(r) = 0.760$ (Fig. 3.8). For the last two cases ($\alpha = 1.25$ and $\alpha = 1.5$), it is *almost surely* that an infinite covered component that spans the entire network will appear when the number of collaborating sensors of a sensor is larger than four ($k \geq 5$). Given the connection function defined for the collaboration between the sensing disks, percolation should be quicker for large disks than for smaller ones. In all cases, the value of the corresponding *critical covered area fraction* will *almost surely* guarantee the appearance of an infinite connected component that spans the underlying network provided that an infinite covered component arises and spans the entire network. Moreover, the value of *critical covered area fraction* depends on the ratio $R/r$. 
3.3 Phase Transition in Network Connectivity

Fig. 3.8 Plot of the function $g_2(A_c(r), \alpha, k)$ for different values of $\alpha$ ($1 \leq \alpha < 2$). For $k = 4$, critical percolation depends on the value of $\alpha$.

Fig. 3.9 Plot of the function $g_2(A_c(r), \alpha, k)$ for different values of $\alpha$ ($1 \leq \alpha < 2$). For $k = 5$, critical percolation depends on the value of $\alpha$. 

+k = 4

$g_2(A_c(r), \alpha, k)$

$A_c(r)$

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

0 -0.2 -0.4 -0.6 -0.8 -1 0 0.2 0.4

$g_2(A_c(r), \alpha, k)$

$A_c(r)$

0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

0 -0.5 0.5 1 1.5

For $k = 4$, critical percolation depends on the value of $\alpha$. For $k = 5$, critical percolation depends on the value of $\alpha$. 
3.4 Discussion

It is worth noting that both values of critical covered area fractions for sensing coverage and network connectivity represent only lower bounds. In other words, if the actual covered area fraction is higher than $A_c(r)$, it is almost surely that there exists an infinite (or single large) covered component that spans the entire network. That is, a large portion of the deployment field is guaranteed to be covered. Otherwise, there are only a few small fragmented regions of the field that are covered. However, there is no guarantee on the size of the region of the field being covered. Similarly, if the actual covered area fraction is higher than $A_c(R)$, it is almost surely that there exists an infinite connected component that spans the entire network. Otherwise, it is almost surely that the network is disconnected. However, there is no guarantee neither on the number of nodes being connected in this infinite component nor whether the sink belongs to the infinite connected component.

There appears to be little disagreement between our theoretical calculation of the critical covered area fraction \( A_c(r) = 0.575 \) compared to the values previously obtained by approximate calculation and Monte Carlo simulation (between 0.62 and 0.688). Our analysis of phase transitions in both sensing coverage (respectively network connectivity) is mainly based on an estimation of the smallest shape enclosing a covered (respectively connected) k-component. We assumed that this shape is a circle. Although it may not be always true that a circle is the smallest shape enclosing covered (and connected) k-component, we used it to simplify the analysis enough and make it mathematically tractable. Also, we considered this shape as an ellipse with minor axis $a_k$ and major axis $b_k$. Maximizing $P(k) = \frac{\lambda \pi (a_k + r)(b_k + r))^k}{k!} e^{-\lambda \pi a_k b_k}$, the probability that an ellipse encloses a covered k-component, leads however to a unique solution $a_k = b_k$ representing a degenerate ellipse or circle.

3.5 Related Work

Adlakha and Srivastava [2] showed that the number of sensors required to cover an area of size $A$ is in the order of $O(A/\hat{r}_2^2)$, where $\hat{r}_2$ is a good estimate of the radius $r$ of the sensing disk of the sensors. Specifically, $r$ lies between $\hat{r}_1$ and $\hat{r}_2$, where $\hat{r}_1$ overestimates the total number of sensor required to cover an area of size $A$ while $\hat{r}_2$ underestimates it. Franceschetti et al. [85] investigated the number of disks of given radius $r$, centered at the vertices of an infinite square grid, which are required to entirely cover an arbitrary disk of radius $r$ placed on the plane. Their result depends on the ratio of $r$ to the grid spacing.

Kumar et al. [128] proved that for random deployment with uniform distribution, if there exists a slowly growing function $\phi(np)$ such that
3.5 Related Work

\[ n \pi r^2 \geq \log(np) + k \log \log(np) + \phi(np), \] then a square unit area is \( k \)-covered with high probability when \( n \) sensors are deployed in it, where \( p \) is the probability that a sensor is active. It is worth noting that \( n \) also represents the sensor spatial density given that the area of the square region is equal to 1. Hence, the above inequality can be written as \( n \pi r^2 \geq \frac{\log(np) + k \log \log(np) + \phi(np)}{r^2} \), which means that the minimum sensor density required for \( k \)-coverage of a unit square region is equal to \( \frac{\log(n) + k \log \log(n) + \phi(n)}{r^2} \). If we set \( p = 1 \) (i.e., every sensor is active), we obtain \( \frac{\log(n) + k \log \log(n) + \phi(n)}{r^2} \). Recently, Balister et al. [38] computed the sensor density necessary to achieve both sensing coverage and network connectivity in finite region, such as thin strips (or annuli) whose lengths are finite. Balister et al. [38] applied this result to achieve barrier coverage [127] and connectivity in thin strips. In this type of deployment, the sensors act as a barrier to ensure that any moving object or phenomenon that crosses the barrier of sensors will be detected.

Zhang and Hou [217, 219] proved that the required density for \( k \)-coverage of a square field, where sensors are distributed according to a Poisson point process and always active, depends on both the side length of the field and \( k \). Precisely, Zhang and Hou [217, 219] found that a necessary and sufficient condition of complete \( k \)-coverage of a square field with side length \( l \) is that the sensor density is equal to \( \lambda = \log l^2 + (k + 1) \log \log l^2 + c(l) \), where \( c(l) \to +\infty \) as \( l \to +\infty \). Also, given a wireless sensor network deployed as a Poisson point process with density \( \lambda \) and every sensor is active, Zhang and Hou [216] provided a sufficient condition for \( k \)-coverage of a square region with area \( A \). Precisely, they proved that assuming \( \lambda = \log A + 2 \log \log A + c(A) \), if \( c(A) \to +\infty \) as \( A \to +\infty \), then the probability of \( k \)-coverage of the square region approaches 1. Furthermore, Zhang and Hou [216] provided the same result in the case where sensors are deployed according to a uniformly random distribution. Both results are based on the following statement: the square region is divided into square grids with side length \( s = \frac{\sqrt{2} r}{\log A} \), where \( r \) stands for the radius of the sensing range of the sensors. For a grid \( i \) to be completely \( k \)-covered, it is sufficient that there are at least \( k \) sensors within a disk centred at the centre of the grid and with radius \( (1-u) r \), denoted by \( B_i ((1-u) r) \), where \( u = 1/\log A \).

Wan and Yi [189] showed that with boundary effect, the asymptotic \((k+1)\)-coverage of a square with area \( s \) by Poisson point process with unit-area coverage range requires that the sensor density be equal to \( \log s + 2 (k + 1) \log \log s + \xi(s) \) with \( \lim_{s \to +\infty} \xi(s) = +\infty \). Without the boundary effect, however, the asymptotic \((k+1)\)-coverage requires that the sensor density be computed as \( \log s + (k + 2) \log \log s + \xi(s) \) with \( \lim_{s \to +\infty} \xi(s) = +\infty \).
The concept of *continuum percolation* originally due to Gilbert [91], is to find the critical density of a Poisson point process at which an unbounded connected component *almost surely* appears so the network can provide long-distance multi-hop communication. For random plane networks, Gilbert claimed that the filling factor, representing the critical value of the expected number of points in a circle of radius $R$ should be around 3.2. Since then, Gilbert’s model has become the basis for studying continuum percolation in wireless networks. Booth et al. [45] discussed different classes of covering algorithms and determined the critical density of a Poisson point process (centres of spheres of radius $r$) above which an unbounded connected component arises. They also discussed the almost sure existence of an unbounded connected component based on the ratio of the connectivity range of the base stations to the connectivity range of the clients. Bertin et al. [42] proved the existence of site percolation and bond percolation in the Gabriel graph [87] for both Poisson and hard-core stationary point processes. The critical bounds corresponding to the existence of a path of opens sites and a path of open bonds were found by simulation. Glauche et al. [92] proposed a distributed protocol, which guarantees strong connectivity *almost surely* of ad hoc nodes. They translated the problem of finding the critical communication range of mobile devices to that of determining the critical node neighbourhood degree above which an ad hoc network graph is *almost surely* connected. To achieve a little above this degree, each node needs to adjust its transmission power locally. Jiang and Bruck [115] proposed the concept of monotone percolation based on the local adjustment of the communication radii of the nodes for efficient topology control of the network. Their proposed algorithms guarantee the existence of relatively short paths between any pair of source and destination nodes, which makes monotonic progress. Liu and Towsley [146] considered both Boolean and general sensing models, each with a variety of network scenarios, to characterize fundamental coverage properties of large-scale sensor networks, namely area coverage, node coverage, and detectability. According to their simulation setting, the critical density is about $3.53 \times 10^{-3}$.

### 3.6 Summary

In this chapter, we investigated two phase transition problems for *sensing-coverage* and *network-connectivity* in wireless sensor networks using a probabilistic approach [16]. Our goal is to determine when an infinite covered component (SCPT problem) and an infinite connected component (NCPT problem) could form for the first time. To achieve this objective, we computed the covered area fraction for SCPT and NCPT problems at critical percolation. The problem of overlapping disks has been studied extensively in percolation theory and is similar to the SCPT problem. We found that the value of the covered area fraction is close to the one found by Monte Carlo simulations. The specific connection function of the Boolean model associated with the NCPT problem, however, has not been studied.
before and hence no bound exists in the literature. We proposed a *correlated disk model* to study SCPT and NCPT problems in an integrated way from a continuum percolation perspective. Precisely, we considered the physical correlation between them, which is based on the ratio of the radius of the communication disks of the sensors to the radius of their sensing disks. Thus, when an infinite covered component arises for the first time, an infinite connected component will *almost surely* appear based on the ratio $\alpha = R/r$. 
