Visibility Representations of Four-Connected Plane Graphs with Near Optimal Heights

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Abstract. A *visibility representation* of a graph G is to represent the nodes of G with non-overlapping horizontal line segments such that the line segments representing any two distinct adjacent nodes are vertically visible to each other. If G is a plane graph, i.e., a planar graph equipped with a planar embedding, a visibility representation of G has the additional requirement of reflecting the given planar embedding of G. For the case that G is an n-node four-connected plane graph, we give an $O(n)$-time algorithm to produce a visibility representation of G with height at most $\lceil \frac{n}{2} \rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$. To ensure that the first-order term of the upper bound is optimal, we also show an n-node four-connected plane graph G, for infinite number of n, whose visibility representations require heights at least $\frac{n}{2}$.

1 Introduction

Unless clearly specified otherwise, all graphs in the present article are simple, i.e., having no self-loops and multiple edges. A *visibility representation* of a planar graph represents the nodes of the graph by non-overlapping horizontal line segments such that, for any nodes u and v adjacent in the graph, the line segments representing u and v are vertically visible to each other. Observe that if G_1 is a subgraph of G_2 on the save node set, then any visibility representation of G_2 is also a visibility representation of G_1. Therefore, we may assume without loss of generality that the input graph is maximally planar. Let G be an n-node plane triangulation, i.e., a maximally planar graph equipped with a planar embedding. A visibility representation of G has an additional requirement of reflecting the given planar embedding of G. Figure 1(b), for instance, is a visibility representation of the four-connected plane graph shown in Fig. 1(a). Under the conventional restriction of placing the endpoints of horizontal line segments on the integral grid points, any visibility representation of G requires width no more than $3n-7$ and height no more than $n-1$. Otten and van Wijk [7] gave the first known algorithm for constructing a visibility representation for any G. Rosenstiehl and Tarjan [8] and Tamassia

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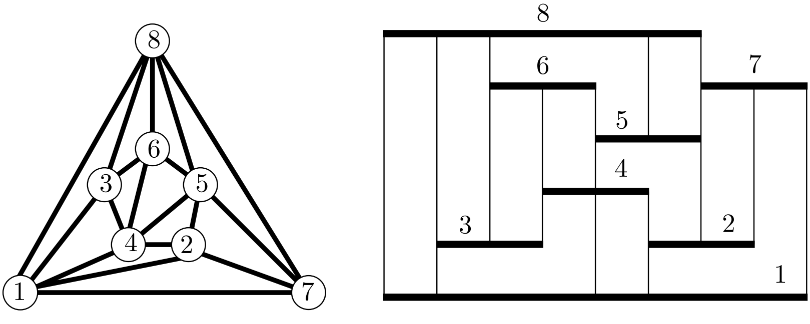


Fig. 1. (a) A four-connected plane triangulation G. (b) A visibility representation of G.

and Tollis [9] independently gave algorithms to compute a visibility representation of G with height at most $2n - 5$. Their work initiated a decade of competition on minimizing the width and height of the output visibility representation. All these algorithms run in linear time. In particular, the results of Fan, Lin, Lu, and Yen [2] and Zhang and He [16] are optimal in that the upper bounds differ from the best known lower bounds by very small constants.

The present article focuses on four-connected plane G. The $O(n)$-time algorithm of Kant and He [5] provides the optimal upper bound $n - 1$ on the width. The best previously known upper bound on the height, ensured by the $O(n)$-time algorithm of Zhang and He [12], is $\lceil \frac{3n}{4} \rceil$. In the present article, we obtain the following result with an improved upper bound on the required height.

Theorem 1. *For any n-node four-connected plane graph G, it takes $O(n)$ time to construct a visibility representation of G with height at most $\lceil \frac{n}{2} \rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$.*

Table 1 compares our upper bound with previous results. All algorithms shown in Table 1 run in $O(n)$ time. Our algorithm follows the approach of Zhang and He [10, 15–17], originating from Rosenstiehl and Tarjan [8] and Tamassia and Tollis [9], that reduces the problem of computing a visibility representation for G with small height to finding an appropriate st-ordering of G. To find such an st-ordering of G, we resort to three linear-time obtainable node orderings:

– four-canonical orderings of four-connected plane graphs (Kant and He [5]),
– consistent orderings of ladder graphs (Zhang and He [15–17]), and
– post-orderings of canonical ordering spanning trees (He, Kao, and Lu [3]).

Our result is near optimal in that we can construct an n-node four-connected plane graph, for infinite number of n, whose visibility representations require heights at least $\lceil \frac{n}{2} \rceil$. That is, the first-order term of our upper bound is optimal.

The remainder of the paper is organized as follows. Section 2 gives the preliminaries. Section 3 describes and analyzes our algorithm. Section 4 ensures that the first-order term of our upper bound on height is optimal. Section 5 concludes the paper.

Table 1. Previous upper bounds and our result for any n-node plane graph G

	general G		four-connected G	
	width	height	width	height
Otten and van Wijk [7]	$3n-7$	$n-1$		
Rosenstiehl and Tarjan [8], Tamassia and Tollis [9]	$2n-5$			
Kant [4]	$\lfloor \frac{3n-6}{2} \rfloor$			
Kant and He [5]			$n-1$	
Lin, Lu, and Sun [6]	$\lfloor \frac{22n-24}{15} \rfloor$			
Zhang and He [10]		$\lceil \frac{15n}{16} \rceil$		
Zhang and He [14]		$\lfloor \frac{5n}{6} \rfloor$		
Zhang and He [11, 13]	$\lfloor \frac{13n-24}{9} \rfloor$			
Zhang and He [12]				$\lceil \frac{3n}{4} \rceil$
Zhang and He [15, 17]	$\frac{4n}{3} + 2\lceil \sqrt{n} \rceil$	$\frac{2n}{3} + 2\lceil \sqrt{\frac{n}{2}} \rceil$		
Zhang and He [16]		$\frac{2n}{3} + O(1)$		
Fan, Lin, Lu, and Yen [2]	$\lfloor \frac{4n}{3} \rfloor - 2$			
This paper				$\lceil \frac{n}{2} \rceil + 2\left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$

2 Preliminaries

2.1 Ordering and st-Ordering

Let G be an n-node plane graph. An *ordering* of G is a one-to-one mapping σ from the nodes of G to $\{1, 2, \ldots, n\}$. A path of G is *σ-increasing* if $\sigma(u) < \sigma(v)$ holds for any nodes u and v such that u precedes v in the path. Let $length(G, \sigma)$ denote the maximum of the lengths of all σ-increasing paths in G. For instance, if G and σ are as shown in Fig. 1(a), then one can verify that $(1, 2, 5, 6, 8)$ is a σ-increasing path with maximum length. Therefore, $length(G, \sigma) = 4$.

Let s and t be two distinct external nodes of G. An *st-ordering* [1] of G is an ordering σ of G such that

- $\sigma(s) = 1$, $\sigma(t) = n$, and
- each node v of G other than s and t has neighbors u and w in G with $\sigma(u) < \sigma(v) < \sigma(w)$.

An example is shown in Fig. 1(a): the node labels form an st-ordering for the graph.

The following lemma reduces the problem of minimizing the height of visibility representation of G to that of finding an st-ordering σ of G with minimum $length(G, \sigma)$.

Lemma 1 (See [2, 8–10, 15, 17]). *If G admits an st-ordering σ for two distinct external nodes s and t of G, then it takes $O(n)$ time to obtain a visibility representation of G with height exactly $length(G, \sigma)$.*

For instance, if G and σ are as shown in Fig. 1(a), then a visibility representation for G with height at most $length(G, \sigma) = 4$, as shown in Fig. 1(b), can be found in linear time.

2.2 Four-Canonical Ordering

Let G be an n-node four-connected plane triangulation. Let v_1, v_2, and v_n be the external nodes of G in counterclockwise order. Since G is a four-connected plane triangulation, G has exactly one internal node adjacent to both v_2 and v_n. Let v_{n-1} be the internal node adjacent to v_2 and v_n in G. A *four-canonical ordering* [5] of G is an ordering ϕ in G such that

- $\phi(v_1) = 1$, $\phi(v_2) = 2$, $\phi(v_{n-1}) = n - 1$, $\phi(v_n) = n$, and
- each node v of G other than v_1, v_2, v_{n-1} and v_n has neighbors u, u', w and w' in G with $\phi(u') < \phi(u) < \phi(v) < \phi(w) < \phi(w')$.

An example is shown in Fig. 2(a): the node labels form a four-canonical ordering of the four-connected plane triangulation.

Lemma 2 (Kant and He [5]). *It takes $O(n)$ time to compute a four-canonical ordering for any n-node G.*

2.3 Consistent Ordering of Ladder Graph

Let L be an $\lceil \frac{n}{2} \rceil$-node path. Let R be an $\lfloor \frac{n}{2} \rfloor$-node path. Let X consist of edges with one endpoint in L and the other endpoint in R. Let (L, R, X) denote the n-node graph $L \cup R \cup X$. We say that (L, R, X) is a *ladder graph* [15, 17] if $L \cup R \cup X$ is outerplanar. A ladder graph is shown in Fig. 3(a).

An ordering σ of ladder graph (L, R, X) is *consistent* [15, 17] with respect to an outerplanar embedding $\mathcal{E}$ of (L, R, X) if L (respectively, R) forms a σ-increasing path in clockwise (respectively, counterclockwise) order according to $\mathcal{E}$. See Fig. 3(a) for an example: The node labels form a consistent ordering of the ladder graph with respect to the displayed outerplanar embedding.

Lemma 3 (He and Zhang [15, 17]). *Let (L, R, X) be an n-node ladder graph. It takes $O(n)$ time to compute a consistent ordering σ of (L, R, X) with respect to any given outerplanar embedding of (L, R, X) such that $length((L, R, X), \sigma) \leq \lceil \frac{n}{2} \rceil + 2 \left\lceil \sqrt{\frac{n}{2}} \right\rceil - 1$.*

For technical reason, we need a consistent ordering with additional properties, as stated in the next lemma, which is also illustrated by Fig. 3(a).

Lemma 4. *Let (L, R, X) be an n-node ladder graph. It takes $O(n)$ time to compute a consistent ordering σ of (L, R, X) with respect to any given outerplanar embedding $\mathcal{E}$ of (L, R, X) such that*

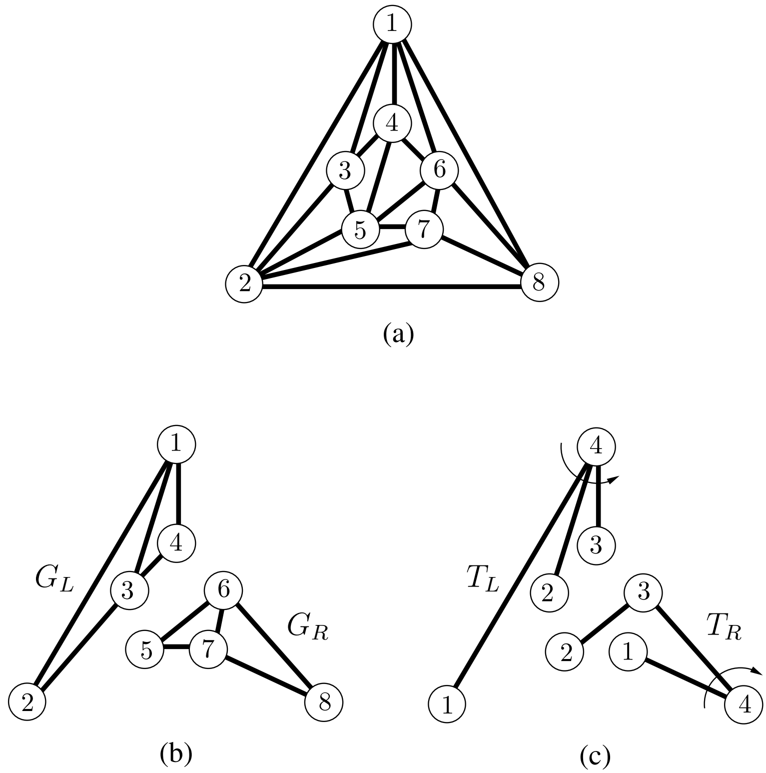


Fig. 2. (a) A four-canonical ordering ϕ of the four-connected plane triangulation G. (b) G_L is the subgraph induced by the nodes v with $1 \leq \phi(v) \leq 4$ and G_R is the subgraph induced by the nodes v with $5 \leq \phi(v) \leq 8$. (c) The counterclockwise post-ordering ψ_L of T_L and the clockwise post-ordering ψ_R of T_R.

– $\sigma(\ell_1) = 1$, $\sigma(r_1) = 2$, *and*
– *length*$((L, R, X), \sigma) \leq \lceil \frac{n}{2} \rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil$,

where ℓ_1 (*respectively,* r_1) *is the first* (*respectively, last*) *node of* L (*respectively,* R) *in clockwise order around the external boundary of* (L, R, X) *with respect to* $\mathcal{E}$.

Proof. Let $L' = L \setminus \{\ell_1\}$. Let $R' = R \setminus \{r_1\}$. Let $X' = X \setminus \{\ell_1, r_1\}$. Clearly, (L', R', X') is a ladder graph of $n - 2$ nodes. Let σ' be the consistent ordering of (L', R', X') with respect to $\mathcal{E}$ ensured by Lemma 3. We have

$$\textit{length}((L', R', X'), \sigma') \leq \left\lceil \frac{n}{2} \right\rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil - 2.$$

Let σ be the ordering of (L, R, X) such that

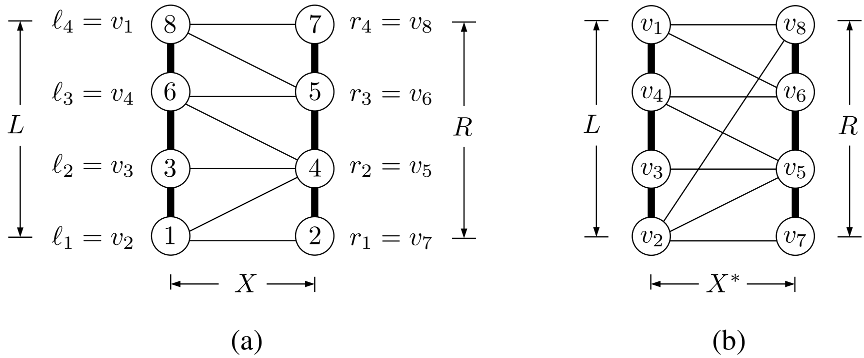


Fig. 3. (a) A consistent ordering of a ladder graph (L, R, X) with respect to the displayed outer-planar embedding. (b) $H^* = L \cup R \cup X^*$, where $X^* = X \cup \{(v_2, v_8)\}$.

- $\sigma(\ell_1) = 1$, $\sigma(r_1) = 2$, and
- $\sigma(u) = \sigma'(u) + 2$ holds for each node u other than ℓ_1 and r_1.

One can easily verify that the lemma holds. □

3 Our Algorithm

Let G be the input n-node four-connected plane triangulation. According to Lemma 1, it suffices to describe our algorithm for computing an st-ordering σ for G in the following four steps.

3.1 Step 1

Let ϕ be a four-canonical ordering of G ensured by Lemma 2.

- Let G_L be the subgraph of G induced by the nodes v with $1 \leq \phi(v) \leq \lceil \frac{n}{2} \rceil$.
- Let G_R be the subgraph of G induced by the nodes v with $\lceil \frac{n}{2} \rceil < \phi(v) \leq n$.

Figure 2(b) illustrates this step, which runs in $O(n)$ time. Observe that each edge of G not in $G_L \cup G_R$ has one endpoint on the external boundary of G_L and the other endpoint on the external boundary of G_R.

3.2 Step 2

For each $i = 1, 2, \ldots, n$, let v_i denote the node of G with $\phi(v_i) = i$. It follows from the definition of ϕ that v_1, v_2, and v_n are the external nodes of G.

- For each $i = 2, 3, \ldots, \lceil \frac{n}{2} \rceil$, let $\pi(i)$ be the index j with $j < i$ such that v_j is the first neighbor of v_i in G_L in counterclockwise order around v_i. Let T_L be the spanning tree of G_L rooted at v_1 such that each $v_{\pi(i)}$ is the parent of v_i in T_L. Let ψ_L be the counterclockwise post-ordering of T_L.

- For each $i = \lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 2, \ldots, n-1$, let $\pi(i)$ be the index j with $j > i$ such that v_j is the first neighbor of v_i in G_R in clockwise order around v_i. Let T_R be the spanning tree of G_R rooted at v_n such that each $v_{\pi(i)}$ is the parent of v_i in T_R. Let ψ_R be the clockwise post-ordering of T_R.

Figure 2(c) illustrates this step, which runs in $O(n)$ time. As a matter of fact, T_L is the canonical ordering spanning tree of G_L with respect to ϕ, as defined by He, Kao, and Lu [3].

Lemma 5. $\psi_L(v_2) = 1$, $\psi_L(v_1) = \lceil \frac{n}{2} \rceil$, $\psi_R(v_{n-1}) = 1$, *and* $\psi_R(v_n) = \lfloor \frac{n}{2} \rfloor$.

Proof. Since ϕ is a four-canonical ordering of G, if (v_2, v_i) with $i \geq 3$ is an edge of G_L, then v_i has to have a neighbor v_k with $2 \neq k < i$ in G_L. Observe that v_2 is the node immediately succeeding v_1 in counterclockwise order around the external boundary of G_L. One can verify that v_2 cannot be the first neighbor of v_i in G_L in counterclockwise order around v_i. That is, we have $\pi(i) \neq 2$. Since v_2 cannot be the parent of v_i in T_L, v_2 has to be a leaf of T_L. By the relative position between v_2 and v_1, it is clear that v_2 is the first node in the counterclockwise post-ordering of T_L, i.e., $\psi_L(v_2) = 1$.

One can prove $\psi_R(v_{n-1}) = 1$ analogously, where v_n (respectively, v_{n-1}, ψ_R, T_R, and G_R) plays the role of v_1 (respectively, v_2, ψ_L, T_L, and G_L). Since v_1 is the root of T_L and ψ_L is a post-ordering of T_L, we have $\psi_L(v_1) = \lceil \frac{n}{2} \rceil$. Since v_n is the root of T_R and ψ_R is a post-ordering of T_R, we have $\psi_R(v_n) = \lfloor \frac{n}{2} \rfloor$. □

3.3 Step 3

Let L, R, and X be defined as follows.

- Let L be the path $(\ell_1, \ell_2, \ldots, \ell_{\lceil n/2 \rceil})$, where ℓ_i is the node of G_L with $\psi_L(\ell_i) = i$.
- Let R be the path $(r_1, r_2, \ldots, r_{\lfloor n/2 \rfloor})$, where r_i is the node of G_R with $\psi_R(r_i) = i$.
- Let $X = X^* \setminus \{(v_2, v_n)\}$, where X^* consists of the edges of G with one endpoint in L and the other endpoint in R.

Figure 3(a) illustrates Lemma 5 and this step, which runs in $O(n)$ time. Figure 3(b) shows the corresponding $L \cup R \cup X^*$.

Lemma 6. (L, R, X) *is an* n*-node ladder graph.*

Proof. Consider any edge (ℓ_i, r_j) of X. By definition of ϕ, ℓ_i has to be on the external boundary of G_L and r_j has to be on the external boundary of G_R. By definition of T_L, ℓ_i is either a leaf of T_L or on the rightmost path of T_L. By definition of ψ_L, if $\ell_{i_1}, \ell_{i_2}, \ldots, \ell_{i_p}$ with $i_1 = 1$ are the nodes on the external boundary of G_L in counterclockwise order, then $i_1 < i_2 < \cdots < i_p$. Similarly, by definition of T_R, r_j is either a leaf of T_R or on the leftmost path of T_R. By definition of ψ_R, if $r_{j_1}, r_{j_2}, \ldots, r_{j_q}$ with $j_1 = 1$ are the nodes on the external boundary of G_R in clockwise order, then $j_1 < j_2 < \cdots < j_q$. Since G is a plane graph and the edges of X do not cross one another in G, the edges of X do not cross one another in (L, R, X). Therefore, (L, R, X) is outerplanar. □

3.4 Step 4

Let $H = (L, R, X)$. Lemma 6 ensures that H is an n-node ladder graph. Consider the outerplanar embedding $\mathcal{E}$ of H such that

$$\ell_1, \ell_2, \ldots, \ell_{\lceil n/2 \rceil}, r_{\lfloor n/2 \rfloor}, r_{\lfloor n/2 \rfloor - 1}, \ldots, r_1$$

are the nodes in clockwise order around the external boundary of H. Let the output σ of our algorithm be the consistent ordering of H with respect to $\mathcal{E}$ ensured by Lemma 4. Figure 3(a) illustrates this step, which also runs in $O(n)$ time.

Lemma 7. *The $O(n)$-time obtainable σ is an st-ordering of G with $\sigma(v_2) = 1$ and $\max(\sigma(v_1), \sigma(v_n)) = n$.*

Proof. We first show that ψ_L is an st-ordering of G_L. Let i be an index with $2 \leq i < \lceil \frac{n}{2} \rceil$. Let k be the index such that ℓ_k is the parent of ℓ_i in T_L. Since ψ_L is a post-ordering of T_L, we know that ℓ_k is a neighbor of ℓ_i in G_L with $i < k$. Let j be the index such that ℓ_j is the neighbor of ℓ_i in G_L immediately succeeding ℓ_k in counterclockwise order around ℓ_i. Recall that ℓ_k is the first neighbor of ℓ_i in G_L with $\phi(\ell_k) < \phi(\ell_i)$ in counterclockwise order around ℓ_i. Since ϕ is a four-canonical ordering of G, we also have $\phi(\ell_j) < \phi(\ell_i)$. Since ψ_L is the counterclockwise post-ordering of T_L, we have $\psi(\ell_j) < \psi(\ell_i)$, i.e., $j < i$. Since ℓ_j and ℓ_k are two neighbors of ℓ_i in G_L with $j < i < k$, we know that ψ_L is an st-ordering of G_L. It can be proved analogously that ψ_R is an st-ordering of G_R.

Since σ is a consistent ordering of H with respect to $\mathcal{E}$, we know that $1 \leq i < j \leq \lceil \frac{n}{2} \rceil$ implies $\sigma(\ell_i) < \sigma(\ell_j)$ and $1 \leq i < j \leq \lfloor \frac{n}{2} \rfloor$ implies $\sigma(r_i) < \sigma(r_j)$. We have the following observations.

- Since ψ_L is an st-ordering of G_L, for each $i = 1, \ldots, \lceil \frac{n}{2} \rceil - 1$, ℓ_i has a neighbor ℓ_k in G_L with $i < k$. Since G_L is a subgraph of G, ℓ_k is a neighbor of ℓ_i in G with $\sigma(\ell_i) < \sigma(\ell_k)$.
- Since ψ_L is an st-ordering of G_L, for each $i = 2, \ldots, \lceil \frac{n}{2} \rceil$, ℓ_i has a neighbor ℓ_j in G_L with $j < i$. Since G_L is a subgraph of G, we know that ℓ_j is a neighbor of ℓ_i in G with $\sigma(\ell_j) < \sigma(\ell_i)$.
- Since ψ_R is an st-ordering of G_R, for each $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor - 1$, r_i has a neighbor r_k in G_R with $i < k$. Since G_R is a subgraph of G, we know that r_k is a neighbor of r_i in G with $\sigma(r_i) < \sigma(r_k)$.
- Since ψ_R is an st-ordering of G_R, for each $i = 2, \ldots, \lfloor \frac{n}{2} \rfloor$, r_i has a neighbor r_j in G_R with $j < i$. Since G_R is a subgraph of G, we know that r_j is a neighbor of r_i in G with $\sigma(r_j) < \sigma(r_i)$.

According to the above observations, it suffices to ensure that edges (ℓ_1, r_1) and $(\ell_{\lceil n/2 \rceil}, r_{\lfloor n/2 \rfloor})$ belong to G. By Lemma 5, $\ell_1 = v_2$, $r_1 = v_{n-1}$, $\ell_{\lceil n/2 \rceil} = v_1$, and $r_{\lfloor n/2 \rfloor} = v_n$. Since v_1 and v_n are external nodes of the plane triangulation G, we know that $(\ell_{\lceil n/2 \rceil}, r_{\lfloor n/2 \rfloor}) = (v_1, v_n)$ is an edge of G. By definition of four-canonical ordering ϕ, we know that v_{n-1} is adjacent to v_2. Therefore, $(\ell_1, r_1) = (v_2, v_{n-1})$ is an edge of G. □

Figure 1(a) shows the resulting st-ordering σ of G computed by our algorithm.

3.5 Proving Theorem 1

Proof. Note that v_1, v_2, and v_n are the external nodes of G. By Lemmas 1 and 7, it suffices to ensure

$$length(G, \sigma) \leq \left\lceil \frac{n}{2} \right\rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil. \tag{1}$$

By Step 4 and Lemmas 4 and 6, we have

$$length(H, \sigma) \leq \left\lceil \frac{n}{2} \right\rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil. \tag{2}$$

Let $H^* = L \cup R \cup X^*$. That is, $H^* = H \cup \{(v_2, v_n)\}$, as illustrated by Fig. 3(a) and 3(b). By definition of σ and Lemma 5, we have $\sigma(v_2) = 1$ and $\sigma(v_n) \geq \max_j \sigma(r_j)$. Therefore, any σ-increasing path of H^* containing edge (v_2, v_n) contains exactly one node of R, i.e., v_n, and thus has length at most $\lceil \frac{n}{2} \rceil$. It follows from Inequality (2) that

$$length(H^*, \sigma) \leq \left\lceil \frac{n}{2} \right\rceil + 2 \left\lceil \sqrt{\frac{n-2}{2}} \right\rceil. \tag{3}$$

To prove Inequality (1), it remains to show that if P is a σ-increasing path of G, then there is a σ-increasing path Q of H^* such that the length of Q is no less than that of P. For each edge (u, v) of P with $\sigma(u) < \sigma(v)$, let $Q(u, v)$ be the σ-increasing path of H^* defined as follows.

- If $u = \ell_i$ and $v = r_j$, then let $Q(u, v) = (u, v)$, which is a σ-increasing path of X^*.
- If $u = r_i$ and $v = \ell_j$, then let $Q(u, v) = (u, v)$, which is a σ-increasing path of X^*.
- If $u = \ell_i$ and $v = \ell_j$, then by $\sigma(\ell_i) < \sigma(\ell_j)$ we know $\psi_L(\ell_i) < \psi_L(\ell_j)$ and thus $i < j$. Let $Q(u, v) = (\ell_i, \ell_{i+1}, \ldots, \ell_j)$. Since σ is a consistent ordering of H with respect to $\mathcal{E}$, $Q(u, v)$ is a σ-increasing path of L.
- If $u = r_i$ and $v = r_j$, then by $\sigma(r_i) < \sigma(r_j)$ we know $\psi_R(r_i) < \psi_R(r_j)$ and thus $i < j$. Let $Q(u, v) = (r_i, r_{i+1}, \ldots, r_j)$. Since σ is a consistent ordering of H with respect to $\mathcal{E}$, $Q(u, v)$ is a σ-increasing path of R.

Let Q be the union of $Q(u, v)$ for all edges (u, v) of P. Since each $Q(u, v)$ is a σ-increasing path of H^*, so is Q. The length of Q is no less than that of P. That is, we have

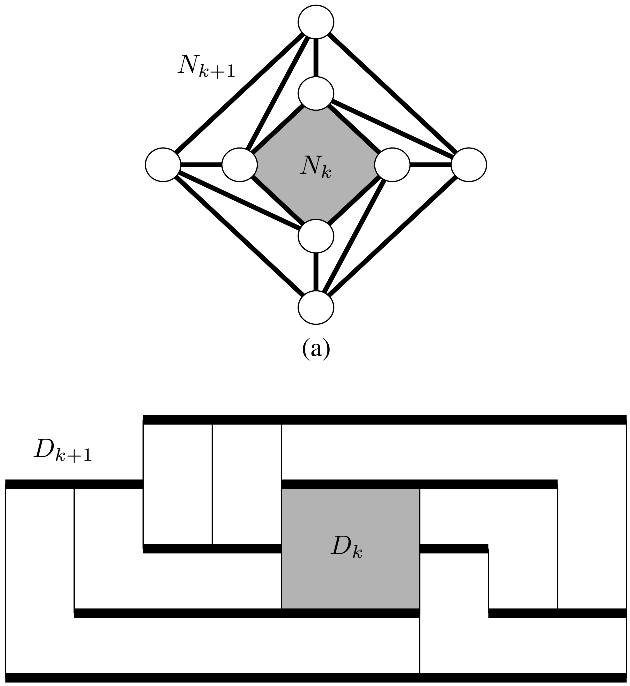
$$length(G, \sigma) \leq length(H^*, \sigma). \tag{4}$$

Since Inequality (1) is immediate from Inequalities (3) and (4), the lemma is proved. □

4 A Lower Bound

Let plane graph N_k be defined recursively as follows.

- Let N_1 be the four-node internally triangulated plane graph with four external nodes.



(b)

Fig. 4. (a) A four-connected plane graph N_{k+1} and its relation with N_k. (b) A visibility representation D_{k+1} of N_{k+1} and its relation with D_k.

– Let N_{k+1} be obtained from N_k by adding four nodes and twelve edges in the way as shown in Fig. 4(a).

One can easily verify that each N_k with $k \geq 1$ is indeed four-connected. The following lemma ensures that the the upper bound provided by Theorem 1 has an optimal first-order term.

Lemma 8. *All visibility representations of N_k have heights at least $2k$.*

Proof. We prove the lemma by induction on k. The lemma holds trivially for $k = 1$. Assume for a contradiction that N_{k+1} admits a visibility representation D_{k+1} with height no more than $2k + 1$. Let D_k be obtained from D_{k+1} by deleting all the horizontal segments representing those four external nodes of N_{k+1}. Since D_{k+1} has to reflect the planar embedding of N_{k+1}, D_k is a visibility representation of N_k. Since the external nodes of N_k are internal in N_{k+1}, the horizontal segments of D_{k+1} representing the external nodes of N_{k+1} have to wrap D_k completely. That is, D_{k+1} must have a horizontal segment above D_k and a horizontal segment below D_k. Therefore, the height of D_{k+1} is at least two more than that of D_k. It follows that the height of D_k is at most $2k - 1$, contradicting the inductive hypothesis. Since N_{k+1} cannot admit a visibility representation with height less than $2k + 2$, the lemma is proved. ☐

5 Concluding Remarks

It would be of interest to close the $\Theta(\sqrt{n})$ gap between the upper and lower bounds on the required height for the visibility representation of any n-node four-connected plane graph. We conjecture that the $\Theta(\sqrt{n})$ term in our upper bound can be reduced to $O(1)$.

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