Unimaximal Sequences of Pairs in Rectangle Visibility Drawing

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Abstract. We study the existence of unimaximal subsequences in sequences of pairs of integers, e.g., the subsequences that have exactly one local maximum in each component of the subsequence. We show that every sequence of $\frac{1}{12}n^2(n^2-1)+1$ pairs has a unimaximal subsequence of length n. We prove that this bound is tight. We apply this result to the problem of the largest complete graph with a 3D rectangle visibility representation and improve the upper bound from 55 to 50.

1 Introduction

A 3D rectangle visibility drawing represents vertices by axis-aligned rectangles lying in planes parallel to the xy-plane. Edges correspond to the z-parallel visibility among these rectangles. This type of graph drawing was studied, for example, in [1,2,5,6,7,8].

We continue in the study of the maximum size of a complete graph with a 3D rectangle visibility representation. The representation of K_{22} given by Rote and Zelle (included in [8]) provides the best known lower bound. On the other hand, Bose et al. [2] showed that no complete graph with 103 or more vertices has such a representation. This result was then improved to 56 by Fekete et al. [1]. Their proof is based on the analysis of unimaximal subsequences in sequences of rectangle coordinates.

A sequence $x_1, x_2, \ldots$ of distinct integers is called *unimaximal* if it has exactly one local maximum, i. e., for all i, j, k with $i < j < k$ we have $x_j > \min\{x_i, x_k\}$. The following lemma (attributed by Chung [3] to V. Chvátal and J.M. Steele, among others) summarizes the most important properties of unimaximal sequences.

Lemma 1. *For all $n > 1$, in every sequence of $\binom{n}{2}+1$ distinct integers, there exists a unimaximal subsequence of length n. On the other hand, there exists a sequence of $\binom{n}{2}$ distinct integers that has no unimaximal subsequence of length n.*

The notion of unimaximality can be generalized to sequences of pairs:

Definition 1. *A sequence $(x_1, y_1), (x_2, y_2), \ldots$ of pairs of integers is called* unimaximal *if it is unimaximal in both components, i. e., if both sequences $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ are unimaximal.*

I.G. Tollis and M. Patrignani (Eds.): GD 2008, LNCS 5417, pp. 61–66, 2009.

If we apply the previous lemma twice on a sequence of pairs then we can see that every sequence of $\binom{\binom{n}{2}+1}{2}+1 \approx \frac{1}{8}n^4$ pairs has a unimaximal subsequence of length n. In fact, the result of Fekete et al. [1] is based on this fact. We show in this paper that we can improve this bound to $\frac{1}{12}n^2(n^2-1)+1$ if we consider both components of a sequence of pairs together. This result allows us to improve the upper bound on the size of the largest complete graph with a 3D rectangle visibility representation from 55 to 50.

2 Upper Bound

The definition of a unimaximal sequence requires distinct values in the sequence. Therefore both components of a unimaximal sequence of pairs must contain distinct values.[1] Hence we consider only sequences with this property in the sequel.

We show that every sufficiently long sequence of pairs contains a unimaximal subsequence of a given length. The following relations turn out to be useful in the analysis of this problem.

Definition 2. *Let* $(x_1, y_1), (x_2, y_2), \ldots$ *be a sequence of pairs of integers. We say that two pairs* $(x_i, y_i), (x_j, y_j), i < j$ *have a* ⇗-relation *if* $x_i < x_j$ *and* $y_i < y_j$*. The pairs have a* ⇘-relation *if* $x_i > x_j$ *and* $y_i > y_j$*.*

If both relations are forbidden then our problem becomes a simple consequence of the Erdős-Szekeres theorem [4].

Lemma 2. *If a sequence of* $(n-1)^2+1$ *pairs of integers doesn't contain pairs with* ⇗- *and* ⇘-*relations then it has a unimaximal subsequence of length* n*.*

Proof. Let $((x_i, y_i))_i$ be a sequence of length $(n-1)^2+1$. The sequence $(x_i)_i$ contains a monotone subsequence $(x_{i_j})_j$ of length n according to the Erdős-Szekeres theorem. The sequence $(y_{i_j})_j$ is monotone as well because the original sequence doesn't have pairs with ⇗- and ⇘-relations, e.g., if the sequence $(x_{i_j})_j$ is increasing then $(y_{i_j})_j$ is decreasing and vice versa.

Hence the subsequence $((x_{i_j}, y_{i_j}))_{j=1}^{n}$ is unimaximal. □

Lemma 3 shows how the situation changes if only one relation is forbidden.

Lemma 3. *If a sequence of* $f_n = \frac{1}{6}(n-1)n(2n-1)+1$ *pairs of integers doesn't contain pairs with a* ⇘-*relation then it has a unimaximal subsequence of length* n*.*

Proof. The lemma holds for $n = 1$. Let's suppose that it holds for $n = k \in \mathbb{N}$ and let $P = ((x_i, y_i))_{i=1}^{f_{k+1}}$ be a sequence that doesn't contain pairs with a ⇘-relation. Let S be the set of pairs (x, y) such that P contains a unimaximal subsequence

[1] Both components $(x_i)_i$ and $(y_i)_i$ of $((x_i, y_i))_i$ must contain distinct values, but it may happen that $x_i = y_j$.

of length k starting at (x, y). We know that every sequence of length f_k contains at least one such a subsequence. Therefore $|S| \geq f_{k+1} - f_k + 1 = k^2 + 1$.

If there are two pairs $(x_i, y_i), (x_j, y_j), i < j$ in S that have a $\nearrow\!\!\nearrow$-relation then we can prepend (x_i, y_i) to the unimaximal subsequence of length k starting at (x_j, y_j) and obtain a unimaximal subsequence of length $k + 1$.

On the other hand, if there are no pairs in S that have a $\nearrow\!\!\nearrow$-relation then S contains a unimaximal subsequence of length $k + 1$ according to the previous lemma. Hence the lemma holds also for $n = k + 1$. □

The idea of the previous proof can be reused to analyze sequences with both relations allowed.

Theorem 1. *For all $n \in \mathbb{N}$, in every sequence of $g_n = \frac{1}{12}n^2(n^2 - 1) + 1$ pairs of integers, there exists a unimaximal subsequence of length n.*

Proof. We proceed in the same way as in the previous proof. The theorem holds for $n = 1$. Let's suppose that it holds for $n = k \in \mathbb{N}$ and let $P = ((x_i, y_i))_i$ be a sequence of length g_{k+1}. Let E be the set of pairs (x, y) such that P contains a unimaximal subsequence of length k ending at (x, y). We know that every sequence of length g_k contains at least one such a subsequence. Therefore $|E| \geq g_{k+1} - g_k + 1 = f_{k+1}$.

If there are two pairs $(x_i, y_i), (x_j, y_j), i < j$ in E that have a $\searrow\!\!\searrow$-relation then we can append (x_j, y_j) to the unimaximal subsequence of length k ending at (x_i, y_i) and obtain a unimaximal subsequence of length $k + 1$.

On the other hand, if there are no pairs in E that have a $\searrow\!\!\searrow$-relation then E contains a unimaximal subsequence of length $k + 1$ according to the previous lemma. Hence the theorem holds also for $n = k + 1$. □

3 Lower Bound

This section shows that the bounds derived in the previous section are tight.

Lemma 4. *For all $n > 1$ there exists a sequence P_n of $(n-1)^2$ pairs of integers that*

- *doesn't contain pairs with $\nearrow\!\!\nearrow$- and $\searrow\!\!\searrow$-relations,*
- *has no unimaximal subsequence of length n.*

Proof. According to the Erdős-Szekeres theorem there exists a sequence $(x_i)_{i=1}^{(n-1)^2}$ that doesn't contain a monotone subsequence of length n. The sequence $P_n = ((x_i, -x_i))_{i=1}^{(n-1)^2}$ clearly doesn't contain pairs with $\nearrow\!\!\nearrow$- and $\searrow\!\!\searrow$-relations.

A unimaximal subsequence of P_n (or any other sequence that doesn't contain pairs with $\nearrow\!\!\nearrow$- and $\searrow\!\!\searrow$-relations) must be monotone in both components. Therefore P_n cannot have a unimaximal subsequence of length n because otherwise $(x_i)_i$ would contain a monotone subsequence of this length. □

Let $P = ((x_i, y_i))_i$ be a sequence of pairs of integers and $m \in \mathbb{N}$. We denote the sequence $((x_i + m, y_i + m))_i$ by $P + m$ in the sequel.

Lemma 5. *For all $n > 1$ there exists a sequence Q_n of $\frac{1}{6}(n-1)n(2n-1)$ pairs of integers that*

- *doesn't contain pairs with a $\searrow\searrow$-relation,*
- *has no unimaximal subsequence of length n.*

Proof. Let $P_i, i = 2, \ldots, n$ be the sequences from the previous lemma. Let $P'_i = P_i + m_i$. The shifts m_i are selected such that for all $n \geq i > j \geq 2$ the pairs from P'_i have to pairs in P'_j $\nearrow\nearrow$-relations. Finally, let Q_n be a concatenation of the sequences $P'_n, \ldots, P'_2$.

The length of Q_n is $\sum_{i=2}^{n}(i-1)^2 = \frac{1}{6}(n-1)n(2n-1)$.

Q_n doesn't contain a $\searrow\searrow$-relation because this relation is not present among pairs from the individual subsequences P'_i and there are $\nearrow\nearrow$-relations among pairs from the different subsequences.

Let U be a unimaximal subsequence of Q_n and k be the minimal index such that U contains a pair $(\overline{x}, \overline{y})$ from P'_k. Each pair from P'_l, $l > k$ has a $\nearrow\nearrow$-relation to $(\overline{x}, \overline{y})$. If (x_i, y_i) and (x_j, y_j), $i < j$ are two pairs from a fixed P'_l, $l > k$ then they cannot be both in U because the triple (x_i, y_i), (x_j, y_j), $(\overline{x}, \overline{y})$ is unimaximal only if (x_i, y_i) has a $\nearrow\nearrow$-relation to (x_j, y_j), but this cannot happen due to the definition of P'_l.

Therefore U contains at most one pair from each P'_l, $l > k$ and at most $k-1$ pairs from P'_k (P'_k has no unimaximal subsequence of length k). Hence $|U| \leq (n-k) + (k-1) = n-1$ and Q_n has no unimaximal subsequence of length n. □

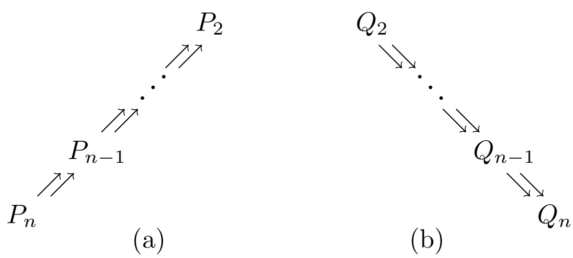

Fig. 1. Construction of (a) Q_n and (b) R_n

Lemmas 4 and 5 provide the lower bounds that match the upper bounds given by Lemmas 2 and 3. Finally, the following theorem shows that the bound in Theorem 1 is tight as well.

Theorem 2. *For all $n > 1$ there exists a sequence R_n of $\frac{1}{12}n^2(n^2-1)$ pairs of integers that has no unimaximal subsequence of length n.*

Proof. The proof is very similar to the proof of the previous lemma.

Let $Q_i, i = 2, \ldots, n$ be the sequences from the previous lemma. Let $Q'_i = Q_i + m_i$. The shifts m_i are selected such that for all $2 \leq i < j \leq n$ the pairs from Q'_i have to pairs in Q'_j $\searrow\searrow$-relations. Finally, let R_n be a concatenation of the sequences $Q'_2, \ldots, Q'_n$.

The length of R_n is $\sum_{i=2}^{n} \frac{1}{6}(i-1)i(2i-1) = \frac{1}{12}n^2(n^2-1)$.

Let U be a unimaximal subsequence of R_n and k be the minimal index such that U contains a pair $(\overline{x}, \overline{y})$ from Q'_k. $(\overline{x}, \overline{y})$ has a $\searrow\searrow$-relation to each pair from Q'_l, $l > k$. If (x_i, y_i) and (x_j, y_j), $i < j$ are two pairs from a fixed Q'_l, $l > k$ then they cannot be both in U because the triple $(\overline{x}, \overline{y})$, (x_i, y_i), (x_j, y_j) is unimaximal only if (x_i, y_i) has a $\searrow\searrow$-relation to (x_j, y_j), but this cannot happen due to the definition of Q'_l.

Therefore U contains at most one pair from each Q'_l, $l > k$ and at most $k-1$ pairs from Q'_k (Q'_k has no unimaximal subsequence of length k). Hence $|U| \leq (n-k) + (k-1) = n-1$ and R_n has no unimaximal subsequence of length n. □

4 Application in 3D Rectangle Visibility Graphs

Fekete et al. [1] showed that every 3D rectangle visibility representation can be described using integer 4-tuples that denote perpendicular distances of sides of individual rectangles to the origin. They also proved the following lemma.

Lemma 6. *In a representation of K_5 by five rectangles $((e_i, n_i, w_i, s_i))_{i=1}^5$, it is impossible that both sequences $(n_i)_{i=1}^5$ and $(s_i)_{i=1}^5$ are unimaximal.*

Lemma 6 and Theorem 1 allow us to improve the best known upper bound on the size of the largest complete graph with a 3D rectangle visibility representation.

Theorem 3. *No complete graph K_n has a 3D rectangle visibility representation for $n \geq 51$.*

Proof. Let's assume we have a representation of K_n with $n \geq 51$ rectangles (e_i, n_i, w_i, s_i). Theorem 1 implies that the sequence $((n_i, s_i))_{i=1}^{51}$ has a unimaximal subsequence $(n'_i, s'_i)_i$ of length 5. Remove the rectangles not associated with the subsequence. The five remaining rectangles represent K_5, but this contradicts the previous lemma because both sequences $(n'_i)_{i=1}^5$ and $(s'_i)_{i=1}^5$ are unimaximal. □

5 Conclusion

We show that every sequence of $\frac{1}{12}n^2(n^2-1)+1$ pairs of integers has a unimaximal subsequence of length n. On the other hand, there are sequences of $\frac{1}{12}n^2(n^2-1)$ pairs that do not contain such a sequence.

The analysis of unimaximal sequences of pairs allows us to improve the best known upper bound on the size of the largest complete graph with a 3D rectangle visibility representation from 55 to 50. The original bound by Fekete el al. [1] is also based on the study of unimaximal subsequences in the sequences of rectangle coordinates but they consider each coordinate independently. It remains an open problem how to analyze all four coordinates together to obtain a better bound.

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