# Hamiltonian Alternating Paths on Bicolored Double-Chains*

Josef Cibulka[1], Jan Kynčl[2], Viola Mészáros[3,4], Rudolf Stolař[1], and Pavel Valtr[2]

[1] Department of Applied Mathematics, Charles University, Faculty of Mathematics and Physics, Malostranské nám. 25, 118 00 Prague, Czech Republic
cibulka@kam.mff.cuni.cz, ruda@kam.mff.cuni.cz
[2] Department of Applied Mathematics and Institute for Theoretical Computer Science (ITI), Charles University, Faculty of Mathematics and Physics, Malostranské nám. 25, 118 00 Prague, Czech Republic
kyncl@kam.mff.cuni.cz
[3] Department of Applied Mathematics and Institute for Theoretical Computer Science, Charles University, Malostranské nám. 25, 118 00 Prague, Czech Republic
[4] Bolyai Institute, University of Szeged, Aradi vértanúk tere 1, 6720 Szeged, Hungary
viola@math.u-szeged.hu

**Abstract.** We find arbitrarily large finite sets $S$ of points in general position in the plane with the following property. If the points of $S$ are equitably 2-colored (i.e., the sizes of the two color classes differ by at most one), then there is a polygonal line consisting of straight-line segments with endpoints in $S$, which is Hamiltonian, non-crossing, and alternating (i.e., each point of $S$ is visited exactly once, every two non-consecutive segments are disjoint, and every segment connects points of different colors).

We show that the above property holds for so-called double-chains with each of the two chains containing at least one fifth of all the points. Our proof is constructive and can be turned into a linear-time algorithm. On the other hand, we show that the above property does not hold for double-chains in which one of the chains contains at most $\approx 1/29$ of all the points.

## 1 Introduction

### 1.1 Previous Results

One of the basic problems in geometric graph theory is to decide if a given graph can be drawn on a given planar point set using pairwise non-crossing straight-line edges. In a more demanding version, the points and the vertices of the graph are colored and each vertex has to be placed in a point of the same color (see the survey [5] for further references). Interesting and non-trivial questions arise already if we want to embed a 2-colored path on a 2-colored point set. The authors of several papers have focused on embeddings of so-called alternating paths, which are paths with no monochromatic

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edge. Since the colors on a 2-colored alternating path must alternate along the path, a 2-colored point set $S$ may admit a Hamiltonian alternating path only if the coloring of $S$ is equitable, i.e., the sizes of the color classes differ by at most one.

Let $S$ be an equitably 2-colored set of points in general position in the plane. It is known that if the two color classes of $S$ can be separated by a line then there is a non-crossing Hamiltonian alternating path on $S$ [1]. The same result holds if one of the color classes is exactly the set of vertices of the convex hull [1]. Kaneko et al. [6] proved that any equitably 2-colored set $S$ of at most 12 points or of 14 points admits a non-crossing Hamiltonian alternating path. On the other hand, Kaneko et al. [6] gave examples of equitably 2-colored sets $S$ of $n$ points admitting no non-crossing Hamiltonian alternating path for any $n > 12$, $n \neq 14$.

The above result on sets with color classes separated by a line easily implies that any equitably 2-colored set $S$ of size $n$ admits a non-crossing alternating path on at least $n/2$ points of $S$. It is an open problem if this lower bound can be improved to $n/2 + f(n)$, where $f(n)$ is unbounded (see also the book [3]). On the other hand, there are equitably 2-colored sets admitting no non-crossing alternating path of length more than $\approx 2n/3$ [2,7]. This upper bound is proved for certain colorings of sets in convex position. The above general lower bound $n/2$ can be slightly improved to $n/2 + \Omega(\sqrt{n/\log n})$ for sets in convex position [7].

In this paper we find arbitrarily large "universal" sets for which any equitable 2-coloring admits a non-crossing Hamiltonian alternating path. We prove the "universality" for so-called double-chains with each chain containing at least one fifth of all the points. Double-chains were first considered in [4].

## 1.2 Our Results

A *convex* or a *concave chain* is a finite set of points in the plane lying on the graph of a strictly convex or a strictly concave function, respectively. A *double-chain* $(C_1, C_2)$ consists of a convex chain $C_1$ and a concave chain $C_2$ such that each point of $C_2$ lies strictly below every line determined by $C_1$ and similarly, each point of $C_1$ lies strictly above every line determined by $C_2$ (see Fig. 1). Note that we allow different sizes of the chains $C_1$ and $C_2$.

Let $(C_1, C_2)$ be a double-chain, and let $p_1, p_2, \ldots, p_k \in C_1 \cup C_2$ be distinct points of $C_1 \cup C_2$. The polygonal line $p_1p_2 \ldots p_k$ consisting of the $k-1$ straight-line segments $p_1p_2, p_2p_3, \ldots, p_{k-1}p_k$ is shortly called *the path* $p_1p_2 \ldots p_k$. The path $p_1p_2 \ldots p_k$ is

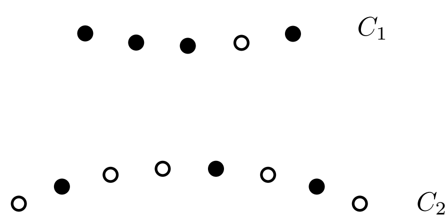


**Fig. 1.** An equitably 2-colored double-chain $(C_1, C_2)$

*non-crossing* if any two non-consecutive segments in it are disjoint. The path $p_1p_2 \dots p_k$ is *Hamiltonian* (*for the double-chain* $(C_1, C_2)$) if it visits all the points of $C_1 \cup C_2$ (i.e., $k = |C_1| + |C_2|$).

Suppose that the points of a double-chain $(C_1, C_2)$ are colored by two colors. Then a path $p_1p_2 \dots p_k$ is *alternating* if the endpoints of each segment are colored by different colors. A path on $C_1 \cup C_2$ is a *good path* if it is non-crossing, Hamiltonian and alternating.

An *equitable* $2$-*coloring* of a double-chain $(C_1, C_2)$ is a coloring of $C_1 \cup C_2$ by two colors such that the sizes of the color classes differ by at most one. We use *black* and *white* as the colors in the colorings. Here is our main result:

**Theorem 1.** *Let $(C_1, C_2)$ be a double-chain whose points are colored by an equitable 2-coloring, and let $|C_i| \geq \frac{1}{5}(|C_1| + |C_2|)$ for $i = 1, 2$. Then $(C_1, C_2)$ has a good path. Moreover, a good path on $(C_1, C_2)$ can be found in linear time.*

On the other hand, we show that double-chains with highly unbalanced sizes of chains do not admit a good path for some equitable 2-colorings:

**Theorem 2.** *Let $(C_1, C_2)$ be a double-chain whose points are colored by an equitable 2-coloring, and let $C_1$ be periodic with the following period of length 16: 2 black, 4 white, 6 black and 4 white points. If $|C_1| \geq 28(|C_2| + 1)$, then $(C_1, C_2)$ has no good path.*

## 2 Proof of Theorem 1

This section contains only the proof for double-chains with an even number of points. The proof for the odd number of points can be found in the Appendix.

The main idea of our proof is to cover the chains $C_i$ by a special type of pairwise non-crossing paths, so called hedgehogs, and then to connect these hedgehogs into a good path by adding some edges between $C_1$ and $C_2$.

### 2.1 Notation Used in the Proof

For $i = 1, 2$, let $b_i$ be the number of black points of $C_i$ and let $w_i := |C_i| - b_i$ denote the number of white points of $C_i$.

Since the coloring is equitable, we may assume that $b_1 \geq w_1$ and $w_2 \geq b_2$. Then black is *the major color of* $C_1$ and *the minor color of* $C_2$, and white is *the major color of* $C_2$ and *the minor color of* $C_1$. Points in the major color, i.e., black points on $C_1$ and white points on $C_2$, are called *major points*. Points in the minor color are called *minor points*.

Points on each $C_i$ are linearly ordered according to the $x$-coordinate. An *interval* of $C_i$ is a sequence of consecutive points of $C_i$. An *inner point* of an interval $I$ is any point of $I$ which is neither the leftmost nor the rightmost point of $I$.

A *body* $D$ is a non-empty interval of a chain $C_i$ $(i = 1, 2)$ such that all inner points of $D$ are major. If the leftmost point of $D$ is minor, then we call it a *head* of $D$. Otherwise

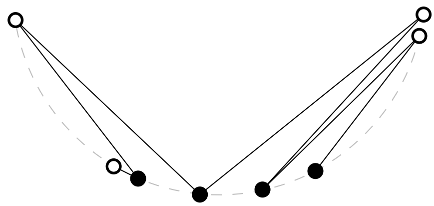

**Fig. 2.** A hedgehog in $C_1$

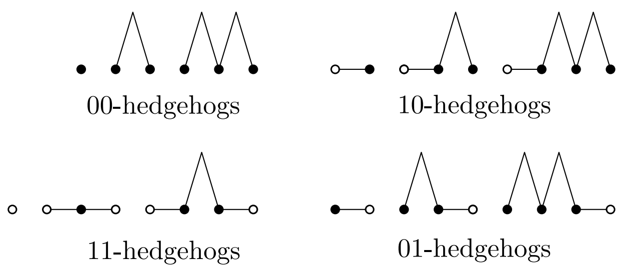


**Fig. 3.** Types of hedgehogs (sketch)

$D$ has no head. If the rightmost point of $D$ is minor, then we call it a *tail* of $D$. Otherwise $D$ has no tail. If a body consists of just one minor point, this point is both the head and the tail.

Bodies are of the following four types. A $00$-*body* is a body with no head and no tail. A $11$-*body* is a body with both head and tail. The bodies of remaining two types have exactly one endpoint major and the other one minor. We will call the body a $10$-*body* or a $01$-*body* if the minor endpoint is a head or a tail, respectively.

Let $D$ be a body on $C_i$. A *hedgehog (built on the body $D \subseteq C_i$)* is a non-crossing alternating path $H$ with vertices in $C_i$ satisfying the following three conditions: (1) $H$ contains all points of $D$, (2) $H$ contains no major points outside of $D$, (3) the endpoints of $H$ are the first and the last point of $D$. A hedgehog built on an $\alpha\beta$-body is an *$\alpha\beta$-hedgehog* ($\alpha, \beta = 0, 1$). If a hedgehog $H$ is built on a body $D$, then $D$ is *the body of* $H$ and the points of $H$ that do not lie in $D$ are *spines*. Note that each spine is a minor point. All possible types of hedgehogs can be seen on Fig. 3 (for better lucidity, we will draw hedgehogs with bodies on a horizontal line and spines indicated only by a "peak" from now on).

On each $C_i$, maximal intervals containing only major points are called *runs*. Clearly, runs form a partition of major points. For $i = 1, 2$, let $r_i$ denote the number of runs in $C_i$.

### 2.2 Proof in the Even Case

Throughout this subsection, $(C_1, C_2)$ denotes a double-chain with $|C_1| + |C_2|$ even. Since the coloring is equitable, we have $b_1 + b_2 = w_1 + w_2$. Set

$$\Delta := b_1 - w_1 = w_2 - b_2.$$

First we give a lemma characterizing collections of bodies on a chain $C_i$ that are bodies of some pairwise non-crossing hedgehogs covering the whole chain $C_i$.

**Lemma 3.** *Let $i \in \{1, 2\}$. Let all major points of $C_i$ be covered by a set $\mathcal{D}$ of pairwise disjoint bodies. Then the bodies of $\mathcal{D}$ are the bodies of some pairwise non-crossing hedgehogs covering the whole $C_i$ if and only if $\Delta = d_{00} - d_{11}$, where $d_{\alpha\alpha}$ is the number of $\alpha\alpha$-bodies in $\mathcal{D}$.*

*Proof.* An $\alpha\beta$-hedgehog containing $t$ major points contains $(t - 1) + \alpha + \beta$ minor points. It follows that the equality $\Delta = d_{00} - d_{11}$ is necessary for the existence of a covering of $C_i$ by disjoint hedgehogs built on the bodies of $\mathcal{D}$.

Suppose now that $\Delta = d_{00} - d_{11}$. Let $F$ be the set of minor points on $C_i$ that lie in no body of $\mathcal{D}$, and let $M$ be the set of the mid-points of straight-line segments connecting pairs of consecutive major points lying in the same body. It is easily checked that $|F| = |M|$. Clearly $F \cup M$ is a convex or a concave chain. Now it is easy to prove that there is a non-crossing perfect matching formed by $|F| = |M|$ straight-line segments between $F$ and $M$ (for the proof, take any segment connecting a point of $F$ with a neighboring point of $M$, remove the two points, and continue by induction); see Fig. 4.

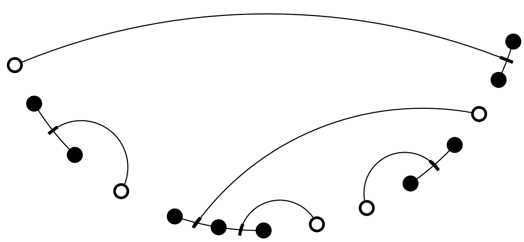

**Fig. 4.** A non-crossing matching of minor points and midpoints (in $C_1$)

If $f \in F$ is connected to a point $m \in M$ in the matching, then $f$ will be a spine with edges going from it to those two major points that determined $m$. Obviously, these spines and edges define non-crossing hedgehogs with bodies in $\mathcal{D}$ and with all the required properties. □

The following three lemmas and their proofs show how to construct a good path in some special cases.

**Lemma 4.** *If $\Delta \geq \max\{r_1, r_2\}$ then $(C_1, C_2)$ has a good path.*

*Proof.* Let $i \in \{1, 2\}$. Since $r_i \leq \Delta \leq \max(b_i, w_i)$, the runs in $C_i$ may be partitioned into $\Delta$ 00-bodies. By Lemma 3, these 00-bodies may be extended to pairwise non-crossing hedgehogs covering $C_i$. This gives us $2\Delta$ hedgehogs on the double-chain. They may be connected into a good path by $2\Delta - 1$ edges between the chains in the way shown in Fig. 5. □

**Lemma 5.** *If $r_1 = r_2$ then $(C_1, C_2)$ has a good path.*

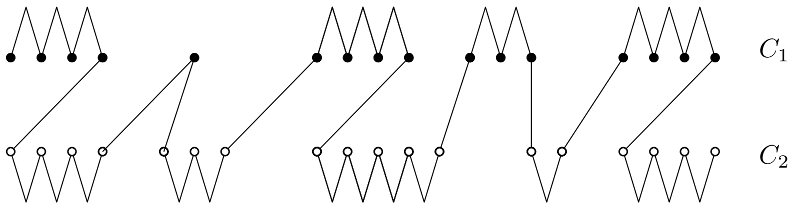


**Fig. 5.** 00-hedgehogs connected to a good path

*Proof.* Set $r := r_1 = r_2$. If $r \leq \Delta$ then we may apply Lemma 4. Thus, let $r > \Delta$.

Suppose first that $\Delta \geq 1$. We cover each run on each $C_i$ by a single body whose type is as follows. On $C_1$ we take $\Delta$ 00-bodies followed by $(r - \Delta)$ 10-bodies. On $C_2$ we take (from left to right) $(\Delta - 1)$ 00-bodies, $(r - \Delta)$ 01-bodies, and one 00-body. By Lemma 3, the $r$ bodies on each $C_i$ can be extended to hedgehogs covering $C_i$. Altogether we obtain $2r$ hedgehogs. They can be connected to a good path by $2r - 1$ edges between $C_1$ and $C_2$ (see Fig. 6).

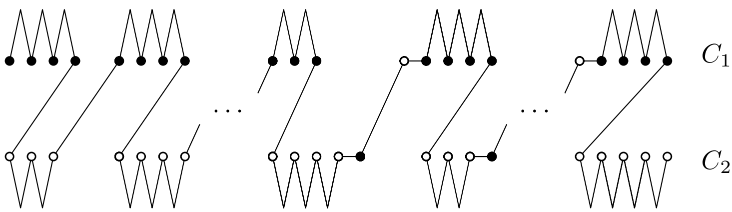


**Fig. 6.** A good path in the case $r_1 = r_2 > \Delta \geq 1$

Suppose now that $\Delta = 0$. We add one auxiliary major point on each $C_i$ as follows. On $C_1$, the auxiliary point extends the leftmost run on the left. On $C_2$, the auxiliary point extends the rightmost run on the right. This does not change the number of runs and increases $\Delta$ to 1. Thus, we may proceed as above. The good path obtained has the two auxiliary points on its ends. We may remove the auxiliary points from the path, obtaining a good path for $(C_1, C_2)$. □

A *singleton* $s \in C_i$ is an inner point of $C_i$ $(i = 1, 2)$ such that its two neighbors on $C_i$ are colored differently from $s$.

**Lemma 6.** *Suppose that $C_1$ has no singletons and $C_2$ can be covered by $r_1 - 1$ pairwise disjoint hedgehogs. Then $(C_1, C_2)$ has a good path.*

*Proof.* For simplicity of notation, set $r := r_1$. We denote the $r - 1$ hedgehogs on $C_2$ by $P_1, P_2, \ldots, P_{r-1}$ in the left-to-right order in which the bodies of these hedgehogs appear on $C_2$. For technical reasons, we enlarge the leftmost run of $C_1$ from the left by an auxiliary major point $\sigma$.

Our goal is to find $r$ hedgehogs $H_1, H_2, \ldots, H_r$ on $C_1 \cup \{\sigma\}$ such that they may be connected with the hedgehogs $P_1, P_2, \ldots, P_{r-1}$ into a good path. For each $j = 1, \ldots, r$, the body of the hedgehog $H_j$ will be denoted by $D_j$. For each $j = 1, \ldots, r$, $D_j$ covers the $j$-th run of $C_1 \cup \{\sigma\}$ (in the left-to-right order). We now finish the definition of the bodies $D_j$ by specifying for each $D_j$ if it has a head and/or a tail. The body $D_1$ is without head. For $j > 1$, $D_j$ has a head if and only if $P_{j-1}$ has a tail. The last body $D_r$ is without tail and $D_j, j < r$, has a tail if and only if $P_j$ has a head.

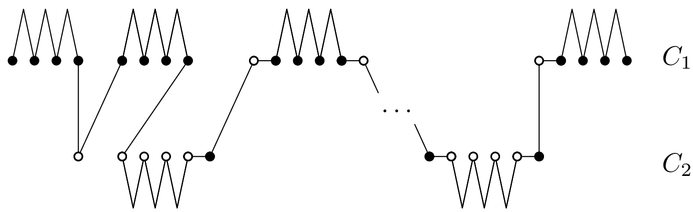


**Fig. 7.** A good path in the case of no singletons on $C_1$

It follows from Lemma 3 that we may add or remove some minor points on $C_1 \cup \{\sigma\}$ so that $D_1, \ldots, D_r$ can then be extended to pairwise non-crossing hedgehogs $H_1, \ldots, H_r$ covering the "new" $C_1$. More precisely, there is a double-chain $(C_1', C_2)$ such that $D_1, \ldots, D_r$ can be extended to pairwise non-crossing hedgehogs $H_1, \ldots, H_r$ covering $C_1'$, where either $C_1' = C_1 \cup \{\sigma\}$ or $C_1'$ is obtained from $C_1 \cup \{\sigma\}$ by adding some minor (white) points on the left of $C_1 \cup \{\sigma\}$ (say) or $C_1'$ is obtained from $C_1 \cup \{\sigma\}$ by removal of some minor (white) points lying in none of the bodies $D_1, \ldots, D_r$. Then the concatenation $H_1 P_1 H_2 P_2 \cdots H_{r-1} P_{r-1} H_r$ shown in Fig. 7 gives a good path on $(C_1', C_2)$. This good path starts with the point $\sigma$. Removal of $\sigma$ from it gives a good path $P$ for the double-chain $(C_1' \setminus \{\sigma\}, C_2)$. The endpoints of $P$ have different colors. Thus, $P$ covers the same number of black and white points. Black points on $P$ are the $\frac{|C_1|+|C_2|}{2}$ black points of $(C_1, C_2)$. Thus, $P$ covers exactly $|C_1| + |C_2|$ points. It follows that $|C_1' \setminus \{\sigma\}| = |C_1|$ and thus $C_1' \setminus \{\sigma\} = C_1$. The path $P$ is a good path on the double-chain $(C_1, C_2)$. □

The following lemma will be used to find a covering needed in Lemma 6.

**Lemma 7.** *Suppose that $|C_i| \geq k$, $r_i \leq k$ and $\Delta \leq k$ for some $i \in \{1, 2\}$ and for some integer $k$. Then $C_i$ can be covered by $k$ pairwise disjoint hedgehogs.*

*Proof.* The idea of the proof is to start with the set $\mathcal{D}$ of $|C_i|$ bodies, each of them being a single point, and then gradually decrease the number of bodies in $\mathcal{D}$ by joining some of the bodies together. We see that $\Delta = d_{00} - d_{11}$, where $d_{\alpha\alpha}$ is the number of $\alpha\alpha$-bodies in $\mathcal{D}$. If we join two neighboring 00-bodies to one 00-body and withdraw a single-point 11-body from $\mathcal{D}$ (to let the minor point become a spine) at the same time, the difference between the number of 00-bodies and the number of 11-bodies remains the same and $|\mathcal{D}|$ decreases by two. We can reduce $|\mathcal{D}|$ by one while preserving the difference $d_{00} - d_{11}$ by joining a 00-body with a neighboring single-point 11-body into a 01- or a 10-body. Similarly we can join a 01- or a 10-body with a neighboring (from the proper side) single-point 11-body into a new 11-body to decrease $|\mathcal{D}|$ by one as well. When we are joining two 00-bodies, we choose the single-point 11-body to remove in such a way to keep as many single-point 11-bodies adjacent to 00-bodies as possible. This guarantees that we can use up to $r_i$ of them for heads and tails.

We start with joining neighboring 00-bodies and we do this as long as $|\mathcal{D}| > k+1$ and $d_{00} > r_i$. Note that by the assumption $\Delta \leq k$, we will have enough single-point 11-bodies to do that. When we end, one of the following conditions holds: $|\mathcal{D}| = k$, $|\mathcal{D}| = k+1$ or $d_{00} = r_i$. In the first case we are done. If $|\mathcal{D}| = k+1$, we just add one head or one tail (we can do this since $d_{00} + d_{11} = |\mathcal{D}| = k+1 \geq d_{00} - d_{11} + 1$, which implies $d_{11} > 0$). If $d_{00} = r_i$, then each run is covered by just one 00-body. We need to add $|\mathcal{D}| - k$ heads and tails. We have enough single-point 11-bodies to do that since $d_{11} = |\mathcal{D}| - d_{00} = |\mathcal{D}| - r_i \geq |\mathcal{D}| - k$. On the other hand, $r_i - d_{11} = \Delta \geq 0$, so the number of heads and tails needed is at most $r_i$. Therefore, all the single-point 11-bodies are adjacent to 00-bodies and we can use them to form heads and tails.

In all cases we get a set $\mathcal{D}$ of $k$ bodies. Now we can apply Lemma 3 to obtain $k$ pairwise disjoint hedgehogs covering $C_i$. □

By a *contraction* we mean removing a singleton with both its neighbors and putting a point of the color of its neighbors in its place instead. It is easy to verify that if there is a good path in the new double-chain obtained by this contraction, it can be expanded to a good path in the original double-chain.

Now we can prove our main theorem in the even case.

*Proof of Theorem 1 (even case).* Without loss of generality we may assume that $r_1 \geq r_2$. In the case $r_1 = r_2$, we get a good path by Lemma 5. In the case $\Delta \geq r_1$, we get a good path by Lemma 4. Therefore, the only case left is $r_1 > r_2$, $r_1 > \Delta$.

If there is a singleton on $C_1$, we make a contraction of it. By this we decrease $r_1$ by one and both $r_2$ and $\Delta$ remain unchanged. If now $r_1 = r_2$ or $r_1 = \Delta$, we again get a good path, otherwise we keep making contractions until one of the previous cases appears or there are no more singletons to contract.

If there is no more singleton to contract on $C_1$ and still $r_1 > r_2$ and $r_1 > \Delta$, we try to cover $C_2$ by $r_1 - 1$ pairwise disjoint paths. Before the contractions, $|C_2| \geq \frac{|C_1|}{4}$ did hold and by the contractions we could just decrease $|C_1|$, therefore it still holds.

All the maximal intervals on the chain $C_1$ (with possible exception of the first and the last one) have now length at least two, which implies that $r_1 \leq \frac{|C_1|}{4} + 1$. Hence $|C_2| \geq \frac{|C_1|}{4} \geq r_1 - 1$, so we can create $r_1 - 1$ pairwise disjoint hedgehogs covering $C_2$ using Lemma 7. Then we apply Lemma 6 and expand the good path obtained by Lemma 6 to a good path on the original double-chain.

There is a straightforward linear-time algorithm for finding a good path on $(C_1, C_2)$ based on the above proof. □

## 3 Unbalanced Double-Chains with No Good Path

In this section we prove Theorem 2. Let $(C_1, C_2)$ be a double-chain whose points are colored by an equitable 2-coloring, and let $C_1$ be periodic with the following period: 2

black, 4 white, 6 black and 4 white points. Let $|C_1| \geq 28(|C_2|+1)$. We want to show that $(C_1, C_2)$ has no good path.

Suppose on the contrary that $(C_1, C_2)$ has a good path. Let $P_1, P_2, \ldots, P_t$ denote the maximal subpaths of the good path containing only points of $C_1$. Since between every two consecutive paths $P_i$, $P_j$ in the good path there is at least one point of $C_2$, we have $t \leq |C_2|+1$. In the following we think of $C_1$ as of a cyclic sequence of points on the circle. Note that we get more intervals in this way. Theorem 2 now directly follows from the following theorem.

**Theorem 8.** *Let $C_1$ be a set of points on a circle periodically 2-colored with the following period of length 16: 2 black, 4 white, 6 black and 4 white points. Suppose that all points of $C_1$ are covered by a set of $t$ non-crossing alternating and pairwise disjoint paths $P_1, P_2, \ldots, P_t$. Then $t > |C_1|/28$.*

*Proof.* Each maximal interval spanned by a path $P_i$ on the circle is called a *base*. Let $b(P_i)$ denote the number of bases of $P_i$. A path with one base only is called a *leaf*. We consider the following special types of edges in the paths. *Long edges* connect points that belong to different bases. *Short edges* connect consecutive points on $C_1$. Note that short edges cannot be adjacent to each other. A maximal subpath of a path $P_i$ spanning two subintervals of two different bases and consisting of long edges only is called a *zig-zag*. A path is *separated* if all of its edges can be crossed by a line. Note that each zig-zag is a separated path. A maximal separated subpath of $P_i$ that contains an endpoint of $P_i$ and spans one interval only is a *rainbow*. We find all the zig-zags and rainbows in each $P_i$, $i = 1, 2, \ldots, t$. Note that two zig-zags, or a zig-zag and a rainbow, are either disjoint or share an endpoint. A *branch* is a maximal subpath of $P_i$ that spans two intervals and is induced by a union of zig-zags.

For each path $P_i$ that is not a leaf construct the following graph $G_i$. The vertices of $G_i$ are the bases of $P_i$. We add an edge between two vertices for each branch that connects the corresponding bases. If $G_i$ has a cycle (including the case of a "2-cycle"), then one of the corresponding branches consists of a single edge that lies on the convex hull of $P_i$. We delete such an edge from $P_i$ and don't call it a branch anymore. By deleting a corresponding edge from each cycle of $G_i$ we obtain a graph $G'_i$, which is a spanning tree of $G_i$. The *branch graph* $G'$ is a union of all graphs $G'_i$.

Let $\mathcal{L}$ denote the set of leaves and $\mathcal{B}$ the set of branches. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_t\}$.

**Observation 1.** *The branch graph $G'$ is a forest with components $G'_i$. Therefore,*

$$|\mathcal{B}| = \sum_{i, P_i \notin \mathcal{L}} (b(P_i) - 1).$$

The branches and rainbows in $P_i$ do not necessarily cover all the points of $P_i$. Each point that is not covered is adjacent to a deleted long edge and to a short edge that connects this point to a branch or a rainbow. It follows that between two consecutive branches (and between a rainbow and the nearest branch) there are at most two uncovered points, that are endpoints of a common deleted edge. By an easy case analysis it can be shown that this upper bound can be achieved only if one of the nearest branches consists of a single zig-zag.

In the rest of the paper, a *run* will be a maximal monochromatic interval of any color. In the following we will count the runs that are spanned by the paths $P_i$. The *weight* of a path $P$, $w(P)$, is the number of runs spanned by $P$. If $P$ spans a whole run, it adds one unit to $w(P)$. If $P$ partially spans a run, it adds half a unit to $w(P)$.

**Observation 2.** *The weight of a zig-zag or a rainbow is at most* $1.5$. *A branch consists of at most two zig-zags, hence it weights at most three units.*

**Lemma 9.** *A path* $P_i$ *that is not a leaf weights at most* $3.5k + 3.5$ *units where* $k$ *is the number of branches in* $P_i$.

*Proof.* According to the above discussion, for each pair of uncovered points that are adjacent on $P_i$ we can join one of them to the adjacent branch consisting of a single zig-zag. To each such branch we join at most two uncovered points, hence its weight increases by at most one unit to at most 2.5 units. The number of the remaining uncovered points is at most $k+1$. Therefore, $w(P_i) \leq 3k+3+0.5\cdot(k+1) = 3.5k+3.5$. □

**Lemma 10.** *A leaf weights at most* $3.5$ *units.*

*Proof.* Let $L$ be a leaf spanning at least two points. Consider the interval spanned by $L$. Cut this interval out of $C_1$ and glue its endpoints together to form a circle. Take a line $l$ that crosses the first and the last edge of $L$. Note that the line $l$ doesn't separate any of the runs. Exactly one of the arcs determined by $l$ contains the gluing point $\gamma$.

Each of the ending edges of $L$ belongs to a rainbow, all of whose edges cross $l$. It follows that if $L$ has only one rainbow, then this rainbow covers the whole leaf $L$ and $w(L) \leq 1.5$. Otherwise $L$ has exactly two rainbows, $R_1$ and $R_2$. We show that $R_1$ and $R_2$ cover all edges of $L$ that cross the line $l$. Suppose there is an edge $s$ in $L$ that crosses $l$ and does not belong to any of the rainbows $R_1$, $R_2$. Then one of these rainbows, say $R_1$, is separated from $\gamma$ by $s$. Then the edge of $L$ that is the second nearest to $R_1$ also has the same property as the edge $s$. This would imply that $R_1$ spans two whole runs, a contradiction. It follows that all the edges of $L$ that are not covered by the rainbows are consecutive and connect adjacent points on the circle. There are at most three such edges; at most one connecting the points adjacent to $\gamma$, the rest of them being short on $C_1$. But this upper bound of three cannot be achieved since it would force both rainbows to span two whole runs. Therefore, there are at most two edges and hence at most one point in $L$ uncovered by the rainbows. The lemma follows. □

**Lemma 11.** $|\mathcal{L}| \geq \sum_{i,P_i \notin \mathcal{L}} (b(P_i) - 2) + 2$.

*Proof.* The number of runs in $C_1$ is at least 4. By Lemma 10, if all the paths $P_i$ are leaves, then at least 2 of them are needed to cover $C_1$ and the lemma follows.

If not all the paths are leaves, we order the paths so that all the leaves come at the end of the ordering. The path $P_1$ spans $b(P_1)$ bases. Shrink these bases to points. These points divide the circle into $b(P_1)$ arcs each of which contains at least one leaf. If $P_2$ is not a leaf then continue. The path $P_2$ spans $b(P_2)$ intervals on one of the previous arcs. Shrink them to points. These points divide the arc into $b(P_2) + 1$ subarcs. At least $b(P_2) - 1$ of them contain leaves. This increased the number of leaves by at least $b(P_2) - 2$. The case of $P_i$, $i > 2$, is similar to $P_2$. The lemma follows by induction. □

**Corollary 12.** $|\mathcal{B}| \leq |\mathcal{P}| - 2$.

*Proof.* Combining Lemma 11 and Observation 1 we get the following:

$$|\mathcal{B}| = \sum_{i, P_i \notin \mathcal{L}} (b(P_i) - 1) = \sum_{i, P_i \notin \mathcal{L}} (b(P_i) - 2) + |\mathcal{P}| - |\mathcal{L}| + 2 - 2 \leq |\mathcal{P}| - 2.$$

□

Now we are in position to finish the proof of Theorem 8. If the whole $C_1$ is covered by the paths $P_i$, then $\sum_{i=1}^{t} w(P_i) \geq \frac{|C_1|}{4}$. Therefore,

$$|C_1| \leq 4 \cdot (3.5|\mathcal{B}| + 3.5(|\mathcal{P}| - |\mathcal{L}|) + 3.5|\mathcal{L}|) < 4 \cdot 7|\mathcal{P}| = 28|\mathcal{P}|.$$

□

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## Appendix: Proof in the Odd Case

In this appendix we prove Theorem 1 for the case when $|C_1| + |C_2|$ is odd. We set $\Delta = w_2 - b_2$ and proceed similarly as in the even case. On several places in the proof we will add one auxiliary point $\omega$ to get the even case (its color will be chosen to equalize the numbers of black and white points). We will be able to apply one of the

Lemmas 4–6 to obtain a good path. The point $\omega$ will be at some end of the good path and by removing $\omega$ we obtain a good path for $(C_1, C_2)$.

Without loss of generality we may assume that $r_1 \geq r_2$. In the case $r_1 = r_2$, we add an auxiliary major point $\omega$, which is placed either as the left neighbor of the leftmost major point on $C_1$ or as the right neighbor of the rightmost major point on $C_2$. Then we get a good path by Lemma 5 and the removal of $\omega$ gives us a good path for $(C_1, C_2)$.

In the case $\Delta \geq r_1$, we add an auxiliary point $\omega$ to the same place and we get a good path by Lemma 4. Again, the removal of $\omega$ gives us a good path for $(C_1, C_2)$.

Now, the only case left is $r_1 > r_2$, $r_1 > \Delta$. If there are any singletons on $C_1$, we make the contractions exactly the same way as in the proof of the even case. If Lemma 4 or 5 needs to be applied, we again add an auxiliary point $\omega$ and proceed as above.

If there is no more singleton to contract on $C_1$ and still $r_1 > r_2$ and $r_1 > \Delta$, we have $|C_2| \geq \frac{|C_1|}{4} \geq r_1 - 1$ as in the proof of the even case and we can use Lemma 7 to get $r_1 - 1$ pairwise disjoint hedgehogs covering $C_2$. Now we need to consider two cases: (1) If $b_1 + b_2 > w_1 + w_2$, then we find a good path for $(C_1, C_2)$ in the same way as in the proof of Lemma 6, except we do not add the auxiliary point $\sigma$. (2) If $b_1 + b_2 < w_1 + w_2$, we add an auxiliary point $\omega$ as the right neighbor of the rightmost major point on $C_1$. The number $r_1$ didn't change so Lemma 6 gives us a good path. Again, the removal of $\omega$ gives us a good path for $(C_1, C_2)$.

There is a straightforward linear-time algorithm for finding a good path on $(C_1, C_2)$ based on the above proof. □