Self-similar Discrete Rotation Configurations and Interlaced Sturmian Words

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Abstract. Rotation configurations for quadratic angles exhibit self-similar dynamics. Visually, it may be considered as quite evident. However, no additional details have yet been published on the exact nature of the arithmetical reasons that support this fact. In this paper, to support the existence of self-similar dynamic in 2d-configuration, we will use the constructive 1-d substitution theory in order to iteratively build quadratic rotation configurations from substitutive Sturmian words. More specifically : the self-similar rotation configurations are first shown to be an interlacing of configurations that are direct product of superposition of Sturmian words.

1 Introduction

Throughout the history of the Digital Geometry, the analysis of the characteristics of discrete lines, discrete planes, discrete spheres, intersections and distances in digital spaces had been a major issue of the field. And, this is generally true for the digital counterparts of all the fundamental objects from Euclidean Geometry Within these studies, regular discrete patterns have occurred quite often. Some of them can be shown to be quasi-periodic, however, the self-similar characteristics that these patterns may have , are generally less obvious.

For instance, the characteristics of the dragon-shape tiles that appear as equivalence class under quasi-affine transform has remained relatively difficult to explain with constructive discrete tools (See [NR95], [LKV04]). Nowadays, through some constructive theories, including continued fraction theory and generalized Sturmian sequences and their generalizations, it seems that we get closer to a wider understanding of connection in-between real dynamics and substitutions occurring in symbolic dynamical systems. Similarly, using above theories, any Sturmian word with specified slope and intercept can be constructed by the mean of S-adic systems. In this paper, we add another example of applications of these theories. We explain how to connect these results to prove the existence of substitutions that underlies the dynamics in discrete rotation configurations.

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The main question that this paper deals with is to test whether knowledge that we have one dimensional self-similar dynamics is useful or not to analyze and generate 2d-self-similar configurations such as rotation configurations ?

This paper provides the details for constructing rotation configurations of quadratic angles from Sturmian words. So far, there was no English publication that really provides the evidences for the self-similar aspects of this process.

At first, we will review the necessary framework : basic definitions, rotation configurations, their properties, constructive results about Sturmian words.Then we will focus on self-similar configurations. We shall empirically notice the self-similarity of the configurations. Then we will considerate some of the properties of quadratic angles. This will provide a useful decomposition of the configurations. This will allow us to emphasize the validity of the approach and to show that the computed substitutions output the correct words.

2 Vocabulary and Notations

We use $\lfloor x \rfloor$ to denote the usual *floor function* (The biggest integer that is equal or smaller to x). The *rounding to the closest integer* point is then defined by $[x] = \lfloor x + \frac{1}{2} \rfloor$. These functions from $\mathbb{R}$ to $\mathbb{Z}$ extend to higher dimensions by independent applications on each component. Let $\{x\} = x - [x]$, $x \mapsto \{x\}$ will be considered as a *canonical projection* to the interval $[-\frac{1}{2}, \frac{1}{2}[$, which is also a representant of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. However, for arithmetical issues, we may tend to prefer projection to the interval $[0, 1[$, another representant of $\mathbb{T}$. We will then use : $\wr x \int = x - \lfloor x \rfloor$.

Generally, we will also work in the complex plane $\mathbb{C}$. For any point $z \in \mathbb{C}$, its real part is denoted $\Re(z)$, and its imaginary parts is denoted $\Im(z)$. They are reals such that $z = \Re(z) + \Im(z)i$. The set of points whose real and imaginary parts are both integers form the set of Gaussian integers, $\mathbb{Z}[i]$.

Let α denote an angle in radians, i.e. an element of $\mathcal{A} = \mathbb{R}/(2\pi\mathbb{Z})$. The *Euclidean rotation* r_α is the one-to-one isometry of $\mathbb{C}$, $z \mapsto ze^{i\alpha}$. *The discretized rotation* $[r_\alpha]$ is the successive computation of the Euclidean rotation of angle α and of the discretization operator $z \mapsto [z]$.

Let Q be a finite set whose elements will be called *letters*. Q is called the *Alphabet*. Any finite sequence of elements of Q is called a *word* on Q . Bi-infinite sequence of letters are *bi-infinite words* ($\mathbb{Z} \to Q$). If all the letters of a word w_f occur consecutively in a word w then w_f is a *factor* of w.

3 Rotation Configurations (C_α)

3.1 Definition

A *configuration* is an application C that maps each Gaussian integer z of $\mathbb{Z}[i]$ to an element $C(z)$ of a finite set Q.

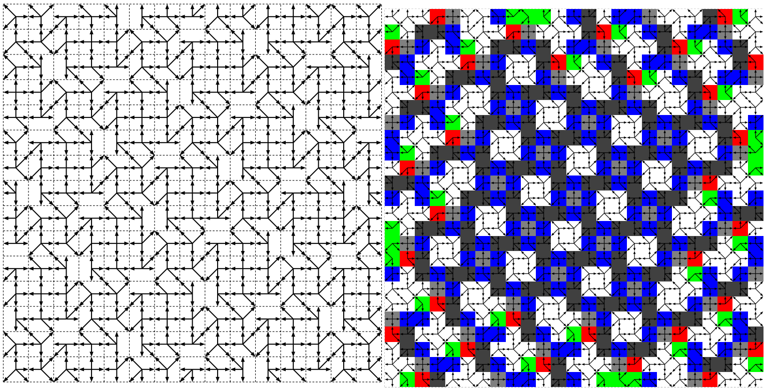


Fig. 1. The configuration $C_{0.4}$. In further drawing, projections of the elements of $2^{\mathbb{A}}$ to arbitrary color-set, in order to observe the dynamic at a broader scale.

For a specified angle α, we are interested in the *rotation configurations* C_α, which is the application that*to any Gaussian integer z associates the relative position of its* 4*-neighbors after rotation.* The set of states Q used in this paper is the set of subsets of the set of arrows $\mathbb{A} = \{-1, 0, 1\}[i]$, $Q = 2^{\mathbb{A}}$. Formally:

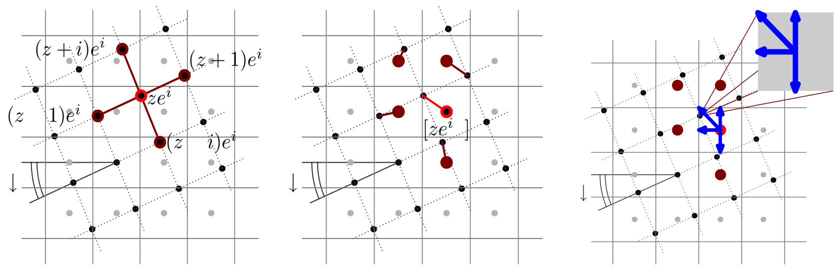
$$C_\alpha : \mathbb{Z}[i] \mapsto 2^{\mathbb{A}} : z \to \bigcup_{\delta \in \{1,-1,i,-i\}} \left\{ [e^{i\alpha}(z+\delta)] - [e^{i\alpha} z] \right\}$$

The details of the construction are illustrated on Figure 2, and can be stated as follows : Figure 2(a), within the Gaussian integers of $\mathbb{Z}[i]$ displayed in gray, a point z has been evinced. Its image under rotation $e^{i\alpha}\mathbb{Z}[i]$ is highlighted. The different discretization cells are separated by thin black dotted lines. These line provide a representation of the frontier in-between the equivalence class for the discretization operator. The Figure 2(b) z, the point that is being considered, and its four 4-neighbors; there we compute their image under discretized rotation. Figure 2(c), we have displayed $C_\alpha(z)$, the code will associate to z. This is the set of arrows that link the image of z with the image of its 4-neighbors. When this code is actually computed for all $z \in \mathbb{Z}[i]$ this leads to figures such as Figure 1.

3.2 General Properties

We can refer to [No06] to ensure that :

- The rotation configurations encode efficiently the discretized rotation $[r_\alpha]$;
- The configurations are bi-periodic if and only if α is such that $cos(\alpha)$ and $\sin(\alpha)$ are rational;
- For any angle α, the resulting configuration C_α is quasi-periodic;



(a) Step 1 (b) Step 2 (c) Step 3

Fig. 2. Step-by-step construction of the C_α configuration as explained in main text. The resulting symbol is designed using a arrows, it directly represents the code associated to z.

– The configuration may be reinterpreted as the coding of a $\mathbb{Z}^2$-action on a labeled torus $(\mathbb{R}/\mathbb{Z})^2$. Formally :

$$\exists l : \mathbb{T}[i] \rightarrow \mathbb{A}, T_x, T_y \in \mathbb{T}[i] \rightarrow \mathbb{T}[i] | \exists T_x^{-1}, T_y^{-1} \wedge \forall \mathbf{v} \in \mathbb{Z}^2, C(\mathbf{v}) = l(T_x^{v_x}(T_y^{v_y}(\mathbf{v})))$$

For a specified angle of rotation α, the following system is precisely known and it has been described in [No06]:

– $T_x : z \mapsto z + e^{i\alpha}$ and $T_y : z \mapsto z + i\, e^{i\alpha}$;
– $l : z \mapsto Q$ is such that there exists a morphism $\phi : Q \rightarrow Q$, for any $z \in \mathbb{T}[i], l(z) = \phi(l(iz))$; moreover $l' : \mathbb{T} \rightarrow Q$ is such that there exists an application $\psi : Q^2 \rightarrow Q$ such that for any $z \in \mathbb{T}[i] l(z) = \psi(l'(\Re(z)), l'(\Im(z)))$.

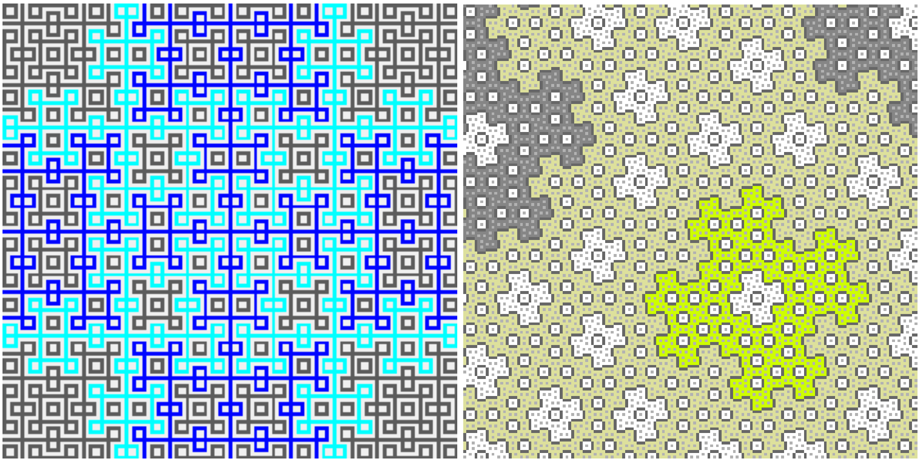
As a consequence of this we retain that C is defined by

$$C(z) = \psi(l'(\Re(\{ze^{i\alpha}\})), l'(\Im(\{ze^{i\alpha}\}))) \tag{1}$$

3.3 Self-similarity

Any angle such that $\cos(\alpha)$ and $\sin(\alpha)$ belong to the same quadratic field is called a *quadratic* angle. Note that if $\cos(\alpha)$ and $\sin(\alpha)$ belong to the same quadratic field then necessarily there exists $p, q, k \in \mathbb{Z}\ p\cos(\alpha) + q\sin(\alpha) = k$.

In this paper, we aim at finding a way to explain the self-similarity of the rotation configurations for quadratic angles. Hence, we have first to conjecture that rotation configurations may be self-similar. In order to get convinced, we show two simple sample configurations for which it is quite easy to notice that the resulting configuration exhibits some self-similar patterns, namely $C_{\pi/4}$ and $C_{\pi/6}$. Please have a look at Figure 3.



(a) $C_{\pi/4}$ (b) $C_{\pi/6}$

Fig. 3. Sample auto-similar configurations. The configurations have been artificially recolored using bucket-fill tool under "the gimp" in order to make more visible the self-similar patterns.

4 Fundamentals for Constructive Sturmian Theory

4.1 Sturmian Words and Rotation Words

According to Morse-Hedlund Theorem [MH38], for any bi-infinite word w, if there is a positive integer n such that the number of factor of size n is smaller than n then w is periodic. If we define the *complexity function* of word w as the function that to $n \in \mathbb{N}$ associates the number of factors of size n, then the contraposition of the theorem is that any aperiodic bi-infinite word is aperiodic has necessarily a complexity function $p(n)$ that is such that $p(n) > n$ for any $n \in \mathbb{N}$. The class of words such that $p(n) = n+1$ for any n, is the class of *Sturmian words*. It widely known that Sturmian words code the trajectory of straight lines of irrational slopes on the discrete grid ([Fo02]). Hence any Sturmian word $\overset{\Rsh}{\mathbf{w}}$ (α, β) is defined by a slope $\alpha\mathbb{R}\backslash\mathbb{Q}$ and by an intercept $\beta \in \mathbb{R}$ and an orientation, in this paper[1]; we set up : $\overset{\Rsh}{\mathbf{w}}\ (\alpha, \beta) = \mathbb{1}_{[1-\alpha,1[}(\lbag n\alpha + \beta \rbag)$. More generally, a rotation word $\overset{\circlearrowleft}{\mathbf{w}}\ (\alpha, \beta, \gamma)$ will be defined here by : $\overset{\circlearrowleft}{\mathbf{w}}\ (\alpha, \beta, \gamma) = \mathbb{1}_{[1-\gamma,1[}(\lbag n\alpha + \beta \rbag)$. Regarding the complexity of rotation words, an extensive study is available in [Al96].

4.2 Constructive Results on Sturmian Words

Due to limited space, we discuss only the essential issues, for real introductions to Sturmian theory, please consult [Fo02], and for more constructive results : [BHZ], [Ad02].

[1] The other case would be : $\overset{\Rsh}{\mathbf{w}}'\ (\alpha, \beta) = \mathbb{1}_{]1-\alpha,1]}(\lbag n\alpha + \beta \rbag)$.

According to a Theorem of Berstel-Séébold, the set of Sturmian morphisms[2] has a monoid structure. Let $\sigma_{0a}, \sigma_{0b}, \sigma_{1a}, \sigma_{1b}$ are four applications from $\{0,1\}$ to $\{0,1\}^*$ that we are going to extend as monoid morphism. It can be shown that they can serve as generators for the Berstel-Séébold monoid :

$$\begin{cases} \sigma_{0a}(0) \to \ \ 0 \\ \sigma_{0a}(1) \to 10 \end{cases} \begin{cases} \sigma_{0b}(0) \to \ \ 0 \\ \sigma_{0b}(1) \to 01 \end{cases} \begin{cases} \sigma_{1a}(0) \to 01 \\ \sigma_{1a}(1) \to \ \ 1 \end{cases} \begin{cases} \sigma_{1b}(0) \to 10 \\ \sigma_{1b}(1) \to \ \ 1 \end{cases}$$

Let $0 < \alpha < 1$ be an irrational real, according to Lagrange theorem, it has an *infinite continued fraction expansion* : $[0; a_1, a_2, \ldots, a_i, \ldots]$, which denotes that $\alpha = \frac{1}{\alpha_1 + \frac{1}{\alpha_2 + \ldots}}$. If α is quadratic then sequence of the partial quotients is ultimately periodic. The rational numbers obtained by truncating the expansion of α after a certain partial quotient are called the *convergents*.

Also the error in-between the convergent and the value of α form an important quantity; we will denote it by ϵ_i. This value is major for understanding odometers-like dynamical system (such as calendars), and also the 3-distances Theorem of V.T. Sós.

As presented in Berthé [Be02], let α be an irrational real, the *α-Ostrowski expansion* (c_i) of a real number β is :

$$\beta = \sum c_{i+1}\epsilon_i, \text{ with } \forall i \in \mathbb{N}, c_i \in \mathbb{N}, \text{, and } c_i \text{ relying in} - \text{between 0 and } a_i$$

Theorem 1 (Valérie Berthé,Charles Holton,Luca Q. Zamboni, [BHZ]). [3] *Any Sturmian sequence S-adic representation of the form :*

$$\overset{\Rsh}{\boldsymbol{w}}(\alpha, \beta) = (T^{c_1} \circ \sigma_{0b}^{a_1}) \circ (T^{c_2} \circ \sigma_{1b}^{a_2}) \circ (T^{c_3} \circ \sigma_{0b}^{a_3}) \circ (T^{c_4} \circ \sigma_{1b}^{a_4}) \ldots \qquad (2)$$

*where T is the usual shift on sequences, where $\{a_i\}_{i\in\mathbb{N}}$ correspond to partial quotients of $\alpha' = \frac{\alpha}{1+\alpha}$, and the sequence $\{c_i\}_{i\in\mathbb{N}}$ matches the α'-*Ostrowski-*expansion of the intercept $\beta' = \frac{\beta}{1+\alpha}$.*

We should then remark that, for any Sturmian word w, one has $T^{c_i} \circ \sigma_{0b}^{a_i}(w)$ (resp. $T^{c_i} \circ \sigma_{1b}^{a_i}(w)$) coincide with $\sigma_{0a}^{c_i} \circ \sigma_{0b}^{a_i - c_i}(w)$ (resp. $\sigma_{1a}^{c_i} \circ \sigma_{1b}^{a_i - c_i}(w)$) on the first $|\sigma_{0b}^{a_i}(w)| - c_1$ letters. Thus Equation 2 may be rewritten as :

$$\overset{\Rsh}{\mathbf{w}}(\alpha, \beta) = (\sigma_{0b}^{c_1} \circ \sigma_{0a}^{a_1 - c_1}) \circ (\sigma_{1b}^{c_2} \circ \sigma_{1a}^{a_2 - c_2}) \circ (\sigma_{0b}^{c_3} \circ \sigma_{0a}^{a_3 - c_3}) \circ (\sigma_{1b}^{c_4} \circ \sigma_{1a}^{a_4 - c_4}) \ldots \quad (3)$$

Referring to [BHZ], we know that a Sturmian sequence of slope α of intercept x is a if and if only if α is a quadratic irrational number and $x \in \mathbb{Q}(\alpha)$.

Also, one should note an important corollary of previous remarks and theorems. Any Sturmian word whose slope is quadratic and whose intercept is in the

[2] Endomorphism on Sturmian sequences.

[3] The fix-up of α, and β into α' and β' through $x \mapsto (x)/1 + \alpha$ is a classical operation. It corresponds to the fact that the system we used for defining Sturmian words is different from traditional odometers.

quadratic extension of the slope is substitutive. It is the image under morphism of a fixed point of substitution. Hence it exhibits some self-similar characteristics.

Generalization are possibles to rotation words: In [Ad02], we may find an interesting constructive results on S-adic systems by the mean of generalized continued fraction like expansions. Using this algorithm, we can find the S-adic expansion associated to a rotation word.

5 Grid-Configurations and Sturmian Words

From now, we shall continuously assume that α is a quadratic angle. The interlacing vector z_i associated with a quadratic angle α is a non null Gaussian integer $z_I = p + qi$ such that $z_I e^{i\alpha}$ has an integer component. The notation does not mention the angle α, the link is however existent and implicit.

5.1 Grid Configurations

Let $z_M \in \mathbb{Z}[i]/(z_I\mathbb{Z}[i])$, in order to analyze the configurations C_α, we introduce the *grid configurations* defined by $D_{\alpha,z_M}(z) = C_\alpha(z_I z + z_M)$.

By application of Equation 1, we immediately have

$$D_{\alpha,z_M}(z) = \psi(l'(\Re(\{(zz_I + z_M)e^{i\alpha}\})), l'(\Im(\{(zz_I + z_M)e^{i\alpha}\}))) \tag{4}$$

Since $z_I e^{i\alpha}$ has one integer coordinate, these configurations will have some useful properties in further discussions. Practically, a configuration D_{α,z_M} contains points regularly extracted from C_α. They extracted points form the set : $z_I\mathbb{Z}[i] + z_M$. Such networks are represented within different symbols on rotated grids in the Figures 6 and 5. Conversely, from D_{α,z_M}, we may redefine $C_\alpha(z) = D_{\alpha,z_M}(z_R)$; with $z_M \in D(\mathbb{Z}[i]/(z_I\mathbb{Z}[i]))$ and $z_R \in \mathbb{Z}[i]$ such that $z = z_I z_R + z_M$, where $D(\mathbb{Z}[i]/(z_I\mathbb{Z}[i]))$ denotes the canonical projection domain of $\mathbb{Z}[i]/(z_I\mathbb{Z}[i])$ on $\mathbb{Z}[i]$.

For a specified α, the number of configurations D_{α,z_M} can be seen as $N = \#(\mathbb{Z}[i]/(z_I\mathbb{Z}[i]))$; recall that $z_I = p + qi$, then $N = p^2 + q^2$, it is the number of integer point in a square[4] of side z_i.

5.2 Grid Configurations from Rotation Words

Since zz_I has an integer component then, from Equation 4, we can observe that the bi-dimensional configuration $l'(\{\Re(zz_I + z_M)\})$ is actually completely defined one by one infinite word. Either all lines or columns are constant. More precisely, we have, either for all $z \in \mathbb{Z}[i]$, $\{\Re(1z_I e^{i\alpha})\} = 0$ or for all $z \in \mathbb{Z}[i]$, $\{\Im(iz_I e^{i\alpha})\} = 0$. The same remark holds also for the imaginary part and the configuration $l'(\{\Im(zz_I + z_M)\})$.

[4] Square with half opened borders.

Let's assume without any loss of generality, since the same argumentation hold in the other case, that for all $z \in \mathbb{Z}[i]$ we have : $\wr\Re(z_I e^{i\alpha})\int = 0$

We can now define the word : $w_v(n) = l'(\{\Re((niz_I + z_M)e^{i\alpha})\})$. This one can be rewritten in : $w_v(n) = l'(\{(\Re(z_m) - nq)\cos(\alpha) - (\Im(z_m) + pn)(\sin(\alpha))\})$ Hence, we have

$$w_v(n) = l'(\{i_0 - n\alpha_0\})$$

with $i_0 = (\Re(z_m)\cos(\alpha) - \Im(z_m)\sin(\alpha))$ and $\alpha_0 = (q\cos(\alpha) + pn(\sin(\alpha)))$. Since l is a function to a finite set whose antecedents are forms connected sets of $\mathbb{T}$, we can rewrite l in such way :

$$l'(x) = \sum_{i=0}^{N-1} i\mathbb{1}_{[k(b(i)),k_{d(i)}[}(x)$$

Where N is the number of areas that are necessary to define l'. S, the set of cuts, is then formally defined by:

$$S = \{\overline{l^{\{-1\}}(i)} \cup \overline{l^{\{-1\}}(j)}, (i,j) \in \{0,\ldots,N-1\}^2\} = \{k(i), i \in \{0,\ldots,N-1\}\}$$

Hence k is a function from $\{0,\ldots,N-1\}$ to that associated to an integer the i^{th} cut of the partition that underlies the l function.

We can now write

$$w_v(n) = \sum_{i=0}^{N-1} i \overset{\circlearrowleft}{\mathbf{w}} (\alpha_0, i_0 - k(b(i)), k(d(i)) - k(b(i)))(n)$$

5.3 From Rotation Words to Sturmian Words

All the cuts are actually copies under translation by $ze^{i\alpha}$ of the discontinuity of the discretization operator located at $\frac{1}{2}$, for some z Gaussian integer z. Therefore, all the cuts belong to the quadratic field as $\cos(\alpha)$ and $\sin(\alpha)$.

The conclusion is that actually all the rotation words expressed in previous section can be expressed as product of Sturmian words where the intercepts of the word belong to the quadratic to the same quadratic field.

Proposition 1. *Let α be a quadratic angle. All the configurations C_α can be constructed by superposing, adding, multiplying substitutive Sturmian words according to the scheme that has been explained in this section. Moreover, the sequence of morphism to generate this word from $(01)^\omega$ is precisely known.*

The proof of this proposition is a corollary of the previous argumentation.

5.4 Application to $C_\pi/4$

To be more concrete, we apply our construction to $C_{\pi/4}$.

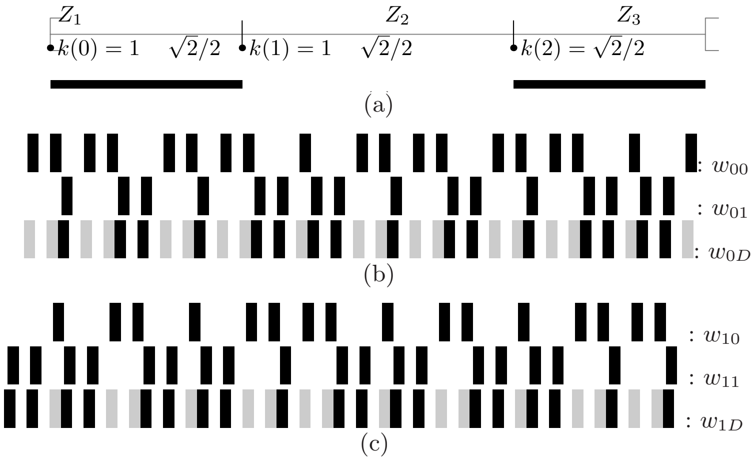


Fig. 4. Decomposition of the words that generate the configurations $D_{\pi/4,0}$,$D_{\pi/4,1}$.(a) : The basic partition function l' that is used for $C_{\pi/4}$ contains only 3 areas.(b) and (c) : The details of the construction of the words W_{1D} and W_{2D} from the words w_{00}, w_{01},w_{10},w_{11} which contains the odd letters of Sturmian words whose characteristics have already been specified.

First, we notice $\sin(\alpha) = \cos(\alpha) = \frac{\sqrt{2}}{2}$. The interlacing vector is $z_I = (1+i)$, we have $\Re((1+i)e^{i\alpha}) = 0$ and $z_i\bar{z}_i = 2$. Thus, two grid-configurations exist: $D_{\pi/4,0}$ and $D_{\pi/4,1}$. From previous argumentation, we have that we can define these configurations from two words : $D_{\pi/4,0} = \psi(w_{0D}(\Im(z)), w_{0D}(\Re(z)))$, (resp. $D_{\pi/4,1} = \psi(w_{1D}(\Im(z)), w_{1D}(\Re(z))))$. It is easy to notice that in this case horizontal and vertical words are the same. We thus have two construct the following words : $w_{1D}(n) = l'(\wr n\sqrt{2} + \frac{\sqrt{2}}{2}\int)$ and $w_{0D}(n) = l'(\wr n\sqrt{2}\int)$.

We choose to compute l', the morphism issued from the characterization of the zones, By using the characteristic function of the zones $Z_1 \cup Z_2$ and $Z_2 \cup Z_3$, (referring to the notations of Figure 4(a)), because they have the appropriate size to make Sturmian words. We obtain that : $w_{0D}(n) = \psi(w_{00}(n), w_{01}(n))$ and $w_{1D}(n) = \psi(w_{10}(n), w_{11}(n))$ where: $w_{01}(n) = \vec{\mathbf{w}}\,(\frac{\sqrt{2}}{2}, \frac{1}{2})(2n)$, $w_{00}(n) = w_{11}(n) = \vec{\mathbf{w}}\,(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} - \frac{1}{2})(2n)$, $w_{10}(n) = \vec{\mathbf{w}}\,(\frac{\sqrt{2}}{2}, \sqrt{2} - \frac{1}{2})(2n)$.

Proposition 2. *The following words may be used to generate $C_\pi/4$, according to the scheme explained :*
$w_{01} = \sigma_{1b}^2(w_B)$, $w_{00} = w_{11} = \sigma_{1a} \circ \sigma_{1b}(w_B)$, $w_{10} = \sigma_{1a}^2(w_B)$, *with*

$$w_B = (\sigma_{0a} \circ \sigma_{0b} \circ \sigma_{1a} \circ \sigma_{1b})^\omega(0)$$

The remaining numerical details that concludes the proof are left to the reader.

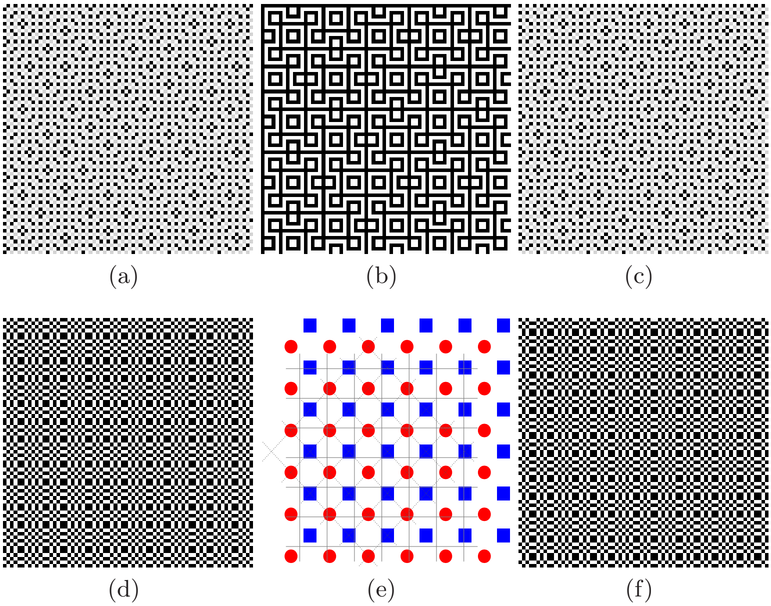


Fig. 5. In (b), the configuration $C_{\pi/4}$. $D_{\pi/4,0}$ and $D_{\pi/4,1}$ on their rotated network have been displayed as (a) and (c) (i.e. has they are inserted in (b)). In (d) and (f), we show the same grid-configurations $D_{\pi/4,0}$ and $D_{\pi/4,1}$ but normally, on an unrotated networks. The sub-figure (e) explains the way the two configurations are interleaved for $C_{\pi/4}$.

6 Perspectives

In this article, we have shown that any quadratic rotation configuration can be reinterpreted as a superposition and an interlacing Sturmian words.

In further work, the question of compatibility of the grid-configuration that are interlaced is to be addressed. Can the substitutions issued from each grid can be rearranged to form a connected rectangular substitutions on the grid ? Also, one may explore the exact limits of the generating power of substitutions as planar substitutive patterns on rotated and interlaced grids may be explored. This generalization of the studied configurations would includes the whole class of configurations that may be obtained by interweaving different grid configurations with different angles. So far, in our experiments, we have noticed that a wide diversity of patterns may appear. The methods that may be used to guess from a configuration whether it belongs to this class are also to be investigated. It may not be possible for all configurations, but it would be nice to know for which configurations this is possible The Rauzy graph, as a tool for the analysis of the topology induced by the patterns of the configurations, seems to be a promising solution, see [No06].

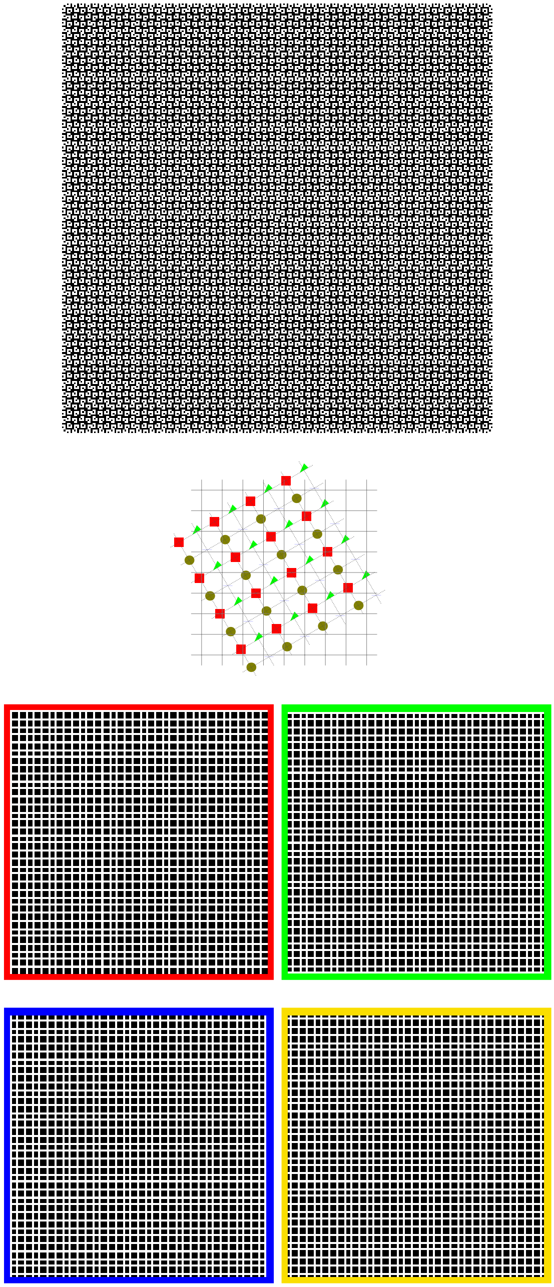


Fig. 6. The largest figure represents the configuration $C_{\frac{\pi}{6}(-)}$, with a very limited color-set. The four underlying figure represents the grid-configurations $\{D_{\frac{\pi}{6},b}\}_{b\in\{0,1,i,1+i\}}$, here $z_I = 2$. In between, the central figures recall visually the way the interlacing of the grid-configurations is done for $C_{\pi/6}$.

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