Analytical Design of Arbitrary Oriented Asteroidal 2-D FIR Filters

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Abstract. Novel approach to the analytical design of asteroidally shaped two-dimensional FIR filters with arbitrary orientation of the frequency response in the frequency plane is presented. The design consists of two steps. In the first step, the asteroidal 2-D FIR filter in the basic position along the frequency coordinates is designed. The design is based on the analytical contour approximation using the parametric representation of the contour. Closed form formulas for the approximation of the contour with asteroidal shape were derived. In the second step, the asteroidally shaped FIR filter is rotated by the desired angle in the frequency plane. Closed form formulas for the calculation of the impulse response of the filter are presented. One example demonstrates both steps of the design procedure.

1 Introduction

Two-dimensional FIR filters with asteroidal contour of the frequency response represent advantageous choice in the processing of rectangular objects in the image area. This is based on the fact, that spectra of rectangular objects exhibit asteroidal shape. Because of the arbitrary orientation of the objects in the image area, filters with general orientation of the frequency response are desired. The image filtering can be accomplished by both the nonlinear [5], [6], [7], [8], [9] and linear [1], [2], [10] filters. In the design of 2-D FIR filters the McClellan transformation technique [1] became popular. Methods for the design of 2-D FIR filters with circular, elliptical, fan and diamond contour based on the McClellan transformation are available, see e.g. [2]-[4]. Here we present novel analytical method for the design of the coefficients of the McClellan transformation for 2-D FIR filters with asteroidal contour. Using the proposed method other shapes of the contour of the 2-D FIR filter are achievable, too. These are for example

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ellipses, circles, epicycloids, hypocycloids and roses. The consecutive rotation allows to orient the frequency response of the filter in the frequency plane as specified.

2 Transformation Method

The McClellan transformation design technique [1] transforms the 1-D frequency response

$$H(e^{j\omega}) = \sum_{n=0}^{N} a(n)cos(n\omega) = \sum_{n=0}^{N} a(n)T_n\left[cos(\omega)\right] \tag{1}$$

using the transformation function $cos(\omega) = F(e^{j\omega_1}, e^{j\omega_2})$ into the 2-D frequency response

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n=0}^{N} a(n)T_n\left[F(e^{j\omega_1}, e^{j\omega_2})\right] = \sum_{p=0}^{PN}\sum_{q=0}^{QN} b(p,q)cos(p\omega_1)cos(q\omega_2) \tag{2}$$

where $T_n(x)$ are Chebyshev polynomials of the first kind. In the transformation function

$$cos(\omega) = F(e^{j\omega_1}, e^{j\omega_2}) = \sum_{p=0}^{P}\sum_{q=0}^{Q} f(p,q)cos(p\omega_1)cos(q\omega_2) \tag{3}$$

the coefficients $f(p,q)$ are called coefficients of the McClellan transformation and the integers P, Q represent the order of the McClellan transformation. For constant transformation function

$$cos(\omega_0) = F(e^{j\omega_1}, e^{j\omega_2}) = const \tag{4}$$

the relation between the frequencies ω_1 and ω_2

$$\omega_2 = g(\omega_1) \tag{5}$$

defines the contour of the transformation function. The transformation function $F(e^{j\omega_1}, e^{j\omega_2})$ affects the shape of the 2-D frequency response $H(e^{j\omega_1}, e^{j\omega_2})$, whereas the 1-D frequency response $H(e^{j\omega})$ determines the selectivity of the 2-D filter. The central part in the design of 2-D FIR filter using McClellan transformation represents the determination of the coefficients $f(p,q)$.

3 Contour Approximation

The contour approximation consists in the design of the coefficients $f(p,q)$ of the McClellan transformation. The contour approximation is solved by the minimization of the error

$$E(\omega_1, \omega_2) = cos(\omega_0) - F(e^{j\omega_1}, e^{j\omega_2}) \tag{6}$$

with respect to the desired shape of the contour (5). The minimization of (6) is accomplished using least square method

$$\int_{\mathcal{D}} \frac{\partial E^2(\omega_1, g(\omega_1))}{\partial f(p,q)} d\omega_1 \tag{7}$$

where $\mathcal{D}$ is the region of definition of the desired contour (5). The minimization (7) is usually solved numerically. Instead of the explicit relation (5), we propose the parametric representation of the contour $\omega_1 = g_1(\varphi)$, $\omega_2 = g_2(\varphi)$ or more specifically

$$\omega_1 = g_1(cos\varphi, sin\varphi)\,,\ \omega_2 = g_2(cos\varphi, sin\varphi)\ . \tag{8}$$

Due to the parametric representation of the contour (8), the numerical solution of (7) can be replaced by the analytical solution of the error

$$\int_0^{k\pi} \frac{\partial E^2(g_1(\varphi), g_2(\varphi))}{\partial f(p,q)} d\varphi\ . \tag{9}$$

For the second-order McClellan transformation ($P = Q = 2$), we have to determine nine coefficients $f(p,q)$. Defining the error function in the form

$$\begin{aligned} E(\omega_1, \omega_2) = {} & A_0 + A_1 f(1,0) + A_2 f(0,1) \\ & + A_3 f(1,1) + A_4 f(2,0) + A_5 f(0,2) \\ & + A_6 f(2,1) + A_7 f(1,2) + A_8 f(2,2) \end{aligned} \tag{10}$$

where

$$\begin{aligned} & A_0 = cos(\omega_0) - s && A_1 = 1 - cos(\omega_1) \\ & A_2 = 1 - cos(\omega_2) && A_3 = 1 - cos(\omega_1)cos(\omega_2) \\ & A_4 = 1 - cos(2\omega_1) && A_5 = 1 - cos(2\omega_2) \\ & A_6 = 1 - cos(2\omega_1)cos(\omega_2) && A_7 = 1 - cos(\omega_1)cos(2\omega_2) \\ & A_8 = 1 - cos(2\omega_1)cos(2\omega_2) \end{aligned}$$

$$s = \sum_{p=0}^{2} \sum_{q=0}^{2} f(p,q)\ , \tag{11}$$

the coefficients of the McClellan transformation $f(p,q)$ are given by the minimization of (9) as a solution of the eight equations

$$\begin{bmatrix} I_1 & I_2 & I_3 & I_4 & I_5 & I_6 & I_7 & I_8 \\ I_2 & I_9 & I_{10} & I_{11} & I_{12} & I_{13} & I_{14} & I_{15} \\ I_3 & I_{10} & I_{16} & I_{17} & I_{18} & I_{19} & I_{20} & I_{21} \\ I_4 & I_{11} & I_{17} & I_{22} & I_{23} & I_{24} & I_{25} & I_{26} \\ I_5 & I_{12} & I_{18} & I_{23} & I_{27} & I_{28} & I_{29} & I_{30} \\ I_6 & I_{13} & I_{19} & I_{24} & I_{28} & I_{31} & I_{32} & I_{33} \\ I_7 & I_{14} & I_{20} & I_{25} & I_{29} & I_{32} & I_{34} & I_{35} \\ I_8 & I_{15} & I_{21} & I_{26} & I_{30} & I_{33} & I_{35} & I_{36} \end{bmatrix} \times \begin{bmatrix} f(1,0) \\ f(0,1) \\ f(1,1) \\ f(2,0) \\ f(0,2) \\ f(2,1) \\ f(1,2) \\ f(2,2) \end{bmatrix} = \begin{bmatrix} I_{37} \\ I_{38} \\ I_{39} \\ I_{40} \\ I_{41} \\ I_{42} \\ I_{43} \\ I_{44} \end{bmatrix}\ . \tag{12}$$

The constants I_i can be for particular shape of the contour (8) expressed analytically by closed form formulas. This analytical solution is based on the expansion of the functions $cos(\alpha cos\varphi)$, $cos(\alpha sin\varphi)$, $sin(\alpha cos\varphi)$, $sin(\alpha sin\varphi)$ into the sum of Bessel functions which enables analytical integration of the terms in the quadratic difference (9). For asteroidal contour defined by

$$\omega_1 = \frac{3}{4}\omega_0 cos\frac{\varphi}{4} + \frac{1}{4}\omega_0 cos\frac{3\varphi}{4} \quad , \quad \omega_2 = \frac{3}{4}\omega_0 sin\frac{\varphi}{4} - \frac{1}{4}\omega_0 sin\frac{3\varphi}{4} \tag{13}$$

the derived constants I_i are summarized in Tab. 1.

4 Rotation of the Frequency Response

The 2-dimensional zero-phase FIR filter is represented by the impulse response $h(m, n)$. Provided the impulse response $h(m, n)$ is of odd length in both directions with central term $h(0, 0)$ and with symmetry $h(m, n) = h(-m, -n)$, then the relations between the impulse and frequency response are as follows

$$H(e^{j\omega_1}, e^{j\omega_2}) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} h(m, n) e^{-jm\omega_1} e^{-jn\omega_2} \tag{14}$$

$$h(m, n) = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} H(e^{j\omega_1}, e^{j\omega_2}) e^{jm\omega_1} e^{jn\omega_2} \, d\omega_1 \, d\omega_2 \ . \tag{15}$$

I order to rotate the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$ in the frequency plane we propose the transformation of the frequency coordinates

$$\Omega_1 = \omega_1 \cos\phi - \omega_2 \sin\phi \quad , \quad \Omega_2 = \omega_2 \cos\phi + \omega_1 \sin\phi \ . \tag{16}$$

The impulse response $h_r(p, q)$ of the filter with the rotated frequency response $H_r(e^{j\omega_1}, e^{j\omega_2})$ is given by substitution of (16) into (14) and evaluating (15) in the form

$$h_r(p, q) = \frac{1}{(2\pi)^2} \sum_{m=-M}^{M} \sum_{n=-N}^{N} h(m, n) \times \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{j(p\omega_1 - m\Omega_1)} e^{j(q\omega_2 - n\Omega_2)} \, d\omega_1 \, d\omega_2 \tag{17}$$

yielding

$$h_R(p, q) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} h(m, n) \frac{\sin \pi k_1}{\pi k_1} \frac{\sin \pi k_2}{\pi k_2} \quad , \quad |p| \leq M \, , \quad |q| \leq N \tag{18}$$

where

$$k_1 = p - (m \cos\phi - n \sin\phi) \quad , \quad k_2 = q - (n \cos\phi + m \sin\phi) \, . \tag{19}$$

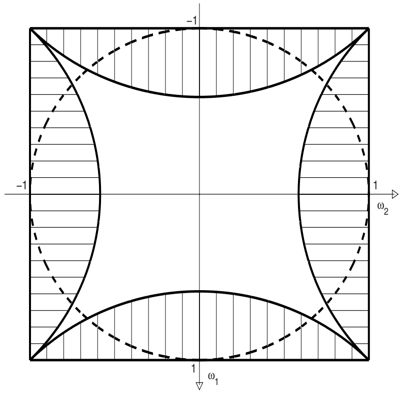

Fig. 1. Limitations of the rotation

The frequency response $H_r(e^{j\omega_1}, e^{j\omega_2})$ given by the impulse response $h_r(p, q)$ is in the frequency domain rotated counterclockwise by the angle ϕ with respect to the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$ given by the impulse response $h(m, n)$. However, we have to take into account two limitations of the rotation denoted in Fig. 1. First, the aliasing from higher spectral periods may occur during the rotation. For aliasing prevention the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$ should be constant in the four dashed border regions demarcated by the arcs with the radius equal $\sqrt{2}$. Second, the regions of the frequency response $H(e^{j\omega_1}, e^{j\omega_2})$ outside the unity circle as indicated in Fig. 1 may disappear during the rotation. The regions of the frequency response inside the unity circle never disappear during the rotation. Due to these limitations it is not possible to rotate the transformation function $F(e^{j\omega_1}, e^{j\omega_2})$ prior the McClellan transformation.

5 Example of the Design

In our example we demonstrate the design of the 2-D FIR low pass filter with asteroidal contour for $\omega_0 = 0.5$ oriented counterclockwise by the angle $\phi = 3\pi/25$ with respect to the frequency coordinates. We assume normalized frequencies in the interval $(-1, 1)$. In the first step we calculate the coefficients of the McClellan transformation

$$f(p, q) = \begin{bmatrix} 0.263308 & 0.092866 & 0.056574 \\ 0.092866 & 0.019525 & 0.075101 \\ 0.056574 & 0.075101 & -0.148859 \end{bmatrix} \quad (20)$$

using formulas summarized in Tab. 1. The corresponding transformation function $F(e^{j\omega_1}, e^{j\omega_2})$ with contours is presented in Fig. 2. The 1-D maximally flat low pass FIR filter with 3dB-decay cut-off frequency $\omega_{stop} = 0.22$ of the length 17 coefficients was designed using the analytical procedure presented in [6]. The impulse response $h(n)$ of the filter is summarized in Tab. 2. Using the transformation coefficients $f(p, q)$ and the 1-D impulse response $h(n)$ we calculate the

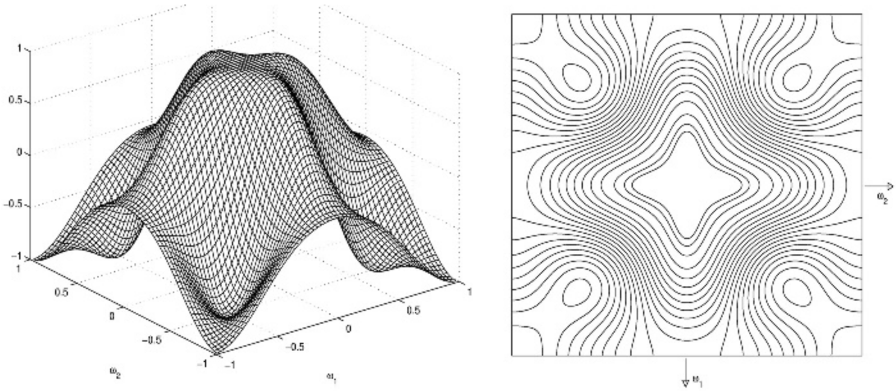

Fig. 2. Transformation function $F(e^{j\omega_1}, e^{j\omega_2})$ with contours

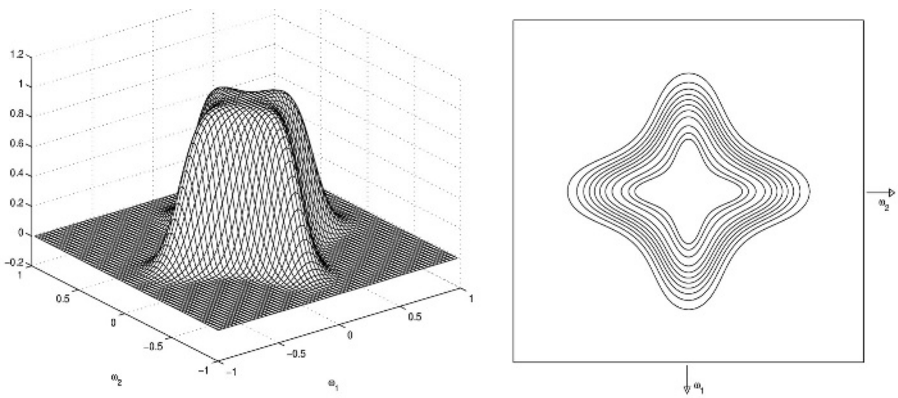

Fig. 3. Amplitude frequency response $|H(e^{j\omega_1}, e^{j\omega_2})|$ with contours

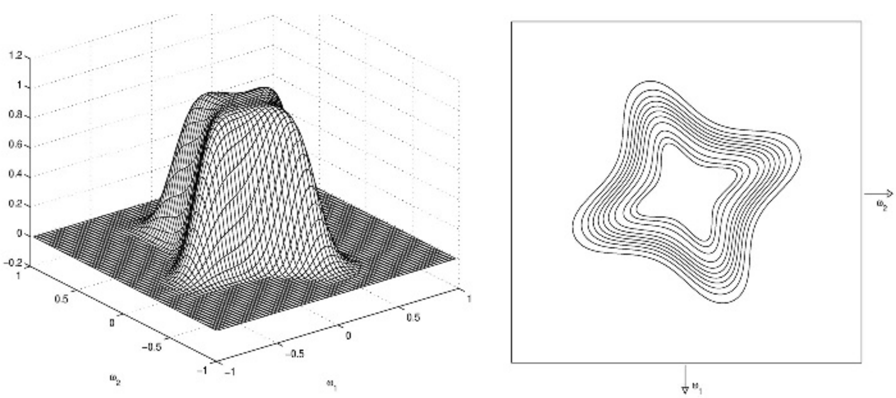

Fig. 4. Rotated amplitude frequency response $|H_r(e^{j\omega_1}, e^{j\omega_2})|$ with contours

Table 1. Constants I_i

$$
\begin{aligned}
I_1 &= 3\pi/2 - 2F(\omega_1) + F(2\omega_1)/2 \\
I_2 &= \pi - F(\omega_1) - F(\omega_2) + E(\omega_1, \omega_2) \\
I_3 &= \pi - F(\omega_1) - E(\omega_1, \omega_2) + F(\omega_2)/2 + E(2\omega_1, \omega_2)/2 \\
I_4 &= \pi - F(\omega_1)/2 - F(2\omega_1) + F(3\omega_1)/2 \\
I_5 &= \pi - F(\omega_1) - F(2\omega_2) + E(\omega_1, 2\omega_2) \\
I_6 &= \pi - F(\omega_1) - E(2\omega_1, \omega_2) + E(3\omega_1, \omega_2)/2 + E(\omega_1, \omega_2)/2 \\
I_7 &= \pi - F(\omega_1) - E(\omega_1, 2\omega_2) + F(2\omega_2)/2 + E(2\omega_1, 2\omega_2)/2 \\
I_8 &= \pi - F(\omega_1) - E(2\omega_1, 2\omega_2) + E(3\omega_1, 2\omega_2)/2 + E(\omega_1, 2\omega_2)/2 \\
I_9 &= 3\pi/2 - 2F(\omega_2) + F(2\omega_2)/2 \\
I_{10} &= \pi - F(\omega_2) - E(\omega_1, \omega_2) + F(\omega_1)/2 + E(\omega_1, 2\omega_2)/2 \\
I_{11} &= \pi - F(\omega_2) - F(2\omega_1) + E(2\omega_1, \omega_2) \\
I_{12} &= \pi - F(\omega_2)/2 - F(2\omega_2) + F(3\omega_2) \\
I_{13} &= \pi - F(\omega_2) - E(2\omega_1, \omega_2) + F(2\omega_1)/2 + E(2\omega_1, 2\omega_2)/2 \\
I_{14} &= \pi - F(\omega_2) - E(\omega_1, 2\omega_2) + E(\omega_1, 3\omega_2)/2 + E(\omega_1, \omega_2)/2 \\
I_{15} &= \pi - F(\omega_2) - E(2\omega_1, 2\omega_2) + E(2\omega_1, 3\omega_2)/2 + E(2\omega_1, \omega_2)/2 \\
I_{16} &= 5\pi/4 - 2E(\omega_1, \omega_2) + [F(2\omega_1) + F(2\omega_2) + E(2\omega_1, 2\omega_2)]/4 \\
I_{17} &= \pi - E(\omega_1, \omega_2)/2 - F(2\omega_1) + E(3\omega_1, \omega_2)/2 \\
I_{18} &= \pi - E(\omega_1, \omega_2)/2 - F(2\omega_2) + E(\omega_1, 3\omega_2)/2 \\
I_{19} &= \pi - E(\omega_1, \omega_2) - E(2\omega_1, \omega_2) + [F(3\omega_1) + E(3\omega_1, 2\omega_2) + F(\omega_1) + E(\omega_1, 2\omega_2)]/4 \\
I_{20} &= \pi - E(\omega_1, \omega_2) - E(\omega_1, 2\omega_2) + [F(3\omega_2) + F(\omega_2) + E(2\omega_1, 3\omega_2) + E(2\omega_1, \omega_2)]/4 \\
I_{21} &= \pi - E(\omega_1, \omega_2) - E(2\omega_1, 2\omega_2) + [E(3\omega_1, 3\omega_2) + E(\omega_1, 3\omega_2) + E(\omega_1, \omega_2)]/4 \\
I_{22} &= 3\pi/2 - 2F(2\omega_1) + F(4\omega_1)/2 \\
I_{23} &= \pi - F(2\omega_1) - F(2\omega_2) + E(2\omega_1, 2\omega_2) \\
I_{24} &= \pi - F(2\omega_1) - E(2\omega_1, \omega_2) + F(\omega_2)/2 + E(4\omega_1, \omega_2)/2 \\
I_{25} &= \pi - F(2\omega_1) - E(\omega_1, 2\omega_2) + F(\omega_1)/2 + E(\omega_1, 4\omega_2)/2 \\
I_{26} &= \pi - F(2\omega_1) - E(\omega_1, 2\omega_2)/2 + E(3\omega_1, 2\omega_2)/2 \\
I_{27} &= 3\pi/2 - 2F(2\omega_2) + F(4\omega_2)/2 \\
I_{28} &= \pi - F(2\omega_2) - E(2\omega_1, \omega_2) + E(2\omega_1, 3\omega_2)/2 + E(2\omega_1, \omega_2)/2 \\
I_{29} &= \pi - F(2\omega_2) - E(\omega_1, 2\omega_2) + E(\omega_1, 4\omega_2)/2 + F(\omega_1)/2 \\
I_{30} &= \pi - F(2\omega_2) - E(2\omega_1, 2\omega_2) + E(2\omega_1, 4\omega_2)/2 + F(2\omega_2)/2 \\
I_{31} &= 5\pi/4 - 2E(2\omega_1, \omega_2) + [E(4\omega_1, 2\omega_2) + F(4\omega_1) + F(2\omega_2)]/4 \\
I_{32} &= \pi - E(2\omega_1, \omega_2) - E(\omega_1, 2\omega_2) + [E(3\omega_1, 3\omega_2) + E(3\omega_1, \omega_2) + E(\omega_1, 3\omega_2) + E(\omega_1, \omega_2)]/4 \\
I_{33} &= \pi - E(2\omega_1, \omega_2) - E(2\omega_1, 2\omega_2) + [E(4\omega_1, 3\omega_2) + E(4\omega_1, \omega_2) + F(3\omega_2) + F(\omega_2)]/4 \\
I_{34} &= 5\pi/4 - 2E(\omega_1, 2\omega_2) + [E(2\omega_1, 4\omega_2) + F(2\omega_1) + F(4\omega_2)]/4 \\
I_{35} &= \pi - E(\omega_1, 2\omega_2) - E(2\omega_1, 2\omega_2) + [E(3\omega_1, 4\omega_2) + E(\omega_1, 4\omega_2) + F(3\omega_1) + F(\omega_1)]/4 \\
I_{36} &= 5\pi/4 - 2E(2\omega_1, 2\omega_2) + [E(4\omega_1, 4\omega_2) + F(4\omega_1) + F(4\omega_2)]/4 \\
I_{37} &= (cos\omega_0 - s)(\pi - F(\omega_1)) \\
I_{38} &= (cos\omega_0 - s)(\pi - F(\omega_2)) \\
I_{39} &= (cos\omega_0 - s)(\pi - E(\omega_1, \omega_2)) \\
I_{40} &= (cos\omega_0 - s)(\pi - F(2\omega_1)) \\
I_{41} &= (cos\omega_0 - s)(\pi - F(2\omega_2)) \\
I_{42} &= (cos\omega_0 - s)(\pi - E(2\omega_1, \omega_2)) \\
I_{43} &= (cos\omega_0 - s)(\pi - E(\omega_1, 2\omega_2)) \\
I_{44} &= (cos\omega_0 - s)(\pi - E(2\omega_1, 2\omega_2))
\end{aligned}
$$

$$F(x) = \omega_0 \left[J_0(x) - 2 \sum_{m=1}^{\infty} J_{2m}(x) \frac{1}{4m^2 - 1} \right]$$

$$E(x, y) = \omega_0 \left[J_0\left(\sqrt{x^2 + y^2}\right) - 2 \sum_{m=1}^{\infty} J_{2m}\left(\sqrt{x^2 + y^2}\right) \frac{cos\left(2m \arccos\left(\frac{x}{\sqrt{x^2 + y^2}}\right)\right)}{4m^2 - 1} \right]$$

Table 2. Impulse Response $h(n)$

n		$h(n)$	n		$h(n)$
0	16	-0.000107	5	11	0.022217
1	15	-0.001221	6	10	0.122192
2	14	-0.005981	7	9	0.244385
3	13	-0.015381	8		0.301117
4	12	-0.016663			

2-D impulse response $h(m,n)$ of the asteroidally shaped 2-D FIR filter of the length 37×37 coefficients. The frequency response of the filter is oriented in the basic position along the frequency coordinates. The amplitude frequency response $|H(e^{j\omega_1}, e^{j\omega_2})|$ of the filter with its contours is shown in Fig. 3. In the second step the basically oriented 2-D low-pass filter with asteroidal shape of the passband is rotated counterclockwise by the angle $\phi = \frac{3}{25}\pi$ using (18). The rotated frequency response $|H_r(e^{j\omega_1}, e^{j\omega_2})|$ of the filter with its contours is shown in Fig. 4.

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