A Closed Form Solution of the Run-Time of a Sliding Bead along a Freely Hanging Slinky

Haiduke Sarafian

The Pennsylvania State University
York, PA 17403, USA
has2@psu.edu

Abstract. The author has applied Lagrangian formalism to explore the kinematics of a bead sliding along a frictionless, freely hanging vertical Slinky. For instance, we derived a closed analytic equation for the run-time of the bead as a function of the traversed coil number. We have applied *Mathematica* to animate the 3-dimensional motion of the bead. The derived run-time is incorporated within the animation to clock the bead's actual motion. With the help of *Mathematica* we have solved the inverse run-time equation and have expressed the traversed coil number as a function of the run-time. The latter is applied to further the analysis of the problem conducive to analytic time-dependent equations for the bead's vertical position, its falling speed and its falling acceleration, and its angular velocity about the symmetry axis of the Slinky. It is also justified that a Slinky is a device capable of converting the gravitational potential energy of a sliding bead into pure rotational energy.

1 Introduction

A Slinky is a massive, soft spring-like object and has curious dynamic and static features. By hanging masses to a freely, vertically suspended Slinky and setting it in motion, the authors of [1] have investigated some of its dynamic features. The Slinky's static characteristics have been studied in [2]. In both references, analytically, it is proven how the Slinky's own weight contributes to the uneven spacing of the adjacent coils along a vertically hung Slinky. The proven equations match the intuitive expectations – the spacing between the adjacent coils for the coils closer to the support are wider than the ones at the bottom. Furthermore, the lower coils are less slanted verses the ones closer to the support. In fact, the bottom coils are almost horizontal.
To incorporate these known characteristic features and to broaden the scope of the Slinky's related issues, we studied a kinematic problem. We considered the effects of the uneven spacing of the Slinky's coils to calculate the run-time of a sliding bead under the gravity pull. We have shown, although the calculation of the run-time of a sliding bead along a theoretical, massless evenly-spaced vertically hung spring is trivial, it is not so for a real Slinky. We were able to solve the Slinky problem exactly and derived an analytic closed form equation to express the run-time of a sliding bead as a function of the traversed coil number.

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We have applied *Mathematica* [3] to animate the 3-dimensional motion of a bead. We considered a typical Slinky and have applied its geometrical lengths to clock the run-time of a sliding bead. The numeric values of the run-time were embedded in the 3-d animation so that one can visually correspond the actual movement of the bead to its run-time and the traversed coil number. Because of the length limitation of the article the *Mathematica* code is not included, the oral presentation will feature the animation.
To further our analysis with the help of *Mathematica* we have solved the inverse run-time equation, we expressed the traversed coil number as a function of the run-time. The detail of the procedure and the specific use of *Mathematica* in achieving this goal is given in section 4. The latter is applied to further the analysis of the problem conducive to analytic time-dependent equations for the bead's vertical position, its falling speed and its falling acceleration, and its angular velocity about the symmetry axis of the Slinky. For comprehensive understanding, the derived equations are plotted v.s. time.

2 The Physics and the Analysis of the Problem

We denote the number of Slinky's coils by N, the radius of the circular coils by R, its un-stretched axial length when laid on a level table by L_0 and its stretched length when freely suspended vertically by L. Figure 1 depicts one such Slinky. We have applied *Mathematica's* ParamtericPlot3D command to display the Slinky. The origin of a right-handed Cartesian coordinate system is set at the bottom of the Slinky with the z-axis pointing to the top of the page. The first coil, $n = 0$, is at the $z = 0$ and the top coil, $N = 35$, is at the support. The height of the individual coil is measured from the bottom, the data is shown in Fig. 2. The size of the data points indicates the accuracy of the measurements along the vertical axis.

According to [1,2] the height of the nth coil is given by; $z_n = an + bn^2$ with $a = \frac{L_0}{N}$ and $b = \frac{L-L_0}{N^2}$. In Fig. 2, z_n, is shown by the solid line – it perfectly fits the data.

The position vector of a bead in the aforementioned Cartesian coordinate system is $\boldsymbol{r} = \{x(t), y(t), z(t)\}$ and can be written as $\boldsymbol{r} = \{R\cos(2\pi n^{'}), R\sin(2\pi n'), z_{n'}\}$ where $n^{'}$, is the number of the traversed coils and equals $n^{'} = N - n$. The kinetic energy $T = \frac{1}{2}m\nu^2$ and the potential energy $V = mgz_{n'}$ of a freely released bead of mass m in terms of the traversed coil number n' are:

$$T = \frac{1}{2}m\dot{n^{'}}^2\{c^2 + [a + 2b(N - n^{'})]^2\} \tag{1}$$

$$V = mg[a(N - n^{'}) + b(N - n^{'})^2] \tag{2}$$

where $\dot{n^{'}} = \frac{d}{dt}n^{'}$, $c = 2\pi R$ and g is the gravity.

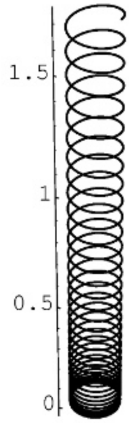

Fig. 1. The display of a typical Slinky with specs of $N = 35$, $R = 5.0cm$, $L_0 = 7.0cm$ and $L = 172.0cm$

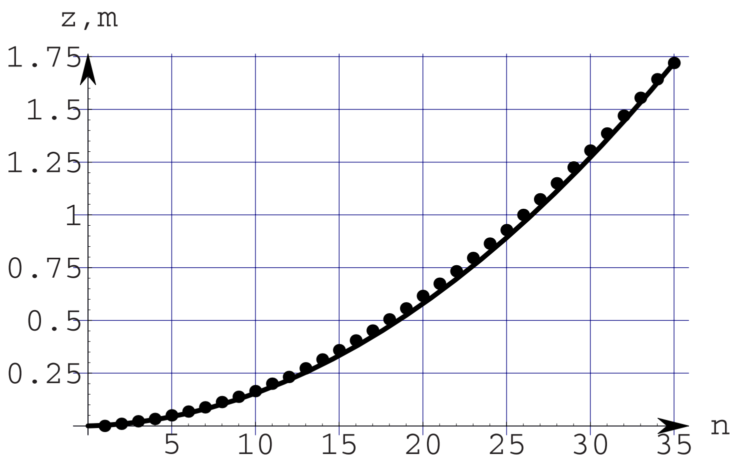

Fig. 2. The dots are the data and the solid line is $z_n = an + bn^2$

The Euler-Lagrange equation, $\frac{d}{dt}\frac{\partial L}{\partial \dot{n'}} = \frac{\partial L}{\partial n'}$ for Lagrangian $L = T - V$ with T and V subject to (1) and (2) is,

$$\ddot{n}'\{c^2 + [a + 2b(N - n')]^2\} - 2b\dot{n}'^2[a + 2b(N - n')] - g[a + 2b(N - n')] = 0. \quad (3)$$

To solve (3) we introduce $\xi = a + 2b(N - n')$. In terms of ξ, (3) becomes,

$$\ddot{\xi}(c^2 + \xi^2) + \dot{\xi}^2\xi + \nu^2\xi = 0 \quad (4)$$

Here, $\nu^2 = 2bg$ and assumes the speedsquared dimension. To solve (4) we set, $\eta = \dot{\xi}^2$. In terms of η and ξ, (4) yields

$$\frac{\dot{\eta}}{2\dot{\xi}}(c^2 + \xi^2) + \xi(\eta + \nu^2) = 0 \tag{5}$$

By separating the variables of (5) and integrating both sides of the resulting equation with appropriate limits we arrive at

$$\int_0^{\eta} \frac{d\eta}{\eta + \nu^2} = -\int_{\xi_0}^{\xi} \frac{2\xi d\xi}{c^2 + \xi^2} \tag{6}$$

here, $\xi_0 = a + 2bN$. The integration of (6) yields,

$$\eta = \nu^2 \frac{{\xi_0}^2 - \xi^2}{c^2 + \xi^2} \tag{7}$$

In (7) we replace η with $(\frac{d\xi}{dt})^2$, and by rearranging the terms and integrating the result, we arrive at

$$t = \frac{1}{\nu} \int_{\xi_0}^{\xi} \sqrt{\frac{c^2 + \xi^2}{{\xi_0}^2 - \xi^2}} d\xi \tag{8}$$

this yields

$$t = \frac{1}{\nu}\sqrt{c^2 + {\xi_0}^2} E(\arccos(\frac{a + 2bn}{\xi_0}), \frac{\xi_0}{\sqrt{c^2 + {\xi_0}^2}}) \tag{9}$$

$E(\delta, r)$ is the Elliptic integral of the second kind [4]. Equation (9)is the run-time; i.e. it is the time a bead starting from the top coil takes to traverse to the nth coil.

We have noticed that (8) is a convoluted analog of the kinematics of one dimensional uniform motion. I.e. (8) can be viewed as $\ell = \nu t$, with ℓ being the Slinky's characteristic length given by

$$\ell = \int_{\xi_0}^{\xi} w(c, \xi_0, \xi) d\xi \tag{10}$$

In other words, the bead slides along the characteristic length, ℓ, given by the weighted differential length $d\xi$. The weight, w, is defined by

$$w(c, \xi_0, \xi) = \sqrt{\frac{c^2 + \xi^2}{{\xi_0}^2 - \xi^2}} \tag{11}$$

As mentioned earlier, the bead's characteristic speed ν is $\nu = \sqrt{2gb}$. Intuitively, one would expect the run-time of a bead, for a skinny Slinky to be the same as the run-time of a freely falling bead released at the same height. To verify this, in (9) we set $R = 0$, this yields,

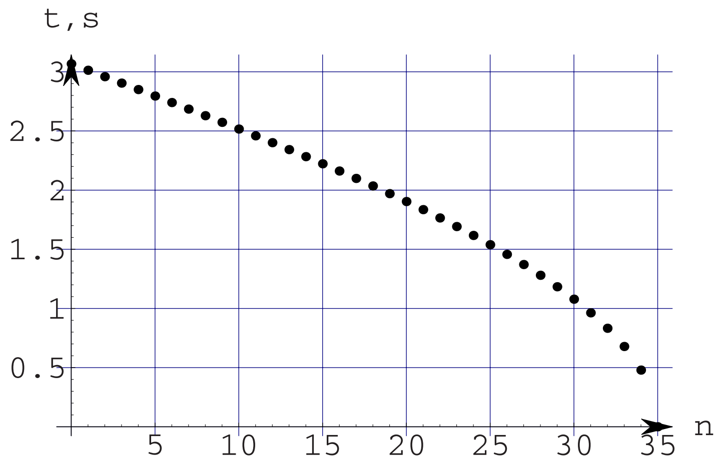

Fig. 3. The run-time of a bead v.s. the coil number n. The specs of the Slinky are the ones used in Fig. 1

$$t = \frac{1}{\nu}\xi_0 E(\arccos(\frac{a+2bn}{\xi_0}), 1) \tag{12}$$

On the other hand, since. $L_0 \ll L$, (12) simplifies further,

$$t \simeq \sqrt{\frac{2L}{g}} E(\arccos(\frac{n}{N}), 1) \tag{13}$$

For $n = 0$, (13) yields the run-time of a bead traversing the entire length of the stretched Slinky, L. The E in (13) for $n = 0$ yields, $E(\frac{\pi}{2}, 1) = 1$ and yields the shortest run-time,

$$t \simeq \sqrt{\frac{2L}{g}} \tag{14}$$

Equation (14) is identical to the time of flight of a free falling object, $L = \frac{1}{2}gt^2$, released at height L. Equation (14) for $L = 172.0cm$, the length of a hanging Slinky gives $t = 0.592s$ and matches the numeric value of (9) for small values of R, e.g. $R = 0.005cm$. It is instructive to display the run-time t, given by (9) v.s. the coil number n. For the Slinky on hand, this is shown in Fig 3.

3 Corollary Topics of Interest

It is curious to find out at any given time how high the bead is from the bottom of the Slinky, how fast the bead is falling and its falling acceleration. To address

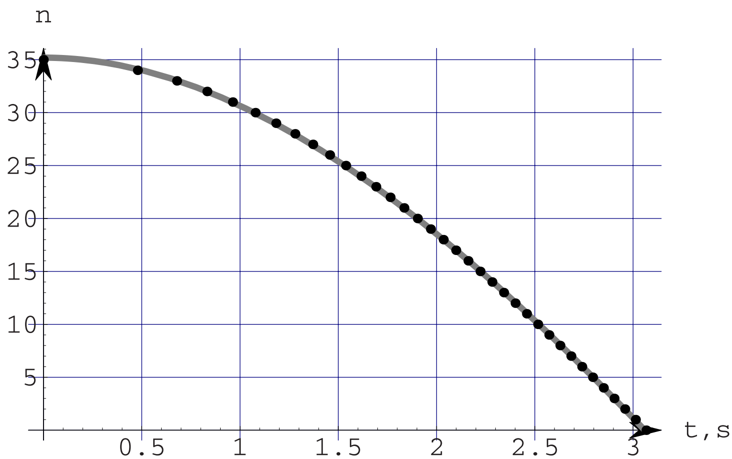

Fig. 4. The plots of the coil number n v.s. t. The solid line is the fitted polynomial

these questions, there is a need to solve (9) for t, this is problematic. However, we pursued the goal graphically. First, by interchanging the horizontal and vertical axes in Fig. 3, we re-plot the data points. We then apply *Mathematica's* Fit command to fit the data with a suitable polynomial. Figure 4 displays the output.

For the Slinky on hand the fitted function is an incomplete third order polynomial with the given fitted coefficients $n(t) = 35.190 - 4.925t^2 + 0.381t^3$. By substituting $n(t)$ in z_n we evaluate the vertical height of the bead, $z(t)$, its falling speed, $\dot{z}(t)$ and its falling acceleration $\ddot{z}(t)$. These quantities are plotted in Fig. 5.

The ordinate of Fig. 5, is calibrated in MKS units, and hence, m, $\frac{m}{s}$, $\frac{m}{s^2}$ are to be used to read the height, the velocity and the acceleration.
According to Fig. 5, the bead starts off with an initial zero velocity and of about $1\frac{m}{s^2}$ acceleration. In $1.6s$, it reaches its maximum, $0.83\frac{m}{s}$ velocity and acquires zero acceleration. In according to Fig. 3, $t = 1.6s$, corresponds to $n = 24$, that is the bead reaches its maximum velocity after traversing 11 coils. From this point on, it takes another $1.4s$ to decelerate to the bottom.
We also noted, a Slinky is a device capable of converting the gravitational potential energy to a pure rotational energy. This is because the bottom coils are horizontal and the bead upon reaching the bottom is to circulate about the Slinky's symmetry axis with no further fall.
We quantify our claim by applying the conservation of energy principle to the two ends of the Slinky; the potential energy at the top and the rotational kinetic energy at the bottom. That is $(PE)^i = (KE)^f_{rot}$, this gives $mgL = \frac{1}{2}I\omega^2_{max}$, here, $I = mR^2$ is the moment of inertia of the bead about the Slink's symmetry axis and ω_{max} is its maximum angular velocity. Solving this equation for ω_{max} yields $\omega_{max} = \frac{1}{R}\sqrt{2gL}$. For the specs of the Slinky on hand this yields

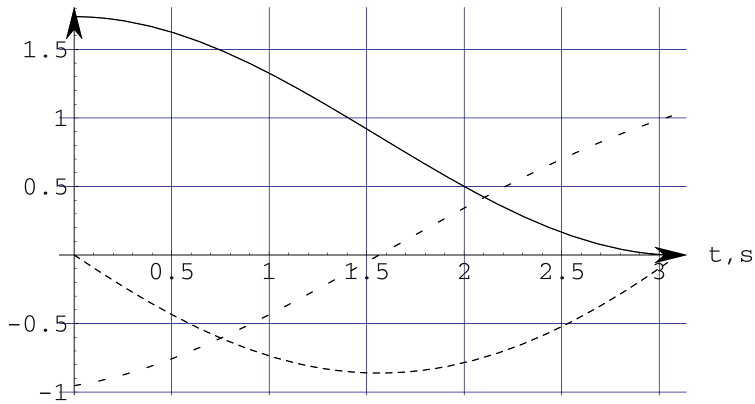

Fig. 5. The plot of $z(t)$(solid-line),the falling velocity,$\dot{z}(t)$(short dashed-line) and the falling acceleration $\ddot{z}(t)$ (long dashed-line)

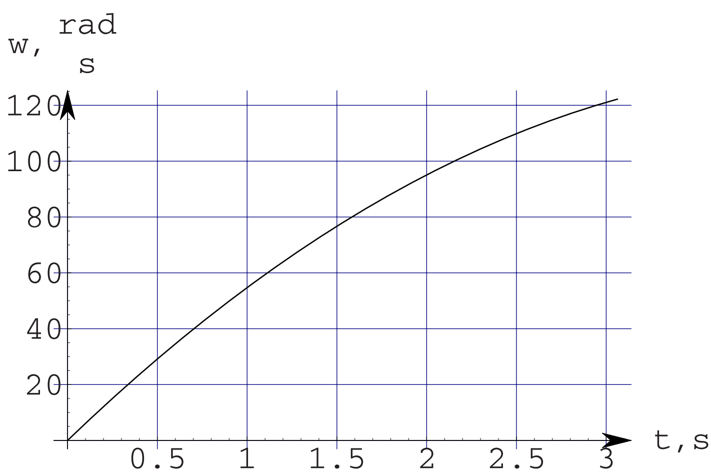

Fig. 6. Plot of angular velocity, ω, v.s. t. The Slinky's specs are the ones used in Fig. 1

$\omega_{max} = 116\frac{rad}{s}$. On the other hand, the angular velocity of the bead is $\omega(t) = \frac{d}{dt}\{2\pi[N - n(t)]\}$. The quantity in the braces is the traversed azimuthal angle about the symmetry axis of the Slinky and $n(t)$ is the aforementioned fitted polynomial.

Figure 6 displays $\omega(t)$ v.s. time. The maximum value of $\omega_{max}(t)$ at the end of the run is $120\frac{rad}{s}$, this is in good agreement with the predicted, $116\frac{rad}{s}$.

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