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Abstraction in Context

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Definition

Abstraction has been the focus of extensive interest in several domains, including Mathematics Education. Many researchers have taken a predominantly theoretical stance and have described abstraction as some type of de-contextualization. *Abstraction in context* (AiC; Hershkowitz et al. 2001) proposes a quite different approach to abstraction. The following is the operational definition of AiC:

> Abstraction in Context is an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure.

The term *activity* above emphasizes that abstraction in context (AiC) is considered to be a process taking place in a specific context; it may capitalize on tools and other artifacts and it occurs in a particular social setting. The phrase *previously constructed mathematics* refers to the outcomes of previous processes of abstraction, which may be used during the current abstraction activity. The phrase *reorganizing into a new structure* implies the establishment of higher unknown mathematical by combining the two sentences. In particular, the word *new* is crucial: it expresses that, as a result of abstraction, learners participating in the activity conceptualize something that was previously inaccessible to them.

The Emergence of the Theoretical Framework

The theoretical framework emerged from interweaving theory and practice. Theory stems from the general development of approaches concerning learning as a whole and abstraction in particular, namely the rejection of the de-contextualization approach to abstraction. Bert van Oers (1998) claimed that "the notion of 'de-contextualization' is a poor concept that provides little explanation for the developmental process toward meaningful abstract thinking"

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(p. 135). van Oers proposed "a conceptualization of the notion of context from an activity point of view and contend[ed] that the conscious process of (re)contextualizing – that is, the continuous process of embedding contexts in contexts – can lead to an explanation of the development of meaningful abstract thinking" (ibid.). This claim fits the view of Freudenthal and his colleagues about learning by *vertically reorganizing* previous mathematical constructs within mathematics and by mathematical means into a construct that is new to the learners (Treffers and Goffree 1985). Davydov's approach is similar. For Davydov (1990), scientific knowledge is not a simple expansion of people's everyday experience. It requires the cultivation of particular ways of thinking, which permit the essence of ideas and their internal connections to emerge, thus enriching rather than impoverishing reality. According to Davydov's "method of ascent to the concrete," abstraction starts from an initial, simple, undeveloped, and vague first form, which often lacks consistency. The development of abstraction proceeds from analysis, at the initial stage of the abstraction, to synthesis. It is a dialectical process leading to a more consistent and elaborated form. It does not proceed from concrete to abstract but from an undeveloped to a developed form.

The above theoretical view served as basis for elaborating tools to observe practices involving abstraction in an innovative junior high school curriculum development and implementation project designed to promote abstraction. The AiC theoretical framework took shape when the project team asked question like: "What did students learn and consolidate, and how?" "What mathematical concepts and strategies remain with them?"

As practitioners, the project team members were overwhelmed and surprised by what they observed in trial classrooms. The need to meaningfully describe and explain the observed learning practices raised the need for an operational definition of abstraction. The researchers adopted and consolidated the approaches mentioned above and translated them into the above operational definition.

Model and Methodology

In order to empirically investigate processes of AiC, the researchers devised a way to make the mental activity of abstraction observable. As activities are composed of actions, and actions are frequently observable, they decided to use *epistemic actions* (Pontecorvo and Girardet 1993), namely mental actions by means of which knowledge is used or constructed. Hershkowitz et al. (2001) defined and described three epistemic actions as constituent of a model. Processes of knowledge construction are expressed in the model through three observable and identifiable epistemic actions*: recognizing, building-with,* and *constructing (RBC)*. Recognizing takes place when the learner recognizes that a specific previous knowledge construct is relevant to the problem he or she is dealing with. Building-with is an action comprising the combination of recognized constructs in order to achieve a localized goal, such as the actualization of a strategy or a justification or the solution of a problem. The model suggests *constructing* as the central epistemic action of mathematical abstraction. Constructing consists of assembling and integrating previous constructs by vertical reorganization. The C-action draws its power from the mathematical connections, made by the learners, which link the R- and B-actions as building blocks, and make them into a single whole unity, often requiring creativity and inspiration from the learners. Hence, R-actions are nested in B-actions, and R- and B-actions with previous constructs are nested in C-actions of new constructs, giving rise to the name *the dynamically nested epistemic actions model of abstraction in context* or briefly, the RBC model. This model is a part of the theoretical framework of AiC and at the same time the methodological micro-analytic lens, through which the mechanism of abstraction in context is observed and analyzed (Hershkowitz 2009).

It is postulated that the genesis of an abstraction passes through a three-stage process: the *need* for a new construct, the *emergence* of the new construct, and the *consolidation* of that construct. The nature of need is well expressed by Kidron and Monaghan (2009) when dealing with the need that pushes students to engage in abstraction, a need that emerges from a suitable design. Emergence is captured by the RBC-model as described above, where constructing is defined as the first emergence of a new construct, and a student may have participated in a C-action without being fully aware of it. This leads to the importance of consolidation, observed via students' successive R- and B-actions with the new construct, for example, during problem solving or successive C-actions of further constructs. Consolidation is expressed in students' increased confidence, immediacy, flexibility, and awareness in using the construct (Dreyfus and Tsamir 2004; Monaghan and Ozmantar 2006). The study of consolidation usually requires data from several subsequent activities. The model it often referred to as RBC+C model using the second C in order to point to the important role of consolidation.

Research Directions

Research studies using AiC may be seen as belonging to two categories, both of which continuously contribute to expanding the power and scope of AiC: Research expanding the range of concepts and research expanding the range of contexts (Dreyfus et al. 2015).

While AiC was originally developed for investigating students' learning at the high school level in topics like functions (Hershkowitz et al. 2001) and probability (e.g., Hershkowitz et al. 2007), researchers' attention soon turned to other topics from elementary to graduate level including fractions (Weiss 2011), limits (Kidron 2008), integrals (Kouropatov and Dreyfus 2014), and bifurcations in dynamic processes (Dreyfus and Kidron 2006). Notably, knowledge construction in cross-content issues such as definition (Gilboa et al. in press) and justification (Kidron and Dreyfus 2010a) were also investigated.

Researchers soon realized the necessity of an a priori analysis of tasks in terms of the opportunities for abstraction, which they offer (Ron et al. 2010); hence, a detailed a priori analysis with general and operational definitions of intended knowledge elements soon became a stable feature of AiC studies.

It is well known that students' correct answers sometimes hide knowing gaps and, on the other hand, incorrect answers often overshadow substantial knowledge students have constructed. These two phenomena have been interpreted in AiC as reflecting aspects of partially correct knowledge constructs – PaCCs (Ron et al. 2010, 2017). PaCCs are, by definition, partial with reference to knowledge intended to be constructed, and not in some absolute sense.

AiC research expanding the range of contexts focused on the roles of technological tools and of social interaction in the construction of knowledge. Artifacts such as manipulatives or computerized learn ware have been shown to possibly have a crucial role in the construction of knowledge. For example, Weiss (2011) has analyzed the role of an analogical model in knowledge construction during model-based tasks. The tasks were in the area of fraction comparison, and the analogical model was the tower of bars. From the methodological point of view, the study by Weiss required not only an RBC analysis but also an analysis based on the emergent models approach of Gravemeijer (1999), which is rooted in the realistic mathematics education philosophy of the Freudenthal tradition and makes use of vertical mathematization. This double analysis has led to the identification of deep theoretical relationships between AiC and RME.

In another study, Kidron and Dreyfus (2010b) describe how instrumentation led to cognitive constructing actions and how the roles of the learner and a computer algebra system (CAS) intertwine, giving the CAS a major influence on interactions between different parallel constructing actions. Specifically, they showed that certain patterns of epistemic actions, such as the combining of constructing actions, have been facilitated by the CAS context.

AiC research on the role of the social context in the construction of knowledge has been extensive. Construction of knowledge by individuals, small groups, and even by peers within a whole class has been considered. For example, the RBC+C model was applied to processes in which groups of individual students construct *shared knowledge* and consolidate it. The data emphasize the interactive flow of knowledge from one student to the others in the group, until they reach a shared knowledge – a common basis of knowledge which allows them to continue the construction of further knowledge in the same topic together (Hershkowitz et al. 2007).

The natural contexts for the observation of AiC are contexts of tutoring and (un-)guided small group work. Tracing abstraction in a classroom full forum demands an adjustment. Documenting Collective Activity (DCA; Rasmussen and Stephan 2008) was a good match with AiC because of environmental, underlying and internal commonalities (Tabach et al. 2017). The combination of AiC and DCA allowed us to investigate learning processes in classrooms taking place into small groups of students as well as teacher-led whole class discussions. The data collected in such investigations is massive and rich, so opportunities for new research avenues opened up. For example, we gained a better understanding of the mechanisms of *knowledge shifts* within and between different settings in a mathematics classroom, as it is expressed in the roles of students as individuals, as groups, and as a community (Tabach et al. 2014). Another example investigates students' collective creativity and teachers' ways of leading them in the above shifts of knowledge (Hershkowitz et al. 2016).

Cross-References

- ▶ Abstraction
- ▶ Actions, Processes, Objects, Schema (APOS) in Mathematics Education
- ▶ Concept Development in Mathematics Education
- ▶ Design Research and Didactic Engineering in Mathematics Education
- ▶ Goals of Mathematics Education
- ▶ Realistic Mathematics Education
- ▶ Recontextualizations in Mathematics Education
- ▶ Situated Cognition in Mathematics Education

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