Chapter 8
Electromagnetic Fields in Meta-Media with Interfacial Surface Admittance

David C. Christie and Robin W. Tucker

Abstract We exploit Clemmow's complex plane-wave representation of electromagnetic fields to construct globally exact solutions of Maxwell's equations in a piecewise homogeneous dispersive conducting medium containing a plane interface that can sustain (possibly dissipative) field-induced surface electric currents. Families of solutions, parametrised by the complex rotation group $SO(3,\mathbb{C})$, are constructed from the roots of complex polynomials with coefficients determined by constitutive properties of the medium and a particular interface admittance tensor. Such solutions include coupled TE and TM-type surface polariton and Brewster modes and offer a means to analyse analytically their physical properties given the constitutive characteristics of bulk meta-materials containing fabricated meta-surface interfaces.

8.1 Introduction

This article is concerned with the behaviour of electromagnetic fields in regions of materials that possess rapid spatial variations in their constitutive properties in the vicinity of two-dimensional surfaces. In such regions certain components of these fields are also expected to exhibit enhanced spatial variations. An exact mathematical treatment of such systems is feasible if one idealises such regions as two-dimensional physical interfaces across which the material constitutive properties and electromagnetic fields become discontinuous. Since the fundamental structure of all materials is molecular and electronic one may consider these physical interfaces to be endowed with active or passive electromagnetic properties described classically by constitutive properties that are distinct from those of the media that they separate. Furthermore, with the recent rapid advances in meta-surface technology one may contemplate

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systems with artificially fabricated meta-surfaces that offer novel possibilities for controlling the behaviour of electromagnetic fields in new meta-materials.

There exists a vast literature on recent technological developments in this field. In particular the search for an effective control of surface polariton excitations in various meta-structures remains an active area of current experimental endeavour (Raether, 1988; Pitarke et al, 2006; Sarid and Challener, 2010; Maier, 2007). The interesting properties of meta-surfaces with a tuneable surface impedance have been discussed in Zhu et al (2014); Nemilentsau et al (2016) and the recent discovery of the novel properties of graphene lends new impetus to exploring surface excitations in meta-structures involving this material (Vakil and Engheta, 2011; Gusynin et al, 2009; Sounas and Caloz, 2011, 2012).

Theoretical approaches to the behaviour of meta-surfaces often rely on simplified models that are bench-marked against various numerical computer codes while descriptions involving Greens' functions inevitably lead to various approximation schemes (Hanson, 2008). In this article we outline an approach based on an exact analysis of Maxwell's equations in media with interfaces with particular attention devoted to developing tools for finding solutions induced by the presence of a single planar-interface with an intrinsic frequency dependent complex rank (1,1) surface admittance tensor. Such a tensor can be employed to invoke surface currents from surface electromagnetic fields. The formulation of the entire theory benefits from the use of exterior differential forms in a 3-dimensional Euclidean space.

In Sect. 8.2 some mathematical preliminaries and notations are given for those unfamiliar with the geometrical language of differential forms. Further introductory information can be found in Benn and Tucker (1987); Burton (2003). It also introduces a complex extension of Rodrigues formula (Cheng and Gupta, 1989) for describing complex rotations that is used extensively in subsequent sections. Sections 8.3 and 8.4 use this language in the formulation of Maxwell's equations for fields in any regular domain of a medium free of interfaces. Section 8.5 introduces complex field structures based on Clemmow's plane-wave representation (Clemmow, 1966) and Sect. 8.6 indicates how a general family of solutions can be constructed in local regions (without interfaces) using elements from the complex rotation group SO(3,$\mathbb{C}$). In Sect. 8.7 interface conditions are given that must be satisfied in order that solutions can be constructed describing electromagnetic fields in a piecewise-homogeneous material system containing a planar-interface possessing a complex admittance tensor. In Sect. 8.8 these interface conditions are reduced to a set involving *complex rotated amplitudes*. This set facilitates their reduction in Sect. 8.9 to the solution of certain polynomials with complex coefficients. Furthermore, certain roots of these polynomials are then shown to lead to electromagnetic field solutions that can be identified with *surface polariton or bulk Brewster mode configurations* in particular materials possessing a complex interface admittance tensor. In particular when its components constitute a 2×2 Hermitian matrix the interface can sustain an anisotropic Ohmic surface current.

8.2 Mathematical Preliminaries

Our formulation is in terms of time or frequency dependent tensors on $\mathbb{R}^3$ endowed with the three-dimensional Euclidean metric

$$g = \sum_{a=1}^{3} e^a \otimes e^a,$$

and inverse metric

$$G = \sum_{a=1}^{3} X_a \otimes X_a$$

where $e^a(X_b) = \delta_a^b$ with $\delta_a^b = 1$ when $a = b = 1,2,3$ and zero otherwise.

For $\mathscr{U} \subset \mathbb{R}^3$, define $\Gamma T_s^r \mathscr{U}$ as the space of complex, time-dependent r (contravariant), s (covariant) type tensors on $\mathscr{U}$ and $\Gamma \Lambda^p \mathscr{U}$ as the space of complex, time-dependent p-forms on $\mathscr{U}$.

A 1-form $\gamma \in \Gamma \Lambda^1 \mathscr{U}$ which is the metric-dual of a vector field $X \in \Gamma T \mathscr{U}$ may be written $\gamma = \widetilde{X} \equiv g(X, -)$. Equivalently, one may write $X = \widetilde{\gamma} \equiv G(\gamma, -)$.[1]

For $\Phi \in \Gamma \Lambda^p \mathscr{U}$, $\Psi \in \Gamma \Lambda^q \mathscr{U}$ and $f \in \Gamma \Lambda^0 \mathscr{U}$, define the exterior derivative $\mathrm{d} : \Gamma \Lambda^p \mathscr{U} \to \Gamma \Lambda^{p+1} \mathscr{U}$ with the properties

$$\mathrm{d}f(X) = Xf, \qquad \mathrm{d}(\Phi \wedge \Psi) = \mathrm{d}\Phi \wedge \Psi + (-1)^p \Phi \wedge \mathrm{d}\Psi, \qquad \mathrm{d} \circ \mathrm{d} = 0 \tag{8.1}$$

and the linear interior contraction operator $\mathrm{i}_X : \Gamma \Lambda^p \mathscr{U} \to \Gamma \Lambda^{p-1} \mathscr{U}$ with the properties

$$\mathrm{i}_X \gamma = \gamma(X), \ \mathrm{i}_X f = 0, \ \mathrm{i}_X(\Phi \wedge \Psi) = \mathrm{i}_X \Phi \wedge \Psi + (-1)^p \Phi \wedge \mathrm{i}_X \Psi, \ \mathrm{i}_X \circ \mathrm{i}_X = 0. \tag{8.2}$$

In these relations, the exterior product $\Psi \wedge \Phi$ satisfies $\Psi \wedge \Phi = (-1)^{pq} \Phi \wedge \Psi$. Furthermore, associated with the metric g one may define the *linear* Hodge map $\# : \Gamma \Lambda^p \mathscr{U} \to \Gamma \Lambda^{3-p} \mathscr{U}$ with the properties[2]

$$\#(\Phi \wedge \gamma) = \mathrm{i}_{\widetilde{\gamma}} \# \Phi, \qquad \#1 = e^1 \wedge e^2 \wedge e^3, \qquad \# \circ \# = \mathbb{1} \tag{8.3}$$

where $\mathbb{1} \in T_1^1 \mathbb{R}^3$ is the unit (1,1) tensor on $\mathbb{R}^3$. From (8.2), one obtains the useful result that if Ψ is any decomposable p-form containing the 1-form γ, then

$$\mathrm{i}_{\widetilde{\gamma}} \# \Psi = 0. \tag{8.4}$$

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[1] In a Cartesian coordinate system (x,y,z) for $\mathbb{R}^3$, a global co-frame $e^1 \equiv \mathrm{d}x$, $e^2 \equiv \mathrm{d}y$, $e^3 \equiv \mathrm{d}z$, and $\widetilde{\mathrm{d}x} = \partial_x$, $\widetilde{\mathrm{d}y} = \partial_y$ and $\widetilde{\mathrm{d}z} = \partial_z$. Similarly, $\widetilde{\partial_x} = \mathrm{d}x$, $\widetilde{\partial_y} = \mathrm{d}y$ and $\widetilde{\partial_z} = \mathrm{d}z$.

[2] A p-form is said to be *decomposable* if it can be written as the exterior product of p 1-forms. Then (8.3) is sufficient to define the Hodge map on any decomposable p-form since recursive application gives $\#(\gamma^1 \wedge \cdots \wedge \gamma^p) = \mathrm{i}_{\widetilde{\gamma^p}} \ldots \mathrm{i}_{\widetilde{\gamma^1}} \#1$ for $\gamma^1, \ldots, \gamma^p \in \Gamma \Lambda^1 \mathscr{U}$, and its action on a non-decomposable p-form follows by linearity. In a Cartesian coframe, $\#1 = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, and (8.3) yields $\#\mathrm{d}x = \mathrm{d}y \wedge \mathrm{d}z$, $\#\mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}x$ and $\#\mathrm{d}z = \mathrm{d}x \wedge \mathrm{d}y$.

For $\mathscr{U} \subset \mathbb{R}^3$ and any *unit norm real* vector field $N \in \Gamma T\mathscr{U}$ satisfying $g(N,N) = 1$, we define the normal and tangential projection operators $\mathfrak{n}_N, \mathfrak{t}_N$ and the *tangential Hodge map* $\#_N$ as

$$\mathfrak{n}_N : \Gamma\Lambda^p\mathscr{U} \to \Gamma\Lambda^p\mathscr{U}, \quad \xi \to \mathfrak{n}_N\xi \equiv \tilde{N} \wedge \mathrm{i}_N\xi \tag{8.5}$$

$$\mathfrak{t}_N : \Gamma\Lambda^p\mathscr{U} \to \Gamma\Lambda^p\mathscr{U}, \quad \xi \to \mathfrak{t}_N\xi \equiv \xi - \mathfrak{n}_N\xi = \mathrm{i}_N\left(\tilde{N} \wedge \xi\right) \tag{8.6}$$

where $p = 0,1,2,3$ and

$$\#_N : \Gamma\Lambda^p\mathscr{U} \to \Gamma\Lambda^{2-p}\mathscr{U}, \quad \xi \to \#_N\xi = (-1)^p \mathrm{i}_N \#\xi \equiv \#\left(\tilde{N} \wedge \xi\right), \tag{8.7}$$

where $p = 0,1,2$. Since N has unit norm, $\mathfrak{n}_N\tilde{N} = \tilde{N}$, $\mathfrak{t}_N\tilde{N} = 0$ and one has the operator relations:

$$\mathfrak{n}_N \circ \mathfrak{n}_N = \mathfrak{n}_N, \qquad \mathfrak{t}_N \circ \mathfrak{t}_N = \mathfrak{t}_N, \tag{8.8}$$

$$\mathfrak{n}_N \circ \mathfrak{t}_N = \mathfrak{t}_N \circ \mathfrak{n}_N = 0, \tag{8.9}$$

$$\mathfrak{n}_N \circ \# = \# \circ \mathfrak{t}_N, \qquad \mathfrak{t}_N \circ \# = \# \circ \mathfrak{n}_N \tag{8.10}$$

$$\mathfrak{n}_N \circ \#_N = \#_N \circ \mathfrak{n}_N = 0, \tag{8.11}$$

$$\mathfrak{t}_N \circ \#_N = \#_N \circ \mathfrak{t}_N = \#_N, \tag{8.12}$$

$$\#_N \circ \#_N = \mathfrak{t}_N \circ \eta, \tag{8.13}$$

$$\mathrm{i}_N \circ \mathfrak{n}_N = \mathrm{i}_N, \quad \mathrm{i}_N \circ \mathfrak{t}_N = \mathrm{i}_N \circ \#_N = 0, \tag{8.14}$$

$$\#_N \circ \mathrm{i}_N = \# \circ \mathfrak{n}_N = \mathfrak{t}_N \circ \#, \tag{8.15}$$

$$\# = \#_N \circ \mathrm{i}_N + \tilde{N} \wedge \#_N \circ \eta, \tag{8.16}$$

where $\eta(\Phi) \equiv (-1)^p(\Phi)$. Furthermore, for $\alpha, \beta \in \Gamma\Lambda^1\mathscr{U}$,

$$\mathfrak{n}_N(\alpha \wedge \beta) = \mathfrak{n}_N\alpha \wedge \beta + \alpha \wedge \mathfrak{n}_N\beta, \tag{8.17}$$

$$\mathfrak{t}_N(\alpha \wedge \beta) = \mathfrak{t}_N\alpha \wedge \mathfrak{t}_N\beta = \#_N\alpha \wedge \#_N\beta \tag{8.18}$$

$$\#_N(\alpha \wedge \beta) = \#_N(\#_N\alpha \wedge \#_N\beta) = G(\#_N\alpha, \beta) = -G(\alpha, \#_N\beta). \tag{8.19}$$

Let φ denote a *complex* angle and introduce the complex rotation operator[3]

$$R_N(\varphi) : \Gamma\Lambda^1\mathscr{U} \to \Gamma\Lambda^1\mathscr{U} : \quad \alpha \to R_N(\varphi)(\alpha) \equiv \mathfrak{n}_N\alpha + \cos\varphi\,\mathfrak{t}_N\alpha - \sin\varphi\,\#_N\alpha. \tag{8.20}$$

From (8.8)-(8.14), one has the relations

$$R_N(\varphi)\tilde{N} = \tilde{N} \tag{8.21}$$

$$\mathrm{i}_N \circ R_N(\varphi) = \mathrm{i}_N \tag{8.22}$$

$$R_N(\varphi) \circ \mathfrak{n}_N = \mathfrak{n}_N \circ R_N(\varphi) = \mathfrak{n}_N, \tag{8.23}$$

$$R_N(\varphi) \circ \mathfrak{t}_N = \mathfrak{t}_N \circ R_N(\varphi) \tag{8.24}$$

[3] This is a complex extension of the Rodrigues rotation formula.

i.e. the operator R_N commutes with the projectors $\mathfrak{t}_N$ and $\mathfrak{n}_N$. For a pair of complex angles φ_1 and φ_2, one has

$$R_N(\varphi_1+\varphi_2) = R_N(\varphi_1)\circ R_N(\varphi_2), \tag{8.25}$$

and hence

$$R_N(-\varphi)\circ R_N(\varphi) = R_N(0) = \mathbb{1}. \tag{8.26}$$

Identity (8.9) and definition (8.20) imply that

$$R_N\left(-\frac{\pi}{2}\right)\circ \mathfrak{t}_N = \#_N, \tag{8.27}$$

so that

$$R_N(\varphi)\circ\#_N = \#_N\circ R_N(\varphi) = R_N(\varphi-\frac{\pi}{2})\circ\mathfrak{t}_N = \mathfrak{t}_N\circ R_N(\varphi-\frac{\pi}{2}). \tag{8.28}$$

Using (8.18) and (8.19), it can be shown that

$$G(R_N(\varphi)(\alpha), R_N(\varphi)(\beta)) = G(\alpha,\beta), \tag{8.29}$$

$$\#_N(R_N(\varphi)(\alpha)\wedge R_N(\varphi)(\beta)) = \#_N(\alpha\wedge\beta) \tag{8.30}$$

i.e. the operator R_N is an isometry on the space of 1-forms on $\mathscr{U}$. Finally, since (8.16), (8.25), (8.28) and (8.30) give

$$\begin{aligned}
&(R_N(-\varphi)\circ\#)\,(R_N(\varphi)(\alpha)\wedge R_N(\varphi)(\beta)) \\
&= R_N(-\varphi)\Big[(\mathrm{i}_N(\alpha))(\#_N\circ R_N(\varphi))(\beta) - (\mathrm{i}_N\beta)(\#_N\circ R_N(\varphi))(\alpha) \\
&\qquad + \#_N\,(R_N(\varphi)(\alpha)\wedge R_N(\varphi)(\beta))\widetilde{N}\Big] \\
&= R_N(-\varphi)\,\Big[(\mathrm{i}_N\alpha)R_N(\varphi)(\#_N\beta) - (\mathrm{i}_N\beta)R_N(\varphi)(\#_N\alpha) + \#_N\,(\alpha\wedge\beta)\widetilde{N}\Big] \\
&= (\mathrm{i}_N\alpha)\#_N\beta - (\mathrm{i}_N\beta)\#_N\alpha + \#_N\,(\alpha\wedge\beta)\widetilde{N} \\
&= \#\,(\alpha\wedge\beta),
\end{aligned}$$

one has the useful identity[4]

$$R_N(\varphi)\#(\alpha\wedge\beta) = \#(R_N(\varphi)\alpha\wedge R_N(\varphi)\beta). \tag{8.31}$$

8.3 General Maxwell Equations and their Fourier Transform

In terms of the electric field $\boldsymbol{e}\in\Gamma\Lambda^1\mathscr{U}$, electric displacement $\boldsymbol{d}\in\Gamma\Lambda^1\mathscr{U}$, magnetic flux density $\boldsymbol{b}\in\Gamma\Lambda^1\mathscr{U}$, magnetic field $\boldsymbol{h}\in\Gamma\Lambda^1\mathscr{U}$, *total* free current $\boldsymbol{j}\in\Gamma\Lambda^1\mathscr{U}$

[4] Analogous to the preservation of the real angle between two vectors under a Rodrigues' real rotation.

and *total* free charge density $\rho \in \Gamma\Lambda^0\mathscr{U}$, these time ($t$)-dependent fields satisfy the Maxwell system:

$$\#\mathrm{d}\boldsymbol{e} + \frac{\partial}{\partial t}\boldsymbol{b} = 0, \tag{8.32}$$

$$\#\mathrm{d}\boldsymbol{h} - \frac{\partial}{\partial t}\boldsymbol{d} - \boldsymbol{j} = 0, \tag{8.33}$$

$$\mathrm{d}\#\boldsymbol{b} = 0, \tag{8.34}$$

$$\#\mathrm{d}\#\boldsymbol{d} - \rho = 0, \tag{8.35}$$

provided

$$\#\mathrm{d}\#\boldsymbol{j} + \partial_t\rho = 0.$$

This system is assumed closed with the addition of (possibly nonlocal and nonlinear) constitutive relations correlating $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{h}$ and $\boldsymbol{j}$ on $\mathscr{U}$.

If a t-parametrised (r,s)-type tensor $T \in \Gamma T^r_s\mathscr{U}$ in a t-independent tensor-basis can be related to an ω-parametrised tensor $\widehat{T} \in \widehat{\Gamma} T^r_s\mathscr{U}$ by the Fourier transform:

$$T(x,y,z,t) = \int_{-\infty}^{\infty} \widehat{T}(x,y,z,\omega)e^{-i\omega t}\mathrm{d}\omega, \tag{8.36}$$

then

$$\widehat{T}(x,y,z,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} T(x,y,z,t)e^{i\omega t}\mathrm{d}t \qquad \text{where } \omega \in \mathbb{R}. \tag{8.37}$$

Similarly, $\widehat{\Gamma}\Lambda^p\mathscr{U}$ is the space of complex, ω-dependent Fourier transformed p-forms on $\mathscr{U}$. The complex ω-domain Maxwell equations then satisfy

$$\#\mathrm{d}\widehat{\boldsymbol{e}} - i\omega\widehat{\boldsymbol{b}} = 0, \tag{8.38}$$

$$\#\mathrm{d}\widehat{\boldsymbol{h}} + i\omega\widehat{\boldsymbol{d}} - \widehat{\boldsymbol{j}} = 0, \tag{8.39}$$

$$\mathrm{d}\#\widehat{\boldsymbol{b}} = 0, \tag{8.40}$$

$$\#\mathrm{d}\#\widehat{\boldsymbol{d}} - \widehat{\rho} = 0 \tag{8.41}$$

together with the Fourier transformed constitutive relations on $\mathscr{U}$. Since

$$\mathrm{d}\#\widehat{\boldsymbol{b}} = -\frac{i}{\omega}\mathrm{d}\#\#\mathrm{d}\widehat{\boldsymbol{e}} = -\frac{i}{\omega}\mathrm{d}(\mathrm{d}\widehat{\boldsymbol{e}}) = 0,$$

(8.38) implies that equation (8.40) is satisfied automatically for $\omega \neq 0$.

8.4 Maxwell Equations in a Source-Free Domain of an Ohmic, Homogeneous, Isotropic, Dispersive, Linear Medium

If $\mathscr{U}$ contains a simple *linear* medium, the fields $\widehat{\boldsymbol{e}}$, $\widehat{\boldsymbol{d}}$, $\widehat{\boldsymbol{b}}$ and $\widehat{\boldsymbol{h}}$ are related by the following linear constitutive relations

$$\widehat{\boldsymbol{d}} = \widehat{\boldsymbol{\epsilon}}(\widehat{\boldsymbol{e}}), \qquad \widehat{\boldsymbol{b}} = \widehat{\boldsymbol{\mu}}(\widehat{\boldsymbol{h}}), \tag{8.42}$$

where $\widehat{\boldsymbol{\epsilon}}, \widehat{\boldsymbol{\mu}} \in \widehat{\Gamma T_1^1}\mathscr{U}$ are respectively complex dispersive permittivity and permeability tensors.

If $\mathscr{U}$ contains a stationary, isothermal, conducting medium, a contribution to the total current 1-form $\widehat{\boldsymbol{j}}$ may depend on the local electromagnetic fields in $\mathscr{U}$. The medium is said to be an Ohmic conductor if $\widehat{\boldsymbol{j}}$ contains a contribution $\widehat{\boldsymbol{\sigma}}(\widehat{\boldsymbol{e}})$ where $\widehat{\boldsymbol{\sigma}} \in \widehat{\Gamma T_1^1}\mathscr{U}$ denotes the Hermitian conductivity tensor of the medium. In an *isotropic, homogeneous* medium, one also has $\widehat{\boldsymbol{\sigma}} = \widehat{\sigma}\mathbb{1}$ where $\mathrm{d}\widehat{\sigma} = 0$. When $\widehat{\boldsymbol{\sigma}} \neq 0$, the *bulk* medium is dissipative. In the following, we assume all bulk regions without interfaces to be homogeneous and isotropic. Thus,

$$\widehat{\boldsymbol{\epsilon}} = \widehat{\epsilon}\mathbb{1}, \qquad \widehat{\boldsymbol{\mu}} = \widehat{\mu}\mathbb{1}, \tag{8.43}$$

where $\widehat{\epsilon}$, $\widehat{\mu}$ are frequency-dependent complex 0-forms satisfying $\mathrm{d}\widehat{\epsilon} = \mathrm{d}\widehat{\mu} = 0$ on $\mathscr{U}$. More generally, any form $\widehat{\alpha} \in \widehat{\Gamma\Lambda^p}\mathscr{U}$ is said to be *closed* on $\mathscr{U}$ if $\mathrm{d}\widehat{\alpha} = 0$. When $\widehat{\boldsymbol{\epsilon}} = \epsilon_0\boldsymbol{\kappa}_r$ or $\widehat{\boldsymbol{\mu}} = \mu_0\boldsymbol{\eta}_r$ where $\boldsymbol{\kappa}_r$ and $\boldsymbol{\eta}_r$ are (possibly complex) dimensionless (1,1) tensors *independent of frequency*, their Fourier transforms do not exist as smooth tensors. In these circumstances, we write $\boldsymbol{\epsilon} \equiv \epsilon_0\boldsymbol{\kappa}_r$, $\boldsymbol{\mu} \equiv \mu_0\boldsymbol{\eta}_r$ in terms of the permittivity ϵ_0 and the permeability μ_0 of free space.

We henceforth assume that the isotropic, homogeneous medium $\mathscr{U}$ is also *source-free*, so that $\widehat{\boldsymbol{j}} = \widehat{\sigma}\widehat{\boldsymbol{e}}$ and the Maxwell equations (8.38)-(8.39) may then be rewritten

$$\widehat{\boldsymbol{h}} = -\frac{i}{\omega\widehat{\mu}}\#\mathrm{d}\widehat{\boldsymbol{e}}, \qquad \widehat{\boldsymbol{e}} = \frac{i}{\omega\widehat{\epsilon'}}\#\mathrm{d}\widehat{\boldsymbol{h}}. \tag{8.44}$$

where

$$\widehat{\epsilon'} \equiv \widehat{\epsilon} + \frac{i\widehat{\sigma}}{\omega}.$$

Since

$$\widehat{\rho} = \widehat{\epsilon}\#\mathrm{d}\#\widehat{\boldsymbol{e}} = -\frac{\widehat{\epsilon}}{i\omega\widehat{\epsilon'}}\#\mathrm{d}\#(\#\mathrm{d}\widehat{\boldsymbol{h}}) = -\frac{\widehat{\epsilon}}{i\omega\widehat{\epsilon'}}\#\mathrm{d}(\mathrm{d}\widehat{\boldsymbol{h}}) = 0,$$

the remaining independent Maxwell equation (8.41) reduces to $\widehat{\rho} = 0$, which is consistent with $\#\mathrm{d}\#\widehat{\boldsymbol{j}} = 0$.

8.5 Electromagnetic Fields in a Source-Free Domain of an Ohmic, Homogeneous, Isotropic, Dispersive, Linear Medium

In the following, we explore particular real solutions to equations (8.44) from the particular *complex* forms

$$\widehat{\boldsymbol{e}} = \widehat{\mathscr{E}} e^{i\widehat{\chi}}, \qquad \widehat{\boldsymbol{h}} = \widehat{\mathscr{H}} e^{i\widehat{\chi}}, \tag{8.45}$$

where $\widehat{\mathscr{E}}$ is a closed complex electric polarisation 1-form on $\mathscr{U}$, $\widehat{\mathscr{H}}$ is a closed complex magnetic polarisation 1-form on $\mathscr{U}$ and $\widehat{\chi}$ is a complex 0-form. Substituting (8.45) into (8.44) and recalling that $\widehat{\mathscr{E}}$ and $\widehat{\mathscr{H}}$ are closed yields

$$\widehat{\mathscr{H}} = \frac{1}{\omega\widehat{\mu}} \#(\widehat{K} \wedge \widehat{\mathscr{E}}), \qquad \widehat{\mathscr{E}} = -\frac{1}{\omega\widehat{\epsilon'}} \#(\widehat{K} \wedge \widehat{\mathscr{H}}), \tag{8.46}$$

where $\widehat{K} = \mathrm{d}\widehat{\chi}$ is a complex propagation 1-form. Using (8.3) and (8.4), it follows from (8.46) that the 1-forms $\widehat{\mathscr{E}}$, $\widehat{\mathscr{H}}$ and $\widehat{K}$ must form a mutually orthogonal triplet, i.e.

$$G(\widehat{\mathscr{E}}, \widehat{\mathscr{H}}) = G(\widehat{K}, \widehat{\mathscr{H}}) = G(\widehat{K}, \widehat{\mathscr{E}}) = 0. \tag{8.47}$$

Eliminating $\widehat{\mathscr{H}}$ from (8.46) gives

$$\widehat{\mathscr{E}} = -\frac{1}{\omega^2 \widehat{\epsilon'} \widehat{\mu}} \#(\widehat{K} \wedge \#(\widehat{K} \wedge \widehat{\mathscr{E}})) = \frac{G(\widehat{K}, \widehat{K})}{\omega^2 \widehat{\epsilon'} \widehat{\mu}} \widehat{\mathscr{E}}. \tag{8.48}$$

since (using (8.3))

$$-\#(\widehat{K} \wedge \#(\widehat{K} \wedge \widehat{\mathscr{E}})) = \#(\#(\widehat{K} \wedge \widehat{\mathscr{E}}) \wedge \widehat{K}) = \mathrm{i}_{\widetilde{\widehat{K}}} \#\#(\widehat{K} \wedge \widehat{\mathscr{E}}) = \mathrm{i}_{\widetilde{\widehat{K}}} (\widehat{K} \wedge \widehat{\mathscr{E}}) = \left(\mathrm{i}_{\widetilde{\widehat{K}}} \widehat{K}\right) \widehat{\mathscr{E}}$$

as

$$\mathrm{i}_{\widetilde{\widehat{K}}} \widehat{\mathscr{E}} = G(\widehat{K}, \widehat{\mathscr{E}}) = 0.$$

Therefore, for nontrivial electromagnetic fields, one requires

$$G(\widehat{K}, \widehat{K}) = \omega^2 \widehat{\epsilon'} \widehat{\mu} = \frac{\omega^2 \widehat{\epsilon'}_r \widehat{\mu}_r}{c^2}, \tag{8.49}$$

where

$$\widehat{\epsilon'}_r = \frac{\widehat{\epsilon'}}{\epsilon_0}, \quad \widehat{\mu}_r = \frac{\widehat{\mu}}{\widehat{\mu}_0}$$

and

$$c = \sqrt{\mu_0 \epsilon_0}$$

is the speed of light in vacuo.

In summary, a 0-form $\widehat{\chi}$ and 1-form $\widehat{\mathscr{E}}$ which satisfy (8.49) and (8.47) may be used to construct a complex Maxwell solution of the form (8.45), where the magnetic field amplitude

$$\widehat{\mathscr{H}} = \frac{1}{\omega\widehat{\mu}}\#(\mathrm{d}\widehat{\chi} \wedge \widehat{\mathscr{E}}), \tag{8.50}$$

provided $\mathrm{d}\widehat{\mathscr{E}} = \mathrm{d}\widehat{\mathscr{H}} = 0$. In the next section, we indicate the form of the scalar field $\widehat{\chi}$ which leads to a class of plane-fronted harmonic solutions.

8.6 Plane Wave Solutions in Terms of the Complex Rotation Group

If the Cartesian components of the vector field $\widetilde{\widehat{K}}$ were real constants (and $\widehat{\boldsymbol{j}} = 0$), the fields (8.45) would give rise to a family of plane harmonic electromagnetic waves with polarisation vectors orthogonal to the direction of propagation $\widetilde{\widehat{K}}$.

In a homogeneous medium, the only constraint on $\widehat{K}$ would be that it has modulus $\omega\sqrt{\widehat{\epsilon}\widehat{\mu}}$. Thus, all such plane wave solutions could be related to each other by an $SO(3,\mathbb{R})$ group action. In the presence of a meta-material with non-zero complex conductivity (leading to possible amplification and attenuation of electromagnetic waves) one expects that the group $SO(3,\mathbb{C})$ should play a role in relating solutions that propagate with attenuation (or amplification)[5]. In this section, we approach a representation of $SO(3,\mathbb{C})$ generated from a group action on differential 1-forms. This will then yield a construction of the fields (8.45) in terms of particular elements $\{\widehat{K}, \widehat{\mathscr{E}}, \widehat{\mathscr{H}}\}$ "rotated" by elements of $SO(3,\mathbb{C})$, satisfying the conditions (8.49) and (8.47), and maintaining the *relation* (8.50), thereby automatically satisfying the Maxwell system on $\mathscr{U}$.

Given a g-orthonormal triplet $\{N_1, N_2, N_3\}$ of vector fields on $\mathscr{U}$ and a triplet $\tau = \{\widehat{\psi}, \widehat{\theta}, \widehat{\phi}\}$ of complex functions of ω, use (8.20) to construct a three-complex-parameter linear map

$$\mathscr{R}_\tau : \widehat{\Gamma}\Lambda^1\mathscr{U} \to \widehat{\Gamma}\Lambda^1\mathscr{U}, \quad \widehat{\alpha} \to \mathscr{R}_\tau\widehat{\alpha} \equiv \left(R_{N_3}(\widehat{\psi}) \circ R_{N_2}(\widehat{\theta}) \circ R_{N_1}(\widehat{\phi})\right)(\widehat{\alpha}), \tag{8.51}$$

such that $\widehat{\psi}$, $\widehat{\theta}$, $\widehat{\phi}$ locally parametrise $SO(3,\mathbb{C})$. Repeated application of identity (8.29) gives

$$G(\mathscr{R}_\tau(\widehat{\alpha}), \mathscr{R}_\tau(\widehat{\beta})) = G(\widehat{\alpha}, \widehat{\beta}), \tag{8.52}$$

i.e., the operator $\mathscr{R}_\tau$ is an isometry on the space of 1-forms on $\mathscr{U}$.

A particularly simple solution satisfying the system (8.44) is

$$\widehat{\boldsymbol{e}}_{\tau_0} = \widehat{\mathscr{E}}_{\tau_0} e^{i\widehat{\chi}_{\tau_0}}, \qquad \widehat{\boldsymbol{h}}_{\tau_0} = \widehat{\mathscr{H}}_{\tau_0} e^{i\widehat{\chi}_{\tau_0}}, \tag{8.53}$$

where

[5] The elements of $SO(3,\mathbb{C})$ connected to the identity element are more familiar as elements of the Lorentz group $SO(3,1,\mathbb{R})$ after parametrisation in terms of real group coordinates.

$$\widehat{K}_{\tau_0} = \mathrm{d}\widehat{\chi}_{\tau_0} = \omega\sqrt{\widehat{\epsilon'}\widehat{\mu}}\,\mathrm{d}x, \quad \widehat{\mathscr{E}}_{\tau_0} = \widehat{A}\mathrm{d}y, \quad \widehat{\mathscr{H}} = \widehat{A}\sqrt{\frac{\widehat{\epsilon'}}{\widehat{\mu}}}\mathrm{d}z \tag{8.54}$$

in a global Cartesian co-frame, where $\widehat{A}$ is a complex constant.

From this solution, one may use a generic element $\mathscr{R}_\tau$ of $SO(3,\mathbb{C})$ to generate an orbit of solutions:

$$\widehat{\boldsymbol{e}}_\tau = \mathscr{R}_\tau \widehat{\mathscr{E}}_{\tau_0} e^{i\widehat{\chi}_\tau}, \qquad \widehat{\boldsymbol{h}}_\tau = \mathscr{R}_\tau \widehat{\mathscr{H}_\tau} e^{i\widehat{\chi}_{\tau_0}}, \tag{8.55}$$

with

$$\widehat{K}_\tau \equiv \mathrm{d}\widehat{\chi}_\tau = \mathscr{R}_\tau \widehat{K}_{\tau_0} = \widehat{K}_{\tau,x}\mathrm{d}x + \widehat{K}_{\tau,y}\mathrm{d}y + \widehat{K}_{\tau,z}\mathrm{d}z \tag{8.56}$$

and

$$\widehat{\chi}_\tau \equiv \widehat{K}_{\tau,x}x + \widehat{K}_{\tau,y}y + \widehat{K}_{\tau,z}z = G\left(\widehat{K}_\tau,\ x\mathrm{d}x + y\mathrm{d}y + z\mathrm{d}z\right), \tag{8.57}$$

since (8.52) implies

$$G(\widehat{K}_\tau, \widehat{K}_\tau) = G(\widehat{K}_{\tau_0}, \widehat{K}_{\tau_0}) = \frac{\omega^2 \widehat{\epsilon'}_r \widehat{\mu}_r}{c^2}, \tag{8.58}$$

$$G(\widehat{K}_\tau, \widehat{\mathscr{E}}_\tau) = G(\widehat{K}_{\tau_0}, \widehat{\mathscr{E}}_{\tau_0}) = 0, \tag{8.59}$$

and repeated application of (8.31) yields

$$\widehat{\mathscr{H}_\tau} = \frac{1}{\omega\widehat{\mu}}\#\left(\widehat{K}_\tau \wedge \widehat{\mathscr{E}}_\tau\right) = \mathscr{R}_\tau \widehat{\mathscr{H}_{\tau_0}} \tag{8.60}$$

This approach can be extended to situations involving multiple regions with different material properties. The associated field components must then be matched at interfaces between such regions using junction conditions as described in the next section.

8.7 Interface Conditions for Media Containing Anisotropic, Homogeneous, Planar Interface Consitutive Relations

Let $f = 0$ be a particular smooth interface S belonging to a foliation of a region $\mathscr{M} \subset \mathbb{R}^3$ and assign a normal unit vector field

$$N = \frac{\widetilde{\mathrm{d}f}}{|\mathrm{d}f|} \neq 0,$$

where $|\mathrm{d}f|^2 \equiv G(\mathrm{d}f, \mathrm{d}f)$ with N oriented from a region I where $f \leq 0$ to the region II where $f \geq 0$.

If regions I and II in $\mathscr{M}$ contain material with different constitutive properties then the electromagnetic fields in these regions will in general exhibit discontinuities in certain of their components across the interface $f = 0$, corresponding to surface charge and current densities.

The interface conditions for general electric and magnetic 1-forms in the frequency domain are given in this language as:

$$S^* \mathrm{i}_N \widehat{\boldsymbol{b}}^{II} - S^* \mathrm{i}_N \widehat{\boldsymbol{b}}^{I} = 0 \tag{8.61}$$

$$S^* \mathrm{t}_N \widehat{\boldsymbol{e}}^{II} - S^* \mathrm{t}_N \widehat{\boldsymbol{e}}^{I} = 0 \tag{8.62}$$

$$S^* \mathrm{i}_N \widehat{\boldsymbol{d}}^{II} - S^* \mathrm{i}_N \widehat{\boldsymbol{d}}^{I} - \widehat{\rho}_S = 0 \tag{8.63}$$

$$S^* \mathrm{t}_N \widehat{\boldsymbol{h}}^{II} - S^* \mathrm{t}_N \widehat{\boldsymbol{h}}^{I} + \#_N \widehat{\boldsymbol{j}}_S = 0, \tag{8.64}$$

where for any p-form $\widehat{\gamma} \in \widehat{\Gamma} \Lambda^p \mathscr{M}$, $S^* \widehat{\gamma}$ denotes the pullback onto the interface S. Since $\#_N$ maps forms on any domain to their tangential parts with respect to N, they have a natural extension to maps on the pullbacks of such forms to any surface $S \subset \mathscr{M}$ with the local normal N. Furthermore, $\widehat{\rho}_S \in \widehat{\Gamma} \Lambda^0 S$ and $\widehat{\boldsymbol{j}}_S \in \widehat{\Gamma} \Lambda^1 S$ are the surface charge density 0-form and surface current density 1-form on S respectively. In general, the interface surface forms $\widehat{\rho}_S$ and $\widehat{\boldsymbol{j}}_S$ in (8.63)-(8.64) correspond to surface charge and current densities (including those produced by possible external sources of surface charge density, possible external sources of interface current density, and surface electromagnetic fields).

At this point, we assume that the surface charge density $\widehat{\rho}_S$ is determined solely by electromagnetic fields in the *bulk* media and is given by (8.63). Furthermore, we assume that $\widehat{\boldsymbol{j}}_S$ is determined solely by the fields satisfying the interface constitutive relation

$$\widehat{\boldsymbol{j}}_S = \frac{1}{Z_0} \widehat{\boldsymbol{\Sigma}} \left(S^* \mathrm{t}_N \widehat{\boldsymbol{e}}^{I} \right) \equiv \frac{1}{Z_0} \widehat{\boldsymbol{\Sigma}} \left(S^* \mathrm{t}_N \widehat{\boldsymbol{e}}^{II} \right) \tag{8.65}$$

where the rank (1,1) complex *surface admittance* tensor $\widehat{\boldsymbol{\Sigma}} \in \widehat{T}_1^1 S$ is defined to act only on the *tangential* components of the electric field on S and

$$Z_0 = \sqrt{\frac{\epsilon_0}{\mu_0}} = \frac{1}{c\mu_0} = c\epsilon_0$$

is the impedance of free space.

Assume that in a Cartesian coordinate system the surface S, ($z = 0$), separates $\mathscr{M}$ into two semi-infinite volume regions $\mathscr{V}^I$ ($z < 0$) and $\mathscr{V}^{II}$ ($z > 0$). For notational simplicity, denote the unit normal vector field on S, $\partial_z|_S$, by ∂_z. We suppose that the domains $\mathscr{V}^I$ and $\mathscr{V}^{II}$ are filled with *isotropic*, *homogeneous* media with distinct complex permittivity, permeability and conductivity constitutive *scalars* $\{\widehat{\epsilon}^I, \widehat{\mu}^I, \widehat{\sigma}^I, \widehat{\epsilon}^{II}, \widehat{\mu}^{II}, \widehat{\sigma}^{II}\}$.

The electric and magnetic fields are given by

$$\widehat{\boldsymbol{e}}^L = \widehat{\mathscr{E}}^L e^{i\widehat{\chi}^L}, \qquad \widehat{\boldsymbol{h}}^L = \widehat{\mathscr{H}}^L e^{i\widehat{\chi}^L} \tag{8.66}$$

in terms of $\{\widehat{\chi}^L, \widehat{\mathscr{E}}^L, \widehat{\mathscr{H}}^L\}$ where $L \in \{I, II\}$. We refer to these fields as modes in $\mathscr{V} \equiv \mathscr{V}^I \cup \mathscr{V}^{II} \cup S$. The interface conditions (8.61)-(8.64) become

$$\widehat{\mu}^{II} \mathrm{i}_{\partial_z} \widehat{\mathscr{H}}^{II} e^{i\widehat{\chi}_0^{II}} - \widehat{\mu}^I \mathrm{i}_{\partial_z} \widehat{\mathscr{H}}^I e^{i\widehat{\chi}_0^I} = 0, \tag{8.67}$$

$$\mathrm{t}_{\partial_z} \widehat{\mathscr{E}}^{II} e^{i\widehat{\chi}_0^{II}} - \mathrm{t}_{\partial_z} \widehat{\mathscr{E}}^I e^{i\widehat{\chi}_0^I} = 0, \tag{8.68}$$

$$\mathrm{i}_{\partial_z} \widehat{\mathscr{E}}^{II} e^{i\widehat{\chi}_0^{II}} - \mathrm{i}_{\partial_z} \widehat{\mathscr{E}}^I e^{i\widehat{\chi}_0^I} = \widehat{\rho}_S, \tag{8.69}$$

$$\mathrm{t}_{\partial_z} \widehat{\mathscr{H}}^{II} e^{i\widehat{\chi}_0^{II}} - \mathrm{t}_{\partial_z} \widehat{\mathscr{H}}^I e^{i\widehat{\chi}_0^I} = -\#_N \widehat{\boldsymbol{j}}_S, \tag{8.70}$$

where

$$\widehat{\chi}_0^L = S^* \widehat{\chi}^L \tag{8.71}$$

depends only on x and y[6]. The interface conditions (8.67), (8.68) and (8.70) imply correlations amongst $\{\widehat{\chi}^L, \widehat{\mathscr{E}}^L, \widehat{\mathscr{H}}^L\}$ and $\widehat{\boldsymbol{j}}_S$. The remaining interface condition (8.69) defines the charge density $\widehat{\rho}_S \in \widehat{\Gamma} \Lambda^0 S$ that will arise on the interface. For (8.68) and (8.70) to be satisfied at every point on S we require

$$\widehat{\chi}_0^{II} = \widehat{\chi}_0^I \tag{8.72}$$

$$\widehat{\rho}_S = \widehat{\rho}_0 e^{i\widehat{\chi}_0^I} = \widehat{\rho}_0 e^{i\widehat{\chi}_0^{II}} \tag{8.73}$$

$$\widehat{\boldsymbol{j}}_S = \widehat{\mathscr{J}} e^{i\widehat{\chi}_0^I} = \widehat{\mathscr{J}} e^{i\widehat{\chi}_0^{II}} \tag{8.74}$$

where $\widehat{\rho}_0 \in \widehat{\Gamma} \Lambda^0 S$ and $\widehat{\mathscr{J}} \in \widehat{\Gamma} \Lambda^1 S$ denote the Fourier amplitudes of the surface charge 0-form and current density 1-form respectively.

From (8.56) and (8.56), it follows that (8.72) is equivalent to

$$\mathrm{t}_{\partial_z} \widehat{K}^{II} - \mathrm{t}_{\partial_z} \widehat{K}^I = 0. \tag{8.75}$$

The complex exponential terms now factor out of all the interface conditions which then reduce to

$$\widehat{\mu}^{II} \mathrm{i}_{\partial_z} \widehat{\mathscr{H}}^{II} - \widehat{\mu}^I \mathrm{i}_{\partial_z} \widehat{\mathscr{H}}^I = 0, \tag{8.76}$$

$$\mathrm{t}_{\partial_z} \widehat{\mathscr{E}}^{II} - \mathrm{t}_{\partial_z} \widehat{\mathscr{E}}^I = 0, \tag{8.77}$$

$$\mathrm{i}_{\partial_z} \widehat{\mathscr{E}}^{II} - \mathrm{i}_{\partial_z} \widehat{\mathscr{E}}^I = \widehat{\rho}_0, \tag{8.78}$$

$$\mathrm{t}_{\partial_z} \widehat{\mathscr{H}}^{II} - \mathrm{t}_{\partial_z} \widehat{\mathscr{H}}^I = -\#_N \widehat{\mathscr{J}} \tag{8.79}$$

with

$$\widehat{\mathscr{J}} = \frac{1}{Z_0} \widehat{\boldsymbol{\Sigma}} \left(\mathrm{t}_N \widehat{\mathscr{E}}^I \right) = \frac{1}{Z_0} \widehat{\boldsymbol{\Sigma}} \left(\mathrm{t}_N \widehat{\mathscr{E}}^{II} \right). \tag{8.80}$$

A *homogeneous* but *anisotropic* interface *admittance tensor* with components $(\widehat{\sigma}_{xx}, \widehat{\sigma}_{xy}, \widehat{\sigma}_{yx}, \widehat{\sigma}_{yy})$ depending only on ω may be written in terms of the induced, fixed-frame $(\mathrm{d}x, \mathrm{d}y)$ on S as

[6] For the 1-form $\widehat{\gamma} = \widehat{\gamma}_x \mathrm{d}x + \widehat{\gamma}_y \mathrm{d}y + \widehat{\gamma}_z \mathrm{d}z$ where $\widehat{\gamma}_x, \widehat{\gamma}_y, \widehat{\gamma}_z \in \widehat{\Gamma} \Lambda_0 \mathscr{U}$, $\mathrm{i}_{\partial_z} \widehat{\gamma} = \widehat{\gamma}_z$ and $\mathrm{t}_{\partial_z} \widehat{\gamma} = \widehat{\gamma}_x \mathrm{d}x + \widehat{\gamma}_y \mathrm{d}y$.

$$\widehat{\boldsymbol{\Sigma}} = \widehat{\sigma}_{xx}\mathrm{d}x \otimes \mathrm{i}_{\partial_x} + \widehat{\sigma}_{xy}\mathrm{d}x \otimes \mathrm{i}_{\partial_y} + \widehat{\sigma}_{yx}\mathrm{d}y \otimes \mathrm{i}_{\partial_x} + \widehat{\sigma}_{yy}\mathrm{d}y \otimes \mathrm{i}_{\partial_y}. \tag{8.81}$$

The junction condition (8.79) then becomes

$$\mathrm{t}_{\partial_z}\widehat{\mathscr{H}^{II}} - \mathrm{t}_{\partial_z}\widehat{\mathscr{H}^{I}} = -\frac{1}{Z_0}\left(\#_{\partial_z} \circ \widehat{\boldsymbol{\Sigma}} \circ \mathrm{t}_{\partial_z}\right)(\widehat{\mathscr{E}}^{L}), \tag{8.82}$$

where one can choose L to be either I or II. It will be useful in the following to rewrite (8.82) using (8.27) as

$$\mathrm{t}_{\partial_z}\widehat{\mathscr{H}^{II}} - \mathrm{t}_{\partial_z}\widehat{\mathscr{H}^{I}} = -\frac{1}{Z_0}\left(R_{\partial_z}\left(-\frac{\pi}{2}\right) \circ \widehat{\boldsymbol{\Sigma}} \circ \mathrm{t}_{\partial_z}\right)(\widehat{\mathscr{E}}^{L}). \tag{8.83}$$

8.8 Consequences of the Interface Conditions

The structure of the modes in $\mathscr{V} = \mathscr{V}^{I} \cup \mathscr{V}^{II} \cup S$ is given by (8.66) in terms of the specific 1-forms $\{\widehat{K}^{I}, \widehat{\mathscr{E}}^{I}, \widehat{\mathscr{H}^{I}}, \widehat{K}^{II}, \widehat{\mathscr{E}}^{II}, \widehat{\mathscr{H}^{II}}\}$ where $\widehat{K}^{L} = \mathrm{d}\widehat{\chi}^{L}$ for $L \in \{I, II\}$. In each region, $\widehat{\mathscr{E}}^{L}$ and $\widehat{K}^{L}$ must satisfy the (nonlinear) dispersion relation and orthogonality conditions (8.49) and (8.47), as well as the interface conditions (8.75), (8.76), (8.77) and (8.83) derived in Sect. 8.7, with $\widehat{\mathscr{H}^{L}}$ given by (8.50).

The construction of solutions parametrised by elements of $SO(3,\mathbb{C})$ acting on particular solutions in a domain free of interfaces developed in Sect. 8.6 is now extended to the parametrisation of solutions in region $\mathscr{M}$ containing a planar interface, in terms of the six complex angles $\{\widehat{\phi}^{I}, \widehat{\theta}^{I}, \widehat{\psi}^{I}, \widehat{\phi}^{II}, \widehat{\theta}^{II}, \widehat{\psi}^{II}\}$ using composite rotation operators $\mathscr{R}_{\tau^I}$ and $\mathscr{R}_{\tau^{II}}$ by writing

$$\widehat{\gamma}_{\tau^L} = \mathscr{R}_{\tau^L}\widehat{\gamma}^{L}_{\tau_0} \tag{8.84}$$

where

$$\begin{aligned} \widehat{\gamma}_{\tau^L} &\in \{\widehat{K}_{\tau^L}, \widehat{\mathscr{E}}_{\tau^L}, \widehat{\mathscr{H}_{\tau^L}}\}, \\ \widehat{\gamma}_{\tau^L_0} &\in \{\widehat{K}_{\tau^L_0}, \widehat{\mathscr{E}}_{\tau^L_0}, \widehat{\mathscr{H}_{\tau^L_0}}\}, \end{aligned} \tag{8.85}$$

$$\mathscr{R}_{\tau^L} = R_{\partial_z}(\widehat{\psi}^{L}) \circ R_{\partial_y}(\widehat{\theta}^{L}) \circ R_{\partial_x}(\widehat{\phi}^{L}), \tag{8.86}$$

$$\begin{aligned} \widehat{K}^{L}_{\tau_0} &= \widehat{K}^{L}_{0}\mathrm{d}x = \omega\sqrt{\widehat{\epsilon'}^{L}}\sqrt{\widehat{\mu}^{L}}\mathrm{d}x, \\ \widehat{\mathscr{E}}^{L}_{\tau_0} &= \widehat{A}^{L}\mathrm{d}y, \\ \widehat{\mathscr{H}^{L}_{\tau_0}} &= \frac{\widehat{A}^{L}\widehat{K}^{L}_{0}}{\omega\widehat{\mu}^{L}}\mathrm{d}z = \frac{\widehat{A}^{L}\sqrt{\widehat{\epsilon'}^{L}}}{\sqrt{\widehat{\mu}^{L}}}\mathrm{d}z, \end{aligned} \tag{8.87}$$

so that

$$\widehat{K}_{\tau^L} = \widehat{K}_0^L \left[\cos\widehat{\theta}^L \cos\widehat{\psi}^L \mathrm{d}x - \cos\widehat{\theta}^L \sin\widehat{\psi}^L \mathrm{d}y + \sin\widehat{\theta}^L \mathrm{d}z\right] \tag{8.88}$$

$$\begin{aligned}\widehat{\mathscr{E}}_{\tau^L} = \widehat{A}^L \Big[&\left(\cos\widehat{\phi}^L \sin\widehat{\psi}^L + \sin\widehat{\theta}^L \sin\widehat{\phi}^L \cos\widehat{\psi}^L\right) \mathrm{d}x \\ &+ \left(\cos\widehat{\phi}^L \cos\widehat{\psi}^L - \sin\widehat{\theta}^L \sin\widehat{\phi}^L \sin\widehat{\psi}^L\right) \mathrm{d}y - \cos\widehat{\theta}^L \sin\widehat{\phi}^L \mathrm{d}z\Big]\end{aligned} \tag{8.89}$$

$$\begin{aligned}\widehat{\mathscr{H}_{\tau^L}} = \frac{\widehat{A}^L \widehat{K}_0^L}{\omega\widehat{\mu}^L} \Big[&\left(\sin\widehat{\phi}^L \sin\widehat{\psi}^L - \sin\widehat{\theta}^L \cos\widehat{\phi}^L \cos\widehat{\psi}^L\right) \mathrm{d}x \\ &+ \left(\sin\widehat{\phi}^L \cos\widehat{\psi}^L + \sin\widehat{\theta}^L \cos\widehat{\phi}^L \sin\widehat{\psi}^L\right) \mathrm{d}y + \cos\widehat{\theta}^L \cos\widehat{\phi}^L \mathrm{d}z\Big].\end{aligned} \tag{8.90}$$

Such 1-forms already satisfy (8.49), (8.47) and (8.50), so one must only address the interface conditions (8.75), (8.76), (8.77) and (8.83) to obtain a suitably correlated solution in both regions. In terms of $\{\widehat{K}_{\tau^L}^L, \widehat{\mathscr{E}}_{\tau^L}^L, \widehat{\mathscr{H}_{\tau^L}^L}\}$, these interface conditions are

$$\mathrm{t}_{\partial_z}\widehat{K}_{\tau^{II}} - \mathrm{t}_{\partial_z}\widehat{K}_{\tau^I} = 0, \tag{8.91}$$

$$\widehat{\mu}^{II}\, \mathrm{i}_{\partial_z}\widehat{\mathscr{H}_{\tau^{II}}} - \widehat{\mu}^I\, \mathrm{i}_{\partial_z}\widehat{\mathscr{H}_{\tau^I}} = 0, \tag{8.92}$$

$$\mathrm{t}_{\partial_z}\widehat{\mathscr{E}}_{\tau^{II}} - \mathrm{t}_{\partial_z}\widehat{\mathscr{E}}_{\tau^I} = 0, \tag{8.93}$$

$$\mathrm{t}_{\partial_z}\widehat{\mathscr{H}_{\tau^{II}}} - \mathrm{t}_{\partial_z}\widehat{\mathscr{H}_{\tau^I}} + \frac{1}{Z_0}\left(R_{\partial_z}\left(-\frac{\pi}{2}\right) \circ \widehat{\mathbf{\Sigma}} \circ \mathrm{t}_N\right)(\widehat{\mathscr{E}}_{\tau^I}) = 0. \tag{8.94}$$

To decouple the system, it will prove expedient to work with a new set of 1-forms “rotated” with respect to ∂_z by the complex angle $-\widehat{\psi}^I$ using the operator (8.20). In terms of (8.84)-(8.85), define the *rotated forms*

$$\widehat{\gamma}_R^L \equiv R_{\partial_z}(-\widehat{\psi}^I)\widehat{\gamma}_{\tau^L} = \left(R_{\partial_z}(-\widehat{\psi}^I) \circ \mathscr{R}_{\tau^L}\right)\left(\widehat{\gamma}_{\tau_0}^L\right). \tag{8.95}$$

The identity (8.25) then gives

$$\begin{aligned}\widehat{\gamma}_R^I = \left(R_{\partial_z}(-\widehat{\psi}^I) \circ \mathscr{R}_{\tau^I}\right)\left(\widehat{\gamma}_{\tau_0}^I\right) &= \left(R_{\partial_z}(-\widehat{\psi}^I) \circ R_{\partial_z}(\widehat{\psi}^I) \circ R_{\partial_y}(\widehat{\theta}^I) \circ R_{\partial_x}(\widehat{\phi}^I)\right)\left(\widehat{\gamma}_{\tau_0}^I\right) \\ &= \left(R_{\partial_y}(\widehat{\theta}^I) \circ R_{\partial_x}(\widehat{\phi}^I)\right)\left(\widehat{\gamma}_{\tau_0}^I\right)\end{aligned} \tag{8.96}$$

$$\begin{aligned}\widehat{\gamma}_R^{II} = \left(R_{\partial_z}(-\widehat{\psi}^I) \circ \mathscr{R}_{\tau^{II}}\right)\left(\widehat{\gamma}_{\tau_0}^{II}\right) &= \left(R_{\partial_z}(-\widehat{\psi}^I) \circ R_{\partial_z}(\widehat{\psi}^{II}) \circ R_{\partial_y}(\widehat{\theta}^{II}) \circ R_{\partial_x}(\widehat{\phi}^{II})\right)\left(\widehat{\gamma}_{\tau_0}^{II}\right) \\ &= \left(R_{\partial_z}(\widehat{\psi}^\Delta) \circ R_{\partial_y}(\widehat{\theta}^I) \circ R_{\partial_x}(\widehat{\phi}^I)\right)\left(\widehat{\gamma}_{\tau_0}^I\right)\end{aligned} \tag{8.97}$$

where

$$\widehat{\psi}^\Delta \equiv \widehat{\psi}^{II} - \widehat{\psi}^I \tag{8.98}$$

and the system of rotated 1-forms in (8.96)-(8.97) now depends on the *five* complex angles $\{\widehat{\phi}^I, \widehat{\theta}^I, \widehat{\phi}^{II}, \widehat{\theta}^{II}, \widehat{\psi}^\Delta\}$ rather than the original six. Equations (8.88)-(8.90) may be replaced by the partially simplified formulae

$$\widehat{K}_R^I = \widehat{K}_0^I \left[\cos\widehat{\theta}^I \mathrm{d}x + \sin\widehat{\theta}^I \mathrm{d}z\right], \tag{8.99}$$

$$\widehat{\mathscr{E}}_R^I = \widehat{A}^I \left[\sin\widehat{\theta}^I \sin\widehat{\phi}^I \mathrm{d}x + \cos\widehat{\phi}^I \mathrm{d}y - \cos\widehat{\theta}^I \sin\widehat{\phi}^I \mathrm{d}z\right], \tag{8.100}$$

$$\widehat{\mathscr{H}}_R^I = \frac{\widehat{A}^I \widehat{K}_0^I}{\omega\widehat{\mu}^I}\left[-\sin\widehat{\theta}^I \cos\widehat{\phi}^I \mathrm{d}x + \sin\widehat{\phi}^I \mathrm{d}y + \cos\widehat{\theta}^I \cos\widehat{\phi}^I \mathrm{d}z\right], \tag{8.101}$$

$$\widehat{K}_R^{II} = \widehat{K}_0^{II} \left[\cos\widehat{\theta}^{II} \cos\widehat{\psi}^{\Delta} \mathrm{d}x - \cos\widehat{\theta}^{II} \sin\widehat{\psi}^{\Delta} \mathrm{d}y + \sin\widehat{\theta}^{II} \mathrm{d}z\right], \tag{8.102}$$

$$\begin{aligned}\widehat{\mathscr{E}}_R^{II} = \widehat{A}^{II} \Big[&\left(\cos\widehat{\phi}^{II} \sin\widehat{\psi}^{\Delta} + \sin\widehat{\theta}^{II} \sin\widehat{\phi}^{II} \cos\widehat{\psi}^{\Delta}\right)\mathrm{d}x \\ &+ \left(\cos\widehat{\phi}^{II} \cos\widehat{\psi}^{\Delta} - \sin\widehat{\theta}^{II} \sin\widehat{\phi}^{II} \sin\widehat{\psi}^{\Delta}\right)\mathrm{d}y - \cos\widehat{\theta}^{II} \sin\widehat{\phi}^{II} \mathrm{d}z\Big],\end{aligned} \tag{8.103}$$

$$\begin{aligned}\widehat{\mathscr{H}}_R^{II} = \frac{\widehat{A}^{II} \widehat{K}_0^{II}}{\omega\widehat{\mu}^{II}} \Big[&\left(\sin\widehat{\phi}^{II} \sin\widehat{\psi}^{\Delta} - \sin\widehat{\theta}^{II} \cos\widehat{\phi}^{II} \cos\widehat{\psi}^{\Delta}\right)\mathrm{d}x \\ &+ \left(\sin\widehat{\phi}^{II} \cos\widehat{\psi}^{\Delta} + \sin\widehat{\theta}^{II} \cos\widehat{\phi}^{II} \sin\widehat{\psi}^{\Delta}\right)\mathrm{d}y + \cos\widehat{\theta}^{II} \cos\widehat{\phi}^{II} \mathrm{d}z\Big].\end{aligned} \tag{8.104}$$

The interface conditions (8.91)-(8.94) may now be readily expressed in terms of the rotated 1-forms. Condition (8.92) is satisfied if

$$\widehat{\mu}^{II}\, \mathrm{i}_{\partial_z} \widehat{\mathscr{H}}_R^{II} - \widehat{\mu}^I\, \mathrm{i}_{\partial_z} \widehat{\mathscr{H}}_R^I = 0, \tag{8.105}$$

since identity (8.22) gives

$$\mathrm{i}_{\partial_z} \widehat{\mathscr{H}}_R^L = \mathrm{i}_{\partial_z} R_{\partial_z}(-\widehat{\psi}^I)(\widehat{\mathscr{H}}^L) = \mathrm{i}_{\partial_z} \widehat{\mathscr{H}}^L.$$

From the identities (8.24) and (8.25), the relation

$$\mathrm{t}_N(R_{\partial_z}(\varphi)(\widehat{\gamma}_{\tau^{II}})) - \mathrm{t}_N(R_{\partial_z}(\varphi)(\widehat{\gamma}_{\tau^I})) - R_{\partial_z}(\varphi)(\widehat{\beta}) = 0, \tag{8.106}$$

for any 1-form $\widehat{\beta}$ is readily converted by the action of $R_{\partial_z}(-\varphi)$ to the condition

$$\mathrm{t}_N \widehat{\gamma}_{\tau^{II}} - \mathrm{t}_N \widehat{\gamma}_{\tau^I} - \widehat{\beta} = 0. \tag{8.107}$$

Thus conditions (8.91) and (8.93) are satisfied by (8.106) with $\varphi = -\widehat{\psi}^I$ and $\widehat{\beta} = 0$ while condition (8.94) is satisfied with $\varphi = (\pi/2) - \widehat{\psi}^I$ and $\widehat{\beta} = \frac{1}{Z_0}\left(R_N(-\frac{\pi}{2}) \circ \widehat{\boldsymbol{\Sigma}} \circ \mathrm{t}_N\right)\left(\widehat{\mathscr{E}}^L\right)$, thereby replacing these with the new interface conditions:

$$\begin{aligned} 0 &= \mathrm{t}_{\partial_z}\left(R_{\partial_z}(\varphi)(\widehat{\gamma}_{\tau^{II}})\right) - \mathrm{t}_{\partial_z}\left(R_{\partial_z}(\varphi)(\widehat{\gamma}_{\tau^{II}})\right) - R_{\partial_z}(\varphi)(\widehat{\beta}) \\ &= R_{\partial_z}(\varphi)\left(\mathrm{t}_{\partial_z}(\widehat{\gamma}_{\tau^{II}})\right) - R_{\partial_z}(\varphi)\left(\mathrm{t}_{\partial_z}(\widehat{\gamma}_{\tau^{II}})\right) - R_{\partial_z}(\varphi)(\widehat{\beta}) \\ &= R_{\partial_z}(-\varphi)\left(R_{\partial_z}(\varphi)\left(\mathrm{t}_{\partial_z}(\widehat{\gamma}_{\tau^{II}})\right) - R_{\partial_z}(\varphi)\left(\mathrm{t}_{\partial_z}(\widehat{\gamma}_{\tau^{II}})\right) - R_{\partial_z}(\varphi)(\widehat{\beta})\right) \\ &= \mathrm{t}_{\partial_z}\widehat{\gamma}_{\tau^{II}} - \mathrm{t}_{\partial_z}\widehat{\gamma}_{\tau^{II}} - \widehat{\beta}. \end{aligned}$$

Hence, taking $\varphi = -\widehat{\psi}^I$, $\widehat{\beta} = 0$ in (8.75), (8.77) and

$$\widehat{\varphi} = \frac{\pi}{2} - \widehat{\psi}^I, \widehat{\beta} = \frac{1}{Z_0}\left(R_{\partial_z}\left(-\frac{\pi}{2}\right) \circ \widehat{\boldsymbol{\Sigma}} \circ \mathrm{t}_{\partial_z}\right)\left(\widehat{\mathscr{E}}^L\right)$$

in (8.83) gives the new conditions

$$\mathrm{t}_{\partial_z}\widehat{K}_R^{II} - \mathrm{t}_{\partial_z}\widehat{K}_R^{I} = 0, \tag{8.108}$$

$$\mathrm{t}_{\partial_z}\widehat{\mathscr{E}}_R^{II} - \mathrm{t}_{\partial_z}\widehat{\mathscr{E}}_R^{I} = 0, \tag{8.109}$$

$$\left[\mathrm{t}_{\partial_z} \circ R_{\partial_z}\left(\frac{\pi}{2}\right)\right]\left(\widehat{\mathscr{H}}_R^{II} - \widehat{\mathscr{H}}_R^{I}\right) + \frac{1}{Z_0}\left[R_{\partial_z}\left(\frac{\pi}{2} - \widehat{\psi}^I\right) \circ R_{\partial_z}\left(-\frac{\pi}{2}\right) \circ \widehat{\boldsymbol{\Sigma}} \circ \mathrm{t}_{\partial_z}\right]\left(\widehat{\mathscr{E}}^I\right) = 0. \tag{8.110}$$

Using identities (8.25), (8.24) and definition (8.95), (8.110) can be rewritten:

$$\left(\mathrm{t}_{\partial_z} \circ R_{\partial_z}\left(\frac{\pi}{2}\right)\right)\left(\widehat{\mathscr{H}}_R^{II} - \widehat{\mathscr{H}}_R^{I}\right) = -\frac{1}{Z_0}\left(R_{\partial_z}\left(-\widehat{\psi}^I\right) \circ \widehat{\boldsymbol{\Sigma}} \circ \mathrm{t}_{\partial_z}\right)\left(\widehat{\mathscr{E}}^I\right). \tag{8.111}$$

To further simplify (8.111), introduce the rank (1,1) tensor $\widehat{\boldsymbol{\Lambda}} : \widehat{\Gamma}\Lambda^1 S \to \widehat{\Gamma}\Lambda^1 S$ where

$$\begin{aligned}\widehat{\boldsymbol{\Lambda}} &= R_{\partial_z}\left(-\widehat{\psi}^I\right) \circ \widehat{\boldsymbol{\Sigma}} \circ R_{\partial_z}\left(\widehat{\psi}^I\right) \circ \mathrm{t}_{\partial_z} \\ &= \widehat{\lambda}_{xx}\mathrm{d}x \otimes \mathrm{i}_{\partial_x} + \widehat{\lambda}_{xy}\mathrm{d}x \otimes \mathrm{i}_{\partial_y} + \widehat{\lambda}_{yx}\mathrm{d}y \otimes \mathrm{i}_{\partial_x} + \widehat{\lambda}_{yy}\mathrm{d}y \otimes \mathrm{i}_{\partial_y}.\end{aligned} \tag{8.112}$$

After some algebra, it can be shown that the components of the tensor $\widehat{\boldsymbol{\Lambda}}$ are given by

$$\widehat{\lambda}_{xx} = \frac{1}{2}\left[(\widehat{\sigma}_{xx} + \widehat{\sigma}_{yy}) + (\widehat{\sigma}_{xx} - \widehat{\sigma}_{yy})\cos 2\widehat{\psi}^I - (\widehat{\sigma}_{xy} + \widehat{\sigma}_{yx})\sin 2\widehat{\psi}^I\right], \tag{8.113}$$

$$\widehat{\lambda}_{xy} = \frac{1}{2}\left[(\widehat{\sigma}_{xy} - \widehat{\sigma}_{yx}) + (\widehat{\sigma}_{xy} + \widehat{\sigma}_{yx})\cos 2\widehat{\psi}^I + (\widehat{\sigma}_{xx} - \widehat{\sigma}_{yy})\sin 2\widehat{\psi}^I\right], \tag{8.114}$$

$$\widehat{\lambda}_{yx} = \frac{1}{2}\left[-(\widehat{\sigma}_{xy} - \widehat{\sigma}_{yx}) + (\widehat{\sigma}_{xy} + \widehat{\sigma}_{yx})\cos 2\widehat{\psi}^I + (\widehat{\sigma}_{xx} - \widehat{\sigma}_{yy})\sin 2\widehat{\psi}^I\right], \tag{8.115}$$

$$\widehat{\lambda}_{yy} = \frac{1}{2}\left[(\widehat{\sigma}_{xx} + \widehat{\sigma}_{yy}) - (\widehat{\sigma}_{xx} - \widehat{\sigma}_{yy})\cos 2\widehat{\psi}^I + (\widehat{\sigma}_{xy} + \widehat{\sigma}_{yx})\sin 2\widehat{\psi}^I\right]. \tag{8.116}$$

Condition (8.111) can thus be rewritten in terms of the tensor $\widehat{\boldsymbol{\Lambda}}$ as

$$\left(\mathrm{t}_{\partial_z} \circ R_{\partial_z}\left(\frac{\pi}{2}\right)\right)\left(\widehat{\mathscr{H}}_R^{II} - \widehat{\mathscr{H}}_R^{I}\right) = -\frac{1}{Z_0}\widehat{\boldsymbol{\Lambda}}\left(\widehat{\mathscr{E}}_R^I\right). \tag{8.117}$$

Having written the interface conditions (8.91)-(8.94) in terms of the rotated 1-forms $\widehat{\gamma}_R^L$, as (8.105), (8.108)-(8.110) they may be further decoupled. Substituting (8.99) and (8.102) into (8.108) gives

$$\left(\widehat{K}_0^{II}\cos\widehat{\theta}^{II}\cos\widehat{\psi}^{\Delta} - \widehat{K}_0^{I}\cos\widehat{\theta}^{I}\right)\mathrm{d}x + \widehat{K}_0^{II}\cos\widehat{\theta}^{II}\sin\widehat{\psi}^{\Delta}\mathrm{d}y = 0, \tag{8.118}$$

One solution of (8.118) is $\cos\widehat{\theta}^I = \cos\widehat{\theta}^{II} = 0$. This implies from equation (8.88) that the tangential components of the wave vector $\widehat{K}^L$ are zero in both regions and the fields can only vary with z, thereby excluding any solutions propagating along the interface. In the following, we instead restrict to solutions with $\widehat{K}_0^I\cos\widehat{\theta}^I \neq 0$ and $\widehat{K}_0^{II}\cos\widehat{\theta}^{II} \neq 0$, which include both single-interface surface polariton and Brewster modes (see below). In this case, the dy component of (8.118) requires $\sin\widehat{\psi}^\Delta = 0$ and therefore

$$\widehat{\psi}^\Delta = m\pi, \tag{8.119}$$

where $m \in \mathbb{Z}$. Thus $\cos\widehat{\psi}^\Delta = (-1)^m$, $\sin\widehat{\psi}^\Delta = 0$, and (8.99)-(8.104) can be written in terms of the scalings $\{\widehat{A}^L\}$ and *four* complex angles $\{\widehat{\phi}^I, \widehat{\theta}^I, \widehat{\phi}^{II}, \widehat{\theta}^{II}\}$:

$$\widehat{K}_R^L = \widehat{K}_0^L\left[\left(\zeta^L\right)^m \cos\widehat{\theta}^L \mathrm{d}x + \sin\widehat{\theta}^L \mathrm{d}z\right], \tag{8.120}$$

$$\widehat{\mathscr{E}}_R^L = \widehat{A}^L\left[\left(\zeta^L\right)^m\left(\sin\widehat{\theta}^L\sin\widehat{\phi}^L\mathrm{d}x + \cos\widehat{\phi}^L\mathrm{d}y\right) - \cos\widehat{\theta}^L\sin\widehat{\phi}^L\mathrm{d}z\right], \tag{8.121}$$

$$\widehat{\mathscr{H}}_R^L = \frac{\widehat{A}^L\widehat{K}_0^L}{\omega\widehat{\mu}^L}\left[\left(\zeta^L\right)^m\left(-\sin\widehat{\theta}^L\cos\widehat{\phi}^L\mathrm{d}x + \sin\widehat{\phi}^L\mathrm{d}y\right) + \cos\widehat{\theta}^L\cos\widehat{\phi}^L\mathrm{d}z\right]. \tag{8.122}$$

where the constant ζ^L is defined in each region as:

$$\zeta^I = 1 \qquad \zeta^{II} = -1. \tag{8.123}$$

The interface condition (8.108) (and hence (8.118)) gives (8.124) below, and (8.105) gives (8.125). Furthermore, the components of (8.109) yield (8.126) and (8.127) and the components of (8.117) give (8.128) and (8.129). The complete set of interface conditions therefore becomes

$$\widehat{K}_0^{II}\cos\widehat{\theta}^{II} = (-1)^m\widehat{K}_0^I\cos\widehat{\theta}^I, \tag{8.124}$$

$$\widehat{K}_0^{II}\widehat{A}^{II}\cos\widehat{\theta}^{II}\cos\widehat{\phi}^{II} = \widehat{K}_0^I\widehat{A}^I\cos\widehat{\theta}^I\cos\widehat{\phi}^I, \tag{8.125}$$

$$\widehat{A}^{II}\sin\widehat{\theta}^{II}\sin\widehat{\phi}^{II} = (-1)^m\widehat{A}^I\sin\widehat{\theta}^I\sin\widehat{\phi}^I, \tag{8.126}$$

$$\widehat{A}^{II}\cos\widehat{\phi}^{II} = (-1)^m\widehat{A}^I\cos\widehat{\phi}^I. \tag{8.127}$$

$$(-1)^m\frac{c\widehat{K}_0^{II}}{\omega\widehat{\mu}_r^{II}}\widehat{A}^{II}\sin\widehat{\phi}^{II} - \frac{c\widehat{K}_0^{I}}{\omega\widehat{\mu}_r^{I}}\widehat{A}^{I}\sin\widehat{\phi}^{I} = -\widehat{\lambda}_{xx}\widehat{A}^I\sin\widehat{\theta}^I\sin\widehat{\phi}^I - \widehat{\lambda}_{xy}\widehat{A}^I\cos\widehat{\phi}^I, \tag{8.128}$$

$$\begin{aligned}(-1)^m\frac{c\widehat{K}_0^{II}}{\omega\widehat{\mu}_r^{II}}\widehat{A}^{II}\sin\widehat{\theta}^{II}\cos\widehat{\phi}^{II} - \frac{c\widehat{K}_0^{I}}{\omega\widehat{\mu}_r^{I}}\widehat{A}^{I}\sin\widehat{\theta}^{I}\cos\widehat{\phi}^{I} \\ = -\widehat{\lambda}_{yx}\widehat{A}^I\sin\widehat{\theta}^I\sin\widehat{\phi}^I - \widehat{\lambda}_{yy}\widehat{A}^I\cos\widehat{\phi}^I.\end{aligned} \tag{8.129}$$

with

$$\widehat{\mu}_r^L = \frac{\widehat{\mu}^L}{\mu_0} = \frac{\widehat{\mu}^L c}{Z_0}.$$

We observe that (8.125) automatically follows if (8.124) and (8.127) are satisfied.

8.9 Solving the Interface Conditions

We now show that solving the interface conditions (8.124)-(8.129) reduces to finding solutions to a complex polynomial equation. Introduce the complex quantities

$$\widehat{\Psi} \equiv \widehat{A}^I \sin\widehat{\theta}^I \sin\widehat{\phi}^I, \tag{8.130}$$

$$\widehat{\Phi} \equiv \widehat{A}^I \cos\widehat{\phi}^I, \tag{8.131}$$

$$\widehat{Q} \equiv \frac{c\widehat{K}_0^I}{\omega} \cos\widehat{\theta}^I \neq 0, \tag{8.132}$$

$$\widehat{\alpha} \equiv \frac{c}{2\omega}\left(\widehat{K}_0{}^I \sin\widehat{\theta}^I + \widehat{K}_0{}^{II} \sin\widehat{\theta}^{II}\right), \tag{8.133}$$

$$\widehat{\beta} \equiv \frac{c}{2\omega}\left(\widehat{K}_0{}^I \sin\widehat{\theta}^I - \widehat{K}_0{}^{II} \sin\widehat{\theta}^{II}\right). \tag{8.134}$$

Equation (8.108) is then satisfied with

$$\cos\widehat{\theta}^L = \left(\zeta^L\right)^m \frac{\omega\widehat{Q}}{c\widehat{K}_0^L} \tag{8.135}$$

and (8.133) and (8.134) yield

$$\sin\widehat{\theta}^L = \frac{\omega}{c\widehat{K}_0^L}\left(\widehat{\alpha} + \zeta^L\widehat{\beta}\right). \tag{8.136}$$

Equations (8.133)-(8.134) give

$$\begin{aligned}
\widehat{\alpha}\widehat{\beta} &= \frac{c^2}{4\omega^2}\left(\widehat{K}_0{}^{I\,2} \sin^2\widehat{\theta}^I - \widehat{K}_0{}^{II\,2} \sin^2\widehat{\theta}^{II}\right) \\
&= \frac{c^2}{4\omega^2}\left(\widehat{K}_0{}^{I\,2} - \widehat{K}_0{}^{II\,2} - \widehat{K}_0{}^{I\,2}\cos^2\widehat{\theta}^I + \widehat{K}_0{}^{II\,2}\cos^2\widehat{\theta}^{II}\right) \\
&= \frac{c^2}{4\omega^2}\left(\widehat{K}_0{}^{I\,2} - \widehat{K}_0{}^{II\,2}\right) = \frac{1}{4}\left(\widehat{\epsilon'}^I_r\widehat{\mu}^I_r - \widehat{\epsilon'}^{II}_r\widehat{\mu}^{II}_r\right),
\end{aligned}$$

since $\widehat{K}_0{}^{I\,2}\cos^2\widehat{\theta}^I = \widehat{K}_0{}^{II\,2}\cos^2\widehat{\theta}^{II}$ from (8.124). Hence,

$$\widehat{\beta} = \frac{\widehat{\nu}}{\widehat{\alpha}} \tag{8.137}$$

where

$$\widehat{\nu} \equiv \frac{1}{4}\left(\widehat{\epsilon'}^I_r\widehat{\mu}^I_r - \widehat{\epsilon'}^{II}_r\widehat{\mu}^{II}_r\right) \tag{8.138}$$

depends only on the constitutive properties of the two bulk regions. Equation (8.136) is now written in terms of $\widehat{\alpha}$ as

$$\sin\widehat{\theta}^L = \frac{\omega}{c\widehat{K}_0^L}\left(\widehat{\alpha}+\zeta^L\frac{\widehat{\nu}}{\widehat{\alpha}}\right). \tag{8.139}$$

Definition (8.130) and junction condition (8.126) imply

$$A^L\sin\widehat{\phi}^L = \left(\zeta^L\right)^m\frac{\widehat{\Psi}}{\sin\widehat{\theta}^L} = \left(\zeta^L\right)^m\frac{c\widehat{K}_0^L}{\omega}\frac{\widehat{\Psi}}{\widehat{\alpha}+\zeta^L\frac{\widehat{\nu}}{\widehat{\alpha}}}, \tag{8.140}$$

while (8.131) and junction condition (8.127) yield

$$A^L\cos\widehat{\phi}^L = \left(\zeta^L\right)^m\widehat{\Phi}. \tag{8.141}$$

Substituting equations (8.135) and (8.139) into the complex identity $\cos^2\widehat{\theta}^L + \sin^2\widehat{\theta}^L = 1$ for either $L = I$ or $L = II$ and solving for $\widehat{Q}$ gives

$$\widehat{Q} = \pm\sqrt{\frac{\widehat{\epsilon'}^I_r\widehat{\mu}^I_r}{2}+\frac{\widehat{\epsilon'}^{II}_r\widehat{\mu}^{II}_r}{2}-\widehat{\alpha}^2-\frac{\widehat{\nu}^2}{\widehat{\alpha}^2}}. \tag{8.142}$$

In summary, the trigonometric functions of $\widehat{\theta}^L$ and $\widehat{\phi}^L$ are expressed in terms of $\widehat{\alpha}$, $\widehat{\Phi}$, $\widehat{\Psi}$ and the constitutive properties of $\mathscr{V}^I$ and $\mathscr{V}^{II}$ by the relations

$$\sin\widehat{\theta}^L = \frac{\omega}{c\widehat{K}_0^L}\left(\widehat{\alpha}+\zeta^L\frac{\widehat{\nu}}{\widehat{\alpha}}\right), \qquad \cos\widehat{\theta}^L = \left(\zeta^L\right)^m\frac{\omega\widehat{Q}}{c\widehat{K}_0^L},$$
$$\widehat{A}^L\sin\widehat{\phi}^L = \left(\zeta^L\right)^m\frac{c\widehat{K}_0^L}{\omega}\frac{\widehat{\Psi}}{\widehat{\alpha}+\zeta^L\frac{\widehat{\nu}}{\widehat{\alpha}}}, \quad \widehat{A}^L\cos\widehat{\phi}^L = \left(\zeta^L\right)^m\widehat{\Phi}, \tag{8.143}$$

where ζ^L is defined by (8.123) and $\widehat{Q}$ is given by (8.142). It may be readily confirmed that the interface conditions (8.124)-(8.127) are satified by the equations (8.143). We now turn to the final two conditions (8.128) and (8.129).

Substituting (8.143) into (8.128) and using (8.49) gives

$$\widehat{X}(\widehat{\alpha})\widehat{\Psi} - \widehat{\lambda}_{xy}\widehat{\Phi} = 0, \tag{8.144}$$

where

$$\widehat{X}(\widehat{\alpha}) \equiv \frac{\widehat{\alpha}\left(\widehat{\epsilon'}^I_r-\widehat{\epsilon'}^{II}_r\right)-\frac{\widehat{\nu}}{\widehat{\alpha}}\left(\widehat{\epsilon'}^I_r+\widehat{\epsilon'}^{II}_r\right)-\widehat{\lambda}_{xx}\left(\widehat{\alpha}^2-\frac{\widehat{\nu}^2}{\widehat{\alpha}^2}\right)}{\widehat{\alpha}^2-\frac{\widehat{\nu}^2}{\widehat{\alpha}^2}}. \tag{8.145}$$

Substituting (8.143) into (8.129) then implies

$$\widehat{Y}(\widehat{\alpha})\widehat{\Phi} - \widehat{\lambda}_{yx}\widehat{\Psi} = 0, \tag{8.146}$$

where

$$\widehat{Y}(\alpha) \equiv \left(\frac{1}{\widehat{\mu}_r^I} - \frac{1}{\widehat{\mu}_r^{II}}\right)\widehat{\alpha} + \left(\frac{1}{\widehat{\mu}_r^I} + \frac{1}{\widehat{\mu}_r^{II}}\right)\frac{\widehat{v}}{\widehat{\alpha}} - \widehat{\lambda}_{yy}. \tag{8.147}$$

Equations (8.144) and (8.146) admit different classes of solution for $\widehat{\boldsymbol{\Phi}}$ and $\widehat{\boldsymbol{\Psi}}$ depending on the values of the constitutive functions and their dependence on frequency.

For solutions with $\widehat{\boldsymbol{\Phi}} = 0$ and $\widehat{\boldsymbol{\Psi}} \neq 0$, these equations degenerate to the set

$$\widehat{X}(\widehat{\alpha}) = 0, \qquad \widehat{\lambda}_{yx} = 0 \tag{8.148}$$

where the first equation is a complex *quartic* equation for $\widehat{\alpha}$. We will call the fields determined by such solutions "type-$\widehat{\boldsymbol{\Psi}}$ modes".

When $\widehat{\boldsymbol{\Psi}} = 0$ and $\widehat{\boldsymbol{\Phi}} \neq 0$ one obtains

$$\widehat{Y}(\widehat{\alpha}) = 0, \qquad \widehat{\lambda}_{xy} = 0 \tag{8.149}$$

where the first equation is a complex *quadratic* equation for $\widehat{\alpha}$. We will call the fields determined by such solutions "type-$\widehat{\boldsymbol{\Phi}}$ modes".

In the general case (with both $\widehat{\boldsymbol{\Phi}}$ and $\widehat{\boldsymbol{\Psi}}$ non-zero), one obtains coupled type-$\widehat{\boldsymbol{\Phi}}$-$\widehat{\boldsymbol{\Psi}}$ modes from solutions $\widehat{\alpha}$ satisfying the determinantal condition

$$\widehat{X}(\widehat{\alpha})\widehat{Y}(\widehat{\alpha}) - \widehat{\lambda}_{xy}\widehat{\lambda}_{yx} = 0 \tag{8.150}$$

i.e. the *degree six complex polynomial*

$$\begin{aligned} 0 = &\left(\widehat{\alpha}^3\left(\widehat{\epsilon'}^I_r - \widehat{\epsilon'}^{II}_r\right) - \widehat{v}\widehat{\alpha}\left(\widehat{\epsilon'}^I_r + \widehat{\epsilon'}^{II}_r\right) - \widehat{\lambda}_{xx}\left(\widehat{\alpha}^4 - \widehat{v}^2\right)\right) \\ &\times \left(\left(\frac{1}{\widehat{\mu}_r^I} - \frac{1}{\widehat{\mu}_r^{II}}\right)\widehat{\alpha}^2 + \left(\frac{1}{\widehat{\mu}_r^I} + \frac{1}{\widehat{\mu}_r^{II}}\right)\widehat{v} - \widehat{\lambda}_{yy}\widehat{\alpha}\right) - \widehat{\lambda}_{xy}\widehat{\lambda}_{yx}\widehat{\alpha}\left(\widehat{\alpha}^4 - \widehat{v}^2\right). \end{aligned} \tag{8.151}$$

Then for each root $\widehat{\alpha}$,

$$\frac{\widehat{\boldsymbol{\Psi}}}{\widehat{\boldsymbol{\Phi}}} = \frac{1}{\widehat{\lambda}_{yx}}\left\{\frac{1}{\widehat{\mu}_r^I}\left(\widehat{\alpha} + \frac{\widehat{v}}{\widehat{\alpha}}\right) - \frac{1}{\widehat{\mu}_r^{II}}\left(\widehat{\alpha} - \frac{\widehat{v}}{\widehat{\alpha}}\right) - \widehat{\lambda}_{yy}\right\}. \tag{8.152}$$

and the rotated field 1-forms are given by substituting (8.143) into (8.120)-(8.122), for each root $\widehat{\alpha}$ of (8.151), using (8.49) to rewrite $\widehat{K}_0^2$:

$$\widehat{K}_R^L = \frac{\omega}{c}\left\{\widehat{Q}\mathrm{d}x + \left(\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}\right)\mathrm{d}z\right\}, \tag{8.153}$$

$$\widehat{\mathscr{E}}_R^L = \widehat{\Psi}\mathrm{d}x + \widehat{\Phi}\mathrm{d}y - \frac{\widehat{Q}\widehat{\Psi}}{\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}}\mathrm{d}z, \tag{8.154}$$

$$\widehat{\mathscr{H}}_R^L = \frac{1}{Z_0}\left\{-\left(\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}\right)\frac{\widehat{\Phi}}{\widehat{\mu}_r^L}\mathrm{d}x + \frac{\widehat{\epsilon'}_r\widehat{\Psi}}{\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}}\mathrm{d}y + \frac{\widehat{Q}\widehat{\Phi}}{\widehat{\mu}_r^L}\mathrm{d}z\right\}, \tag{8.155}$$

where $\widehat{v}$ is given by (8.138) and $\widehat{Q}$ is given by (8.142). Finally,

$$\{\widehat{K}^L, \widehat{\mathscr{E}}^L, \widehat{\mathscr{H}}^L\} = R_{\partial_z}(\widehat{\psi}^I)\{\widehat{K}_R^L, \widehat{\mathscr{E}}_R^L, \widehat{\mathscr{H}}_R^L\},$$

where

$$\widehat{K}^L = \frac{\omega}{c}\left\{\widehat{Q}\left(\cos\widehat{\psi}^I\mathrm{d}x - \sin\widehat{\psi}^I\mathrm{d}y\right) - \left(\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}\right)\mathrm{d}z\right\}, \tag{8.156}$$

$$\widehat{\mathscr{E}}^L = \left(\widehat{\Psi}\cos\widehat{\psi}^I + \widehat{\Phi}\sin\widehat{\psi}^I\right)\mathrm{d}x + \left(\widehat{\Phi}\cos\widehat{\psi}^I - \widehat{\Psi}\sin\widehat{\psi}^I\right)\mathrm{d}y + \frac{\widehat{Q}\widehat{\Psi}}{\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}}\mathrm{d}z, \tag{8.157}$$

$$\begin{aligned}\widehat{\mathscr{H}}^L = \frac{1}{Z_0}\Bigg\{&\left(\frac{\widehat{\epsilon'}_r\widehat{\Psi}}{\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}}\sin\widehat{\psi}^I - \left(\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}\right)\frac{\widehat{\Phi}}{\widehat{\mu}_r^L}\cos\widehat{\psi}^I\right)\mathrm{d}x,\\ &+\left(\left(\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}\right)\frac{\widehat{\Phi}}{\widehat{\mu}_r^L}\sin\widehat{\psi}^I + \frac{\widehat{\epsilon'}_r\widehat{\Psi}}{\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}}\cos\widehat{\psi}^I\right)\mathrm{d}y + \frac{\widehat{Q}\widehat{\Phi}}{\widehat{\mu}_r^L}\mathrm{d}z\Bigg\}.\end{aligned} \tag{8.158}$$

The complex parameter $\widehat{\psi}^I$ remains an arbitrary complex angle. Thus (8.156)-(8.158) constitute a family of solutions parametrised by $\widehat{\psi}^I$ along with the bulk and surface constitutive scalars.

For each m, $\widehat{\psi}^I$ and root $\widehat{\alpha}$, the final electric and magnetic field configurations are given by

$$\widehat{\boldsymbol{e}}^L = \widehat{\mathscr{E}}^L e^{i\widehat{\chi}^L}, \qquad \widehat{\boldsymbol{h}}^L = \widehat{\mathscr{H}}^L e^{i\widehat{\chi}^L},$$

where

$$\widehat{\chi}^L = \frac{\omega}{c}\left\{\widehat{Q}\left(\cos\widehat{\psi}^I x - \sin\widehat{\psi}^I y\right) + \left(\widehat{\alpha} + \frac{\zeta^L\widehat{v}}{\widehat{\alpha}}\right)z\right\}. \tag{8.159}$$

8.10 Conclusion

We have developed a strategy for analysing a large class of solutions to Maxwell's equations for the electromagnetic fields in piecewise homogeneous material media containing a plane interface. The constitutive properties on either side of the interface have been assumed dispersive but isotropic while the interface has been endowed with a complex homogeneous anisotropic admittance tensor relating surface currents to electric fields in the interface. Such a model accommodates as a special case both active and passive (including Ohmic) interface electromagnetic characteristics. The analysis yields a family of solutions to this problem characterised by both constitutive properties of the system and a number of arbitrary, complex (frequency-dependent) constants. Different choices of these constants and constitutive scalars determine the physical characteristics of the solutions.

For a planar interfaces $f = 0$ with surface normal unit vector $N = \mathrm{d}f/|\mathrm{d}f|$, bounded solutions with $\widehat{K}^I$ and $\widehat{K}^{II}$ complex and

$$\Im(\mathrm{i}_N(\widehat{K}^L)) \neq 0,$$

exhibit mode attenuation in directions where $|f| \leftrightarrow \infty$ and are referred to as *surface polariton* modes. If the bulk constitutive permittivities or permeabilities are complex, with bulk conductivity non-zero or the interface possesses surface admittance, the *tangential* components of $\widehat{K}^I$ (or $\widehat{K}^{II}$) may also become complex. In these circumstances, the physically acceptable plane-fronted polariton modes will propagate (in half-spaces) with attenuation in directions orthogonal to N. When $\widehat{K}^I$ and $\widehat{K}^{II}$ are real 1-forms with

$$\mathrm{sign}\,(\mathrm{i}_N\widehat{K}^I) = \mathrm{sign}\,(\mathrm{i}_N\widehat{K}^{II}),$$

one speaks of plane-fronted *Brewster* modes.

Particular solutions may be classified further by introducing the notion of a *real* plane of propagation at any point as the span of the *real* vector fields N and $\widetilde{\widehat{\Upsilon}}$ where

$$\widehat{\Upsilon} \equiv \Re(\mathrm{t}_N\widehat{K}^I) = \Re(\mathrm{t}_N\widehat{K}^{II}). \tag{8.160}$$

Such solutions are said to generate TE-type modes in the domain L if

$$\mathrm{i}_{\widetilde{\widehat{\mathscr{E}}}^L}\left(\widetilde{N}\wedge\widehat{\Upsilon}\right) = 0, \tag{8.161}$$

and TM-type modes in the domain L if

$$\mathrm{i}_{\widetilde{\widehat{\mathscr{H}}}^L}\left(\widetilde{N}\wedge\widehat{\Upsilon}\right) = 0. \tag{8.162}$$

By definition the complex propagation vectors for surface polaritons without attenuation in any plane orthogonal to N can be written in the form

$$\widetilde{\widehat{K}}^L_{SP} = \widetilde{\widehat{\Upsilon}}_{SP} + i\widehat{K}^L_N N \tag{8.163}$$

for some $\widehat{K}^L_N \in \mathbb{R}$ and the real propagation vectors for Brewster modes as $\widehat{K}^L_B$. Since

$$\mathrm{i}_{\widetilde{\widehat{\mathscr{E}}}^L}\widehat{K}^L = \mathrm{i}_{\widetilde{\widehat{\mathscr{H}}}^L}\widehat{K}^L = 0 \tag{8.164}$$

by construction, it follows that TE-type solutions in this class also satisfy

$$0 = \mathrm{i}_{\widetilde{\widehat{\mathscr{E}}^L}} \left(\widetilde{N} \wedge \widehat{\Upsilon}\right) = \mathrm{i}_{\widetilde{\widehat{\mathscr{E}}^L}} \left(\widetilde{N} \wedge \widehat{K}^L\right) = \mathrm{i}_N \widehat{\mathscr{E}}^L. \tag{8.165}$$

From (8.157), with $N = \partial_z$ and $\widehat{Q} \neq 0$, this implies that

$$\widehat{\Psi} = 0. \tag{8.166}$$

Similarly, (8.164) implies that TM-type solutions in this class also satisfy

$$0 = \mathrm{i}_{\widetilde{\widehat{\mathscr{H}}^L}} \left(\widetilde{N} \wedge \widehat{\Upsilon}\right) = \mathrm{i}_{\widetilde{\widehat{\mathscr{H}}^L}} \left(\widetilde{N} \wedge \widehat{K}^L\right) = \mathrm{i}_N \widehat{\mathscr{H}}^L. \tag{8.167}$$

From (8.158), this implies that

$$\widehat{\Phi} = 0. \tag{8.168}$$

Thus type-Φ modes in this class correspond to TE polarised fields in the presence of a planar interface, and type-Ψ modes to TM polarised fields[7].

One may observe in this way how, for example, the presence of a non-zero surface conductivity can dramatically change the standard mode structure of surface polariton and Brewster electromagnetic field configurations. These and other effects will be presented elsewhere (Christie and Tucker, 2018).

The systematic strategy outlined in the paper for finding analytic expressions describing the electromagnetic fields in a dispersive medium containing a planar meta-interface can be generalised to accommodate more intricate interface conditions where surface currents are induced by electric and/or magnetic fields that are normal and/or transverse to the planar interface. Such conditions have been contemplated in Epstein and Eleftheriades (2016) in efforts to construct a "tunable meta-surface".

Our geometric formulation in terms of differential forms offers a consistent and compact way to approach more challenging problems involving media with inhomogeneous constitutive components and curved interfaces. Such problems will be discussed more fully elsewhere.

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[7] More generally, if a region $\mathscr{U}$ contains a real unit vector field N we may *define* field components $\{\widehat{\mathscr{E}}, \widehat{\mathscr{H}}\} \in \hat{\Gamma}\Lambda^1\mathscr{U}$ to be of type TE with respect to N if they satisfy $\mathrm{i}_N\widehat{\mathscr{E}} = 0$ and of type TM with respect to N if they satisfy $\mathrm{i}_N\widehat{\mathscr{H}} = 0$.

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