Chapter 11
Interpolation of Morrey Spaces

11.1 Stampacchia-Peetre interpolation; Interpolation via the new duality

Now we turn our attention to interpolation of linear operators on Morrey spaces, say

$$T : L^{p,\lambda} \longrightarrow L^{q,\mu}$$

for various $p,\ q,\ \lambda$, and $\mu;\ 1 < p, q < \infty,\ 0 < \lambda,\ \mu < n$.

First, with regard to the history of such attempts. It must surely start with Stampacchia [S] in 1965 when he showed that interpolation works quite easily when the Morrey Spaces lie only in the range of the operator. In fact, if

$$T : L^{q_i} \longrightarrow L^{p_i,\lambda_i},\ i = 0, 1$$

then

$$T : L^{q_\theta} \longrightarrow L^{p_\theta,\lambda_\theta}$$

where

$$\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},\ \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \tag{11.1}$$

and

$$\frac{\lambda_\theta}{p_\theta} = (1-\theta)\frac{\lambda_0}{p_0} + \theta\frac{\lambda_1}{p_1}, \tag{11.2}$$

and $0 < \theta < 1$.

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Proof. The hypotheses imply

$$\sup_{x,r>0} \left(r^{\lambda_i - n} \int_{B(x,r)} |Tf|^{p_i} \, dy \right)^{1/p_i} \le A_i \left(\int |f|^{q_i} \, dy \right)^{1/q_i},$$

for $i = 0, 1$. Hence for any fixed ball $B(x, r)$,

$$\int_{B(x,r)} |Tf|^{p_i} \, dy \le A_i^{p_i} \, ||f||_{L^{q_i}}^{p_i} \cdot r^{n-\lambda_i}.$$

So now by standard interpolation between Lebesgue spaces, one has

$$\left(\int_{B(x,r)} |Tf|^{p_\theta} \, dy \right)^{1/p_\theta} \le M_\theta \, ||f||_{L^{q_\theta}}$$

with $M_i = A_i \, r^{(n-\lambda_i)/p_i}, \quad i = 0, 1.$

Hence $M_\theta = M_0^{1-\theta} M_1^\theta = A_0^{1-\theta} A_1^\theta r^{(n-\lambda_\theta)/p_\theta}$.

Which gives

$$||Tf||_{L^{p_\theta,\lambda_\theta}} \le M_\theta \, ||f||_{L^{q_\theta}}.$$

□

But this argument does not work for

$$T : L^{p_i,\lambda_i} \longrightarrow L^{q_i}, \; i = 0, 1.$$

So we will concentrate on this case now. And, of course, to prove this, it is now natural to use the duality that we have established in Chapter 5. Thus let $T^* =$ adjoint of T and suppose that

$$T^* : L^{q_i'} \longrightarrow H^{p_i',\lambda_i}, \quad i = 0, 1.$$

The hypothesis implies that there are $\omega_i \in A_1$ such that

$$\int \omega_i \, d\Lambda^{n-\lambda_i} \le 1$$

and

$$\left(\int |T^* f|^{p_i'} \, \omega_i^{1-p_i'} \, dy \right)^{1/p_i'} \le A_i \left(\int |f|^{q_i'} \, dy \right)^{1/q_i'}.$$

We now write

$$\omega_i^{1-p_i'} = (\omega_i^{-1/p_i})^{p_i'}. \tag{11.3}$$

This then allows us to apply "Stein's Interpolation with change of measure" (see [BS]) to get

$$\left(\int |T^* f|^{p'_\theta}\, \omega_\theta^{1-p'_\theta}\, dy\right)^{1/p_\theta{}'} \;\le\; A_\theta \left(\int |f|^{q'_\theta}\, dy\right)^{1/q'_\theta}$$

where

$$\omega_\theta \;=\; \omega_0^{\frac{p_\theta}{p_0}(1-\theta)} \cdot \omega_1^{\frac{p_\theta}{p_1}\theta}. \tag{11.4}$$

To make our conclusion and then by duality our interpolation result, we must now show that

$$\omega_\theta \in A_1 \text{ and } \int \omega_\theta \, d\Lambda^{n-\lambda_\theta} \le \text{constant}.$$

We first note that this is now very easy in the case where $\lambda_0 = \lambda_1 = \lambda$, i.e., in the known case in the literature [Y^2, $\mathrm{Y}^2\mathrm{Z}$], because (11.4) implies that $\omega_\theta \in A_1$ and

$$\int \omega_\theta \, d\Lambda^{n-\lambda} \;\le\; c\left(\int \omega_0 \, d\Lambda^{n-\lambda}\right)^{\frac{1-\theta}{p_0}p_\theta} \left(\int \omega_1 d\Lambda^{n-\lambda}\right)^{\frac{\theta}{p_1}p_\theta}$$

by the quasi Holder (via $\tilde{\Lambda}_0^{n-\lambda}$), i.e., $\dfrac{p_\theta}{p_0}(1-\theta) + \dfrac{p_\theta}{p_1}\theta = 1$.

Thus we get

$$||T^* f||_{H^{p'_\theta,\lambda}} \;\le\; A_\theta \; ||f||_{L^{q'_\theta}}$$

and then for our result

$$T : L^{p_\theta,\lambda} \to L^{q_\theta}$$

for the case $\lambda_0 = \lambda_1 = \lambda$.

However, we are trying for more. And to do this we now invoke the atomic decompositions of [AX2], we have

Lemma 11.1. $\omega \in L^1(\Lambda^d)$ iff $\omega = \sum_k c_k\, a_k$, where $\{c_k\} \in l^1$ and the a_k are (∞, d)- atoms, i.e.

(i) supp $a_k \subset$ cube Q_k

and

(ii) $||a_k||_{L^\infty(Q_k)} \;\le\; |Q_k|^{-d/n}$.

And the norm of $L^1(\Lambda^d)$ (actually a quasi-norm for we must go through $\tilde{\Lambda}^d_0$ - see Chapter 3), is equivalent to

$$\inf \sum_k |c_k|$$

with the infimum over all such representations.

With this lemma, we argue as follows: write

$$\omega_i = \sum_k C_k^{(i)}\, a_k^{(i)}, \quad i = 0, 1, \text{ with}$$
$$\sum_k |C_k^{(i)}| \sim \int \omega_i \, d\Lambda^{n-\lambda_i}.$$

Then set

$$C_k = |C_k^{(0)}|^{1-\Theta}\, |C_k^{(1)}|^{\Theta},$$

with

$$\Theta = \frac{p_\theta}{p_1}\theta \quad \text{and} \quad 1-\Theta = 1 - \frac{p_\theta}{p_1}\theta = \frac{p_\theta}{p_0}(1-\theta).$$

Then

$$\begin{aligned}\sum_k C_k &\le \Big(\sum_k |C_k^{(0)}|\Big)^{1-\Theta}\Big(\sum_k |C_k^{(1)}|\Big)^{\Theta}\\ &\le A\Big(\int \omega_0 \, d\Lambda^{n-\lambda_0}\Big)^{1-\Theta}\Big(\int \omega_1 \, d\Lambda^{n-\lambda_1}\Big)^{\Theta}\\ &\le A = \text{absolute constant.}\end{aligned}$$

Thus setting $a_k^{(\theta)} = |a_k^{(0)}|^{1-\Theta} \cdot |a_k^{(1)}|^{\Theta}$, we see that it is indeed an $(\infty, n-\lambda_\theta)$ atom.

This then is our argument for the

Theorem 11.2. If the linear operator T satisfies

$$T : L^{p_i,\lambda_i} \longrightarrow L^{q_i}, \; i = 0, 1$$

with $1 < p_i < \infty$ and $0 < \lambda_i < n$, then there is a constant such that

$$T : L^{p_\theta,\lambda_\theta} \longrightarrow L^{q_\theta}$$

with $p_\theta, q_\theta, \lambda_\theta$ given by (11.1) and (11.2).

Proof. Proof of Lemma 11.1

On one hand, if $f = \sum_k c_k\, a_k$, then by the quasi-sublinearity

$$\begin{aligned} ||f||_{L^1(\Lambda^d)} &\leq \int \sum_k |c_k|\ |a_k|\ d\Lambda^d \\ &\leq c\ \sum_k |c_k| \int |a_k|\ d\Lambda^d \\ &\leq c\ ||\{c_k\}||_{l^1}. \end{aligned}$$

Conversely, suppose $||f||_{L^1(\Lambda^d)} < \infty$, then using the construction given in [AX2], we can write

$$f = \sum_{j,k} c_{j,k} a_{j,k}$$

where

$$\begin{aligned} c_{j,k} &= l(Q_{j,k})^d \cdot 2^{k+1} \\ a_{j,k}(x) &= f(x) \cdot X_{\Delta_{j,k}(x)} \cdot l(Q_{j,k})^{-d} \cdot 2^{-(k+1)} \end{aligned}$$

$l(Q) =$ edge length of the cube Q. Then $|f(x)| \leq 2^{k+1}$ for $x \in \Delta_{j,k}$ and $\{c_{j,k}\} \in l^1$ since

$$||\{c_{j,k}\}||_{l^1} \leq A \sum l(Q_{j,k})^d\ 2^{k+1} \leq A\ ||f||_{L^1 \Lambda^d}.$$

The $Q_{j,k}$ and $\Delta_{j,k}$ are selectively chosen dyadic cubes; see[AX2]. □

Finally one can argue for

$$T : L^{p,\lambda} \longrightarrow L^{q,\mu}$$

by putting together our argument with that of Stampacchia.

11.2 Counterexamples to interpolation with Morrey Spaces in the domain of the operator

But one should note that there are some counterexamples. A close examination of these examples shows in fact that the Campanato spaces are not stable under interpolation. In [BRV], the authors give an example (in one dimension) where

$0 < \lambda_0 < \lambda_1 = 1 = n$, and then show that interpolation does not achieve a Morrey Space estimate for the intermediate case of $\lambda_\theta : 0 < \lambda_0 < \lambda_\theta < \lambda_1 = 1$. Also, from [SZ] one gets

$$T : C^\alpha \longrightarrow C^\alpha$$
$$T : L^2 \longrightarrow L^2$$

but $T(f) \notin L^q$ for any $q > 2$. One would hope for some Morrey cases as intermediate situations. Also, we note from the proof given for Theorem 11.2, that it is really necessary to have $\lambda \in (0, n)$ for $\int \omega \, d\Lambda^{n-\lambda}$ doesn't make sense when $\lambda = n$.

11.3 Integrability of Morrey Potentials

As an application of Theorem 11.2, we get the following integrability result for Morrey potentials: $I_\alpha f, f \in L^{p,\lambda}, \alpha p < \lambda < n$. Indeed, Lemma 9.1 yields

$$I_\alpha : L^{p,\lambda_0} \longrightarrow \text{BMO} \subset L^{q_0}_{loc}$$

for any $q_0 < \infty$ when $\lambda_0 = \alpha p < n$, $p > 1$. And Theorem 7.1(i)

$$I_\alpha : L^{p,\lambda_1} \longrightarrow L^{q_1}_{loc}$$

for $q_1 = \lambda_1 p/(\lambda_1 - \alpha p),\ \alpha p < \lambda_1 < n,\ p > 1$. Hence by interpolation

$$I_\alpha : L^{p,\lambda} \longrightarrow L^{q}_{loc}$$

for any $q < np/(\lambda - \alpha p),\ \alpha p = \lambda_0 < \lambda < \lambda_1 < n$.

And notice, $I_\alpha f_0 \in L^q_{loc}$ for $f_0(y) = |y|^{-\lambda/p},\ y \in \mathbb{R}^n$.

This all seems to work out here since λ_0 and λ_1 both less that n= dimension of the underlying space, hence no L^p spaces are included in the class of functions being interpolated (see [BRV] and [LR]).