Inquiry-Based Learning in Mathematics

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As Huber (2009) states, inquiry-based learning as a higher education didactic method is subject to strong criticism from many quarters. The issue of the unity of research and teaching, which serves as a central pillar for the educational-theoretical justification of inquiry-based learning, is called into question both in principle and pragmatically. This article will present inquiry-based learning in mathematics as an idealized process on the one hand, and in a real implementation on the other. This will be done against the backdrop of cultural-historical activity theory according to Roth and Radford (2011), which more precisely defines the theory for mathematics learning based on Leontiev (1978).

20.1 Introduction

Inquiry-based learning has become increasingly important as a method of higher education didactics in recent years. As such, inquiry-based learning is also regarded as a bridge between the traditional demand for unity of research and teaching, and newer demands in a pluralistic world that is characterized by the ability and willingness to learn throughout one's life.

Within mathematics, the primary trend in inquiry-based learning consists of very broad attempts to establish inquiry-based learning as part of mathematics instruction. Please refer to the PRIMAS project of the European Union (see Box 20.1) as exemplary for many other projects. Although PRIMAS and similar projects have created a large public for the topic of inquiry-based learning within mathematical didactics so that at conferences of the Gesellschaft für Didaktik der Mathematik (Society for the Didactics of Mathematics), for

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Box 20.1: ***The PRIMAS Project of the European Union***

At 14 institutions of higher learning in 12 European countries, instructional materials are being created, teacher training sessions held, and support provided in the implementation of instruction, all of which serve to introduce inquiry-based learning into instruction in mathematics and the natural sciences. The project describes the goals of PRIMAS as follows (Projekt PRIMAS n.d.):

> PRIMAS has taken on the goal of getting more higher education students, school pupils and graduates interested in school subjects and fields of study from mathematics and the natural sciences. Partners of PRIMAS are convinced that more pupils would be enthusiastic about these subjects if their natural curiosity were stimulated and strengthened through inquiry-based and discovery learning. The pupils will learn to observe phenomena, to ask questions, to look for solutions independently, and to reason. The pupils should not and cannot discover everything themselves. However, they should be allowed to experience how a mathematician or natural scientist proceeds in order to get a fair picture of these subjects and the careers associated therewith

example, regular sections are held on inquiry-based learning in schools, this has found little resonance in higher education. The same is true internationally: although inquiry-based learning is currently widely considered within the school context, it does not play a recognizable role in higher education.

Within the context of the "matheFL: inquiry-based learning right from the start?" project at the University of Bremen, an attempt was made right from the first semester to utilize inquiry-based learning in project groups in order to strengthen and maintain student motivation (Bikner-Ahsbahs et al., 2013). The examples from the implementation used in the text are taken from this project. The matheFL project is sponsored by the University of Bremen within the context of the initiative for inquiry-based learning (cf. Kaufmann & Schelhowe, in this volume).

20.2 Definition of Terms and Theoretical Frameworks

Huber (2009, p. 10, translated) defines inquiry-based learning as follows:

> In contrast to other learning methods, inquiry-based learning is characterized by the fact that learners shape, experience and reflect on the process of a research project, which is aimed at obtaining insights that are of interest to third parties, doing so throughout all the essential phases of said project; from developing questions and hypotheses, selecting and implementing the methods, through testing and presenting the results, either by working independently or in active collaboration with an overarching project.

It already seems difficult to fulfill this definition for courses within a normal course of study. At the start of a course of study, however, it may even seem impossible: much like the current research situation in any scientific discipline, the requirements for independent research or active participation in a project generally demand a much more substantive and, above all, methodical knowledge than can be provided in a school education. Two specific problems become apparent for mathematics. On the one hand, the structure of mathematics is ordered in a strictly deductive manner. There may be no gaps in mathematical proofs. On the other hand, there is currently no research being conducted in the areas directly covered by lectures for beginners; instead, it is frequently conducted in sub-areas motivated by applications, which then require very extensive additional knowledge just in order to understand the research questions in these areas. Obtaining insights that are of interest to third parties within an introductory lecture seems nearly impossible.

Huber (2009) explains the use of inquiry-based learning in various ways: To begin with, he argues at the level of education. Here, the scientific method serves to gain insight by self-reflection, and the engagement with which science is pursued is what makes it possible to go beyond a school education and training (Huber 2009, p. 11 et seq.). At the level of general competencies, the project-like approach of inquiry-based learning most likely offers the opportunity to practice key qualifications such as cooperation, communication, and managing time and work in a natural way (Huber 2009, p. 13 et seq.). Finally, Huber emphasizes the importance of inquiry-based learning for "deeper learning" (for details, see also Wulf, in this volume). Here, deeper learning means that it is not a sluggish knowledge of facts that is being learned, but rather "a living ability that can be actively used in new situations and flexibly modified" (Huber, 2009, p. 15, translated). This does not require completed knowledge, but rather more generalizable and generative action strategies (ibid.). As Huber points out, the approach to authentic problems such as those that may arise within the context of inquiry-based learning provides the best prerequisites for this.

In activity theory according to Wolff-Michael Roth and Luis Radford (2011), activities are social processes that occur within the context of a culture. Each activity consists of individual actions, each of which relate to objects and goals or motives. Roth and Radford describe knowledge as the awareness of possible actions, crystallized as an abstract concept which must be recreated by the individual in concrete situations. Thus, learning can be understood as a process of objectification, meaning an increasing awareness of what one is doing at the moment, and what that means. In so doing, the abstract concept allows one to gain insights through the respective specific manifestation in which action is taken. This view is particularly suitable for the understanding of mathematics learning as this is about examining abstract concepts, such as those of a triangle or vectors, in specific manifestations.

It will be useful here to take a closer look at the processes involved in inquiry-based learning using this concept of learning from activity theory, which is based on the work of Leontiev (1978). In so doing, "researching," "inquiry-based learning" and "completing tutorials" are the activities that are relevant in this context.

To begin with, it is immediately apparent that Huber's above-mentioned definition of inquiry-based learning is formulated as an activity. The motive for the activity is to (help) design, experience and reflect on a research project, and the activity includes activities such as developing hypotheses, choosing the method, presenting the findings, etc., so that, in the end, insights are obtained that are of interest to third parties.

In order to understand research in mathematics as an activity, we must first establish that the motive for mathematical research is to identify generally valid correlations and to demonstrate their validity, thus to formulate and prove mathematical propositions. Although there are areas of mathematics that have other motives, for example applying or algorithmizing, these are not a part of the mainstream of mathematical research. Mathematics is therefore described as the "proving discipline." Boero (1999) examined the activity of producing proofs in greater detail and, by analyzing the working processes of mathematicians, developed a model comprised of six stages:

1. non-specific exploration of an area of interest;
2. formulating a generally valid claim pursuant to the conventions in mathematics;
3. specific exploration of the claim;
4. identification and ordering of the steps of the proof in a chain of deductions;
5. writing proofs in the context of mathematics (publication);
6. (optional stage) all parts of the proof formulated according to the rules of formal logic.

It is natural to understand these stages as actions associated with the activity of proving and mathematical research per se, whereby activity is understood within the meaning of Roth and Radford (2011). When considering specific proofs, it is possible to apprehend these broad activities much more precisely and identify the respective goal.

However, in the case of the form of learning that is common in STEM subjects, a lecture with a tutorial, other activities are realized in the tutorials. On the one hand, one is confronted with a concrete task, the solvability of which has already been established, and the essential process consists of repeating and applying processes and methods from the lecture. If one views "completing a tutorial" as an activity unto itself, the motive consists of acquiring routines for problem-solving and to reach a prescribed level of points in order to obtain a proof of performance (the proportion thereof varying, depending on the student). Even if exercises are proof tasks, at least the essential first two stages in the Boero model are omitted. Tasks that are more open-ended, which include processing these two stages, are virtually non-existent.

According to Schneider and Wildt (2009), the activities "inquiry-based learning" and "researching" can be described in a manner that is consistent with the categories of research activity, although the frames of reference are different. All activities of the Boero model – which of course represent the specifics of mathematical research – can be classified as activities in inquiry-based learning. If we take the intention that the course of study is intended to enable students to conduct independent research as the starting point, it becomes clear that the pure teaching-learning format of a lecture with a tutorial is not suitable for producing a broad repertoire of required sub-activities. Conversely, this does not mean that lectures with tutorials would be pointless; rather, they provide an important part of the set of tools needed in stages 4 and 5 of Boero's model.

20.3 Exemplary Analysis of Projects from "matheFL"

This section will present the implementation of inquiry-based learning in mathematics within the "matheFL" project (Bikner-Ahsbahs et al., 2013) and, at the same time, will analyze the work of one project group from an activity-theory perspective. The examples discussed here occurred in conjunction with the lecture "Linear Algebra 1," within the context of an intensified course on linear algebra for students of secondary-school teaching.

The groups were allowed to select their topics from a list of very general topics, for example perspective, population growth, ranking, mixture, ratios, and resources. Good topics should be able to arouse a certain degree of interest in students, and at the same time should provide the mechanism that makes it possible to connect the mathematical theory gained during the semester to an individual application. The group's first task was to come up with their own research questions on the topic. The instructor then commented on these questions where applicable and, if necessary, the group then narrowed these down, expanded them or discarded them and replaced them with other questions over the course of the project. The only requirement was that the research questions should relate to the lecture on linear algebra.

Every two weeks, the progress of the project was documented in a shared wiki document, both so that the groups could keep one another informed and so that support could be provided. Entries in what was known as the "research wiki" included a brief description, who in the group did what during the period of time in question, what progress was made, what problems occurred or what changes were made, and documentation of the sources read or consulted. The groups received feedback on their entries and, where applicable, help and advice on the documented problems.

Each group received individual feedback on the final presentation from the instructor. In addition, in the last session in the semester, the individual groups reflected on their experiences in the project. The course concluded with the submission of the written papers. As a special incentive, the three best papers were published in an anthology for the course and the three groups with the best presentations were asked to give a slightly more elaborate version of their presentation, again as part of a lecture series for interested pupils.

This raises the question as to whether the aforementioned implementation also satisfies Huber's criteria mentioned in Section 3. The project process is based on the external progress in the research process, i.e. research questions were asked, then investigated in a methodically controlled manner. In so doing, the research questions were potentially changed. At the end, the results of the research had to be presented and written down. Structurally, the procedure thus fulfills Huber's criteria.

The support intensity varied widely, however. To what extent can the groups really act independently, or to what extent is the procedure scientific if it is completely independent? Experience has shown that all groups sought assistance and advice, or received help through the feedback on entries over the course of the research. Nevertheless, significant differences in the need for support were discernable: Approximately half of the groups simply needed confirmation that they were on a reasonable path with their deliberations, and these groups tended to have questions concerning details. These groups independently sought out and analyzed literature, adjusted their research questions, generated examples, and were able to complete their presentation and written paper without much help. This constituted *working very independently*. A third of the groups required stronger incentives, i.e. reference to concrete literature, specific concepts that one could consider, or assistance with helping themselves set up an example. These groups were also limited in their ability to select their topic themselves. Here, we would say this was *working largely independently*. The remaining sixth required substantial assistance. These groups was limited in their ability to independently design and reflect on their topic, even with literature or concrete examples. We would classify these groups as *working largely with support*.

Huber also stipulates that the knowledge generated should be potentially interesting for third parties. The knowledge produced, however, depends on the respective progressions of the group work. In Bikner-Ahsbahs et al. (2013), it is therefore suggested that the approach used in the project be referred to as “research-like learning,” when that approach is similar to a research process, based on research questions and knowledge interest. In any event, in the present round of projects, 8 out of 24 groups generated knowledge that could be of interest to third parties. Five groups modelled practical situations using the methods of linear algebra, and three groups attempted to identify mathematical structures in phenomena. Although, in principle, this knowledge is available in the literature, when linked to a concrete phenomenon, it is nevertheless an interdisciplinary example and in this sense of interest to third parties.

20.4 Exemplary Progression of the “Projection and Perspective” Group

The following presents the progress of a group by way of example. Starting just from the title, “Projection and Perspective,” after some research, the group formulated three questions:

1. How are three-dimensional objects mapped onto a two-dimensional surface?
2. What are perspectives and what perspectives are there? What is the connection between these perspectives and linear algebra?
3. How are projections represented in mathematics?

After another session, an additional question arose: What is the projective space? The group then split up (relevant excerpt from the research wiki):

December 18, 2011

After a lengthy investigation of the research questions, we determined that the questions are too general or are difficult to process. For example, we realized that, given the subject of rotation, the question "3D in 2D" is too complex and would take too much time given the scope of the course.

Then we came up with the idea of putting projections into the context of linear algebra:

What is the relationship between projections and maps?

Are there projections that are maps and vice versa?

December 20, 2011

Member 1: I researched projective space and was able to find out that this is an extension of affine space. Upon researching further, however, I discovered an interesting topic: photogrammetry [...]. This deals with the reconstruction of three-dimensional objects from two-dimensional images and thus falls within projective geometry.

Member 2: I researched how objects are projected onto a graph. In addition, I have dealt with subspaces for projections. Going through the calculations, I determined that I needed to do some catching up on the subject of vectors. So first I refreshed my knowledge of vectors, and at the moment I am still trying to better understand the different types of projections.

As can be gleaned from the above, some group members were working on projections and others were trying to figure out what a projective space was all about, which appears to have been a bridge back to linear algebra for the group. In doing further research, the group came upon the topic of "photogrammetry," which later became the main subject of interest. The entries in the research wiki also clearly show how they still were still working on understanding the terms, however. Finally, the group came up with three content areas: projections (central and parallel projections) as linear transformations, the geometric object of "projective space" and the reconstruction of three-dimensional objects from various two-dimensional projections.

In its presentation, the group limited itself to the first two areas due to time constraints. During the reflection session at the last meeting, the students stated that the questions from the audience made them notice that they were still not fully clear on the connection between the areas. The three areas were brought together in the final report, in which the explanation of the projective space was significantly richer than it had been in the presentation.

In the beginning, the research questions were of a type that tried to make something accessible: The first question was not yet necessarily mathematical, but instead posed the technical question, "how does it work?" With the second question, the students attempted to gain an overview of the topic (i.e. that of perspective) and its relationships to linear algebra, which would then be mathematized in the third question. The topic of photogrammetry gained importance as a result of studying the literature and discussions with the lecturers. The wiki even indicates that an object-related interest in photogrammetry developed: students' engagement with linear transformations, equivalence classes and vector subspaces became more intense and their engagement with the mathematical contents gained in depth.

Looking at the project group from the point of view of action theory, one sees that the group clearly shows some of the actions from Boero's model. Group members initially explored their area in a general and unstructured way; for example, they developed ideas about the connection between projections and linear transformations, which came together as concrete hypotheses, which can be found (mostly only implicitly) in the research wiki. They researched in a structured manner, for instance by concentrating on two and three-dimensional projections. Even when the group failed to fully understand the necessary terms in some areas, that understanding improved as a result of reflection. Thus, in this way, not only did learning incorporate methods, research processes and social interactions, but content-related learning was also promoted.

20.5 Outlook

As the example shows, the fundamental idea of inquiry-based learning in the sense of "research-like learning" (Bikner-Ahsbahs et al., 2013) can already be implemented with students within mathematics. In so doing, however, no "small" research results can be expected; instead, one may expect interesting examples, models and applications.

A later point in time for such a course would be appropriate in order to arrive at a "purer" implementation of inquiry-based learning. This is when the "Undergraduate Research Opportunities Program" at the Massachusetts Institute of Technology and similar programs begin and demonstrate the basic feasibility, for example. Nevertheless, to date there have been no large-scale implementations which appeal not only to individual students in a given year of study, but that are, instead, more broadly established. Furthermore, space must also be created in the course of study to accommodate this. This often does not exist in the narrowly controlled curriculum in mathematics, especially for students in teacher education. It is students in teacher education in particular, however, who need these kinds of experiences in their studies so that later, when they are teaching, inquiry-based learning can become a fruitful form of learning in STEM instruction in school as described in the PRIMAS project.

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