

Chapter 2

The French Didactic Tradition in Mathematics



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Abstract This chapter presents the French didactic tradition. It first describes the emergence and development of this tradition according to four key features (role of mathematics and mathematicians, role of theories, role of design of teaching and learning environments, and role of empirical research), and illustrates it through

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two case studies respectively devoted to research carried out within this tradition on algebra and on line symmetry-reflection. It then questions the influence of this tradition through the contributions of four researchers from Germany, Italy, Mexico and Tunisia, before ending with a short epilogue.

Keywords French didactics · Mathematics education · Anthropological theory of the didactic · Theory of didactical situations · Theory of conceptual fields · Didactic research on line symmetry · Didactic research on algebra · Didactic research on proof · Didactic interactions · Influence of French didactics

This chapter is devoted to the French didactic tradition. Reflecting the structure and content of the presentation of this tradition during the Thematic Afternoon at ICME-13, it is structured into two main parts. The first part, the three first sections, describes the emergence and evolution of this tradition, according to the four key features selected to structure the presentation and comparison of the didactic traditions in France, Germany, Italy and The Netherlands at the congress, and illustrates these through two case studies. These focus on two mathematical themes continuously addressed by French researchers from the early eighties, geometrical transformations, more precisely line symmetry and reflection, and algebra. The second part is devoted to the influence of this tradition on other educational cultures, and the connections established, in Europe and beyond. It includes four sections authored by researchers from Germany, Italy, Mexico and Tunisia with first-hand experience of these interactions. Finally, the chapter ends with a short epilogue. Sections 2.1 and 2.8 are co-authored by Artigue and Trouche, Sect. 2.2 by Chesnais and Durand-Guerrier, Sect. 2.3 by Bosch and Chaachoua, Sect. 2.4 by Knipping, Sect. 2.5 by Maschietto, Sect. 2.6 by Romo-Vázquez and Sect. 2.7 by Chellougui.

2.1 The Emergence and Development of the French Didactic Tradition

As announced, we pay specific attention to the four key features that structured the presentation of the different traditions at ICME-13: role of mathematics and mathematicians, of theories, of design of teaching and learning environments, and of empirical research.

2.1.1 A Tradition with Close Relationship to Mathematics

Mathematics is at the core of the French didactic tradition, and many factors contribute to this situation. One of these is the tradition of engagement of French mathematicians in educational issues. As explained in Gispert (2014), this engagement was visible already at the time of the French revolution. The mathematician Condorcet presided over the Committee of Public Instruction, and well-known mathematicians, such as Condorcet, Lagrange, Laplace, Monge, and Legendre, tried to respond to the demand made to mathematicians to become interested in the mathematics education of young people. The role of mathematicians was also prominent at the turn of the twentieth century, in the 1902 reform and the emergence of the idea of *scientific humanities*. Darboux chaired the commission for syllabus revision and mathematicians strongly supported the reform movement, producing books, piloting textbook collections, and giving lectures such as the famous lectures by Borel and Poincaré. Mathematicians were also engaged in the next big curricular reform, that of the New Math period. Lichnerowicz led the commission in charge of the reform. Mathematicians also contributed through the writing of books (see the famous books by Choquet (1964) and Dieudonné (1964) offering contrasting visions on the teaching of geometry), or the organization of courses for teachers, as the APMEP¹ courses by Revuz. Today mathematicians are still active in educational issues, individually with an increasing participation in popularization mathematics activities (see the activities of the association *Animath*² or the website *Images des Mathématiques*³ from the National Centre for Scientific Research), and also through their academic societies as evidenced by the role played nationally by the CFEM,⁴ the French sub-commission of ICMI, of which these societies are active members.

The Institutes of Research on Mathematics Teaching (IREMs⁵) constitute another influential factor (Trouche, 2016). The creation of the IREMs was a recurrent demand from the APMEP that succeeded finally thanks to the events of May 1968. Independent from, but close to mathematics departments, these university structures welcome university mathematicians, teachers, teacher educators, didacticians and historians of mathematics who collaboratively work part-time in thematic groups, developing action-research, teacher training sessions based on their activities and producing material for teaching and teacher education. This structure has strongly influenced the development of French didactic research, nurtured institutional and scientific relationships between didacticians and mathematicians (most IREM directors were and

¹APMEP: Association des professeurs de mathématiques de l'enseignement public. For an APMEP history, see (Barbazo, 2010).

²<http://www.animath.fr> (accessed 2018/01/08).

³<http://images.math.cnrs.fr> (accessed 2018/01/08).

⁴CFEM (Commission française pour l'enseignement des mathématiques) <http://www.cfem.asso.fr> (accessed 2018/01/08).

⁵IREM (Institut de recherche sur l'enseignement des mathématiques) <http://www.univ-irem.fr> (accessed 2018/01/08).

are still today university mathematicians). It has also supported the strong sensitivity of the French didactic community to epistemological and historical issues.

With some notable exceptions such as Vergnaud, the first generation of French didacticists was made of academics recruited as mathematicians by mathematics departments and working part time in an IREM. Didactic research found a natural habitat there, close to the terrain of primary and secondary education, and to mathematics departments. Within less than one decade, it built solid institutional foundations. The first doctorate programs were created in 1975 in Bordeaux, Paris and Strasbourg. A few years later, the *National seminar of didactics of mathematics* was set up with three sessions per year. In 1980, the journal *Recherches en Didactique des Mathématiques* and the biennial *Summer school of didactics of mathematics* were simultaneously created. Later on, in 1992, the creation of the ARDM⁶ complemented this institutionalization process.

2.1.2 A Tradition Based on Three Main Theoretical Pillars

The didactics of mathematics emerged in France with the aim of building a genuine field of scientific research and not just a field of application for other scientific fields such as mathematics or psychology. Thus it required both fundamental and applied dimensions, and needed specific theories and methodologies. Drawing lessons from the innovative activism of the New Math period with the disillusion it had generated, French didacticists gave priority to understanding the complex interaction between mathematics learning and teaching in didactic systems. Building solid theoretical foundations for this new field in tight interaction with empirical research was an essential step. Theories were thus and are still conceived of first as tools for the understanding of mathematics teaching and learning practices and processes, and for the identification of didactic phenomena. It is usual to say that French didactics has three main theoretical pillars: the *theory of didactical situations* due to Brousseau, the *theory of conceptual fields* due to Vergnaud, and the *anthropological theory of the didactic* that emerged from the *theory of didactic transposition*, due to Chevallard. These theories are complex objects that have been developed and consolidated over decades. In what follows, we focus on a few main characteristics of each of them. More information is accessible on the ARDM and CFEM websites, particularly video recorded interviews with these three researchers.⁷

⁶ARDM (Association pour la recherche en didactique des mathématiques) <http://ardm.eu/> (accessed 2018/01/08).

⁷The ARDM website proposed three notes dedicated to these three pioneers: Guy Brousseau, Gérard Vergnaud and Yves Chevallard. On the CFEM website, the reader can access long video recorded interviews with Brousseau, Vergnaud and Chevallard <http://www.cfem.asso.fr/cfem/ICME-13-didactique-francaise> (accessed 2018/01/08).

2.1.2.1 The Theory of Didactical Situations (TDS)

As explained by Brousseau in the long interview prepared for ICME-13 (see also Brousseau, Brousseau, & Warfield, 2014, Chap. 4), in the sixties, he was an elementary teacher interested in the New Math ideas and having himself developed innovative practices. However, he feared the deviations that the implementation of these ideas by elementary teachers without adequate preparation might generate. Brousseau discussed this point with Lichnerowicz (see Sect. 2.1.1) who proposed that he investigate “the limiting conditions for an experiment in the pedagogy of mathematics”. This was the beginning of the story. Brousseau conceived this investigation as the development of what he called an *experimental epistemology* to make clear the difference with Piagetian cognitive epistemology. According to him, this required “to make experiments in classrooms, understand what happens...the conditions of realizations, the effect of decisions”. From that emerged the ‘revolutionary idea’ at the time that the central object of didactic research should be the situation and not the learner, situations being conceived as a system of interactions between three poles: students, teacher and mathematical knowledge.

The theory was thus developed with the conviction that the new didactic field should be supported by methodologies giving an essential role to the design of situations able to make mathematical knowledge emerge from students’ interactions with an appropriate milieu in the social context of classrooms, and to the observation and analysis of classroom implementations. This vision found its expression in the COREM⁸ associated with the elementary school Michelet, which was created in 1972 and would accompany the development of TDS during 25 years, and also in the development of an original design-based methodology named *didactical engineering* (see Sect. 2.1.4) that would rapidly become the privileged methodology in TDS empirical research.

As explained by Brousseau in the same interview, the development of the theory was also fostered by the creation of the doctorate program in Bordeaux in 1975 and the resulting necessity to build a specific didactic discourse. The core concepts of the theory,⁹ those of *adidactical* and *didactical situations*, of *milieu* and *didactic contract*, of *devolution* and *institutionalisation*, the three *dialectics of action, formulation and validation* modelling the different functionalities of mathematics knowledge, and the fundamental distinction made in the theory between “*connaissance*” and “*savoir*” with no equivalent in English,¹⁰ were thus firmly established already in the eighties. From that time, the theory has been evolving for instance with the introduction of the hierarchy of milieus or the refinement of the concept of didactic contract, thanks to the contribution of many researchers. Retrospectively, there is no doubt that the use

⁸COREM: Centre pour l’observation et la recherche sur l’enseignement des mathématiques.

⁹In her text *Invitation to Didactique*, Warfield provides an accessible introduction to these concepts complemented by a glossary: <https://sites.math.washington.edu/~warfield/Inv%20to%20Did66%207-22-06.pdf> (accessed 2018/01/08).

¹⁰“*Connaissance*” labels knowledge engaged by students in a situation while “*savoir*” labels knowledge as an institutional object.

of TDS in the analysis of the functioning of ordinary classrooms from the nineties has played an important role in this evolution.

2.1.2.2 The Theory of Conceptual Fields (TCF)

Vergnaud's trajectory, as explained in the interview prepared for ICME-13, was quite atypical: beginning as a student in a school of economics, he developed an interest in theatre, and more particularly for mime. His interest for understanding the gestures supporting/expressing human activity led him to study psychology at Paris Sorbonne University, where the first course he attended was given by Piaget! This meeting was the source of his interest for analyzing the competencies of a subject performing a given task. Unlike Piaget however, he gave a greater importance to the *content* to be learnt or taught than to the *logic* of the learning. Due to the knowledge he had acquired during his initial studies, Vergnaud chose mathematical learning as his field of research; then he met Brousseau, and engaged in the emerging French community of didactics of mathematics.

In the same interview prepared for ICME-13, he emphasizes a divergence with Brousseau's theoretical approach regarding the concept of situation: while Brousseau is interested in identifying one *fundamental situation* capturing the epistemological essence of a given concept to organize its learning, Vergnaud conceives the process of learning throughout "the confrontation of a subject with a larger and more differentiated set of situations". This point of view led to the development of the *theory of conceptual fields*, a conceptual field being "a space of problems or situations whose processing involves concepts and procedures of several types in close connection" (Vergnaud, 1981, p. 217, translated by the authors). The conceptual fields of additive and multiplicative structures he has especially investigated have become paradigmatic examples. In this perspective, the difference between "connaissance" and "savoir" fades in favour of the notion of *conceptualisation*. Conceptualisation grows up through the development of *schemes*; these are invariant organisations of activity for a class of situations. The operational invariants, *concepts-in-action*, or *theorems-in-action*, are the epistemic components of schemes; they support the subjects' activity on the way to conceptualizing mathematical objects and procedures. In fact, the theory of Vergnaud has extended its influence beyond the field of didactics of mathematics to feed other scientific fields, such as professional didactics, and more generally cognitive psychology.

2.1.2.3 The Anthropological Theory of the Didactic (ATD)

As explained by Chevallard in the interview prepared for ICME-13, the theory of didactic transposition emerged first in a presentation he gave at the Summer school of didactics of mathematics in 1980, followed by the publication of a book (Chevallard, 1985a). Questioning the common vision of taught knowledge as a simple elementarization of scholarly knowledge, this theory made researchers aware of the

complexity of the processes and transformations that take place from the moment it is decided that some piece of knowledge should be taught, to the moment this piece of knowledge is actually taught in classrooms; it helped researchers to make sense of the specific conditions and results of these processes (Chevallard & Bosch, 2014). The anthropological theory of the didactic (ATD) is an extension of this theory. In the interview, Chevallard makes this clear:

The didactic transposition contained the germs of everything that followed it [...] It showed that the shaping of objects for teaching them at a given grade could not be explained only by mathematical reasons. There were other constraints, not of a mathematical nature [...] In fact, the activity of a classroom, of a student, of a scholar is embedded in the anthropological reality. (translated by the authors)

With ATD the perspective became wider (Bosch & Gascón, 2006; Chevallard & Sensevy, 2014). Institutions and institutional relationships to knowledge became basic constructs, while emphasizing their relativity. A general model for human activities (encompassing thus mathematics and didactic practices) was developed in terms of *praxeologies*.¹¹ ATD research was oriented towards the identification of praxeologies, the understanding of their dynamics and their conditions of existence and evolution—their ecology. In order to analyse how praxeologies ‘live’ and ‘die’, and also how they are shaped, modified, disseminated, introduced, transposed and eliminated, different levels of institutional conditions and constraints are considered, from the level of a particular mathematical topic up to the level of a given culture, a civilization or the whole humanity (see the concept of *hierarchy of didactic codetermination*¹²). Such studies are crucial to analyse what kind of praxeologies are selected to be taught and which ones are actually taught; to access the possibilities teachers and students are given to teach and learn in different institutional settings, to understand their limitations and resources, and to envisage alternatives. During the last decade, ATD has been complemented by an original form of didactic engineering in terms of *study and research paths* (see Sect. 2.1.4). It supports Chevallard’s ambition of moving mathematics education from what he calls the *paradigm of visiting works* by analogy with visiting monuments, here mathematics monuments such as the Pythagorean theorem, to the paradigm of *questioning the world* (Chevallard, 2015).

¹¹ A praxeology is a quadruplet made of types of tasks, some techniques used to solve these types of tasks, a discourse (technology) to describe, explain and justify the types of tasks and techniques, and a theory justifying the technology. Types of tasks and techniques are the practical block of the praxeology while technology and theory are its theoretical block.

¹² The hierarchy of didactic codetermination categorizes these conditions and constraints according to ten different levels: topic, theme, sector, domain, discipline, pedagogy, school, society, civilization, humanity.

2.1.3 Theoretical Evolutions

The three theories just presented are the main pillars of the French didactic tradition,¹³ and they are still in a state of flux. Most French researchers are used to combining them in their theoretical research frameworks depending on their research *problématiques*. However, today the theoretical landscape in the French didactic community is not reduced to these pillars and their possible combinations. New theoretical constructions have emerged, reflecting the global evolution of the field. Increasing interest paid to teacher practices and professional development has led to *the double approach of teacher practices* (Robert & Rogalski, 2002, 2005; Vandebrouck, 2013), combining the affordances of cognitive ergonomics and didactics to approach the complexity of teacher practices. Increasing interest paid to semiotic issues is addressed by the Duval's semiotic theory in terms of *semiotic registers of representations* (Duval, 1995) highlighting the decisive role played by conversions between registers in conceptualization. Major societal changes induced by the entrance into the digital era are met by the *instrumental approach to didactics*, highlighting the importance of instrumental geneses and of their management in classrooms (Artigue, 2002), and by the extension of this approach in terms of a *documentational approach to didactics* (Gueudet & Trouche, 2009).

A common characteristic of these constructions however is that they all incorporate elements of the 'three pillars' heritage. The *theory of joint action in didactics* developed by Sensevy is strongly connected with TDS and ATD (Chevallard & Sensevy, 2014); the instrumental and documentational approaches combine ATD and TCF with cognitive ergonomics. The double approach—ergonomics and didactic of teacher practices—reorganizes this heritage within a global activity theory perspective; this heritage plays also a central role in the model of *mathematical working spaces*¹⁴ that emerged more recently (Kuzniak, Tanguay, & Elia, 2016).

These new constructions benefit also from increasing communication between the French didactic community and other research communities in mathematics education and beyond. Communication with the field of cognitive ergonomics, facilitated by Vergnaud, is a good example. Rabardel, Vergnaud's former student, developed a theory of instrumented activity (Rabardel, 1995) just when the didactic community was looking for concepts and models to analyse new phenomena arising in the thread of digitalization, more exactly when symbolic calculators were introduced in classrooms (see Guin & Trouche, 1999). Thus, the notion of scheme of instrumented action developed in a space of conceptual permeability giving birth to the instrumental approach. As this theoretical construction responded to major

¹³Other constructions, for instance the tool-object dialectics by Douady (1986), have also been influential.

¹⁴The purpose of the MWS theory is to provide a tool for the specific study of mathematical work engaged during mathematics sessions. Mathematical work is progressively constructed, as a process of bridging the epistemological and the cognitive aspects in accordance with three different yet intertwined genetic developments, identified in the model as the *semiotic, instrumental and discursive* geneses.

concerns in mathematics education (Lagrange, Artigue, Laborde, & Trouche, 2003), soon its development became an international affair, leading to the development of new concepts, such as instrumental orchestration (Drijvers & Trouche, 2008), which allowed rethinking of the teachers' role in digital environments. The maturation of the instrumental approach, the digitalization of the information and communication supports, the development of the Internet, led to consider, beyond specific artefacts, the wide set of resources that a teacher deals with when preparing a lesson, motivating the development of the documentational approach to didactics (Gueudet, Pepin, & Trouche, 2012).

2.1.4 Relationship to Design

Due to the context in which the French didactics emerged, its epistemological foundations and an easy access to classrooms provided by the IREM network, classroom design has always been conceived of as an essential component of the research work. This situation is reflected by the early emergence of the concept of *didactical engineering* and its predominant methodological role in research for several decades (Artigue, 2014). Didactical engineering is structured into four main phases: preliminary analyses; design and a priori analysis; realization, observation and data collection; a posteriori analysis and validation. Validation is internal, based on the contrast between a priori and a posteriori analyses, not on the comparison between experimental and control groups.

As a research methodology, didactical engineering has been strongly influenced by TDS, the dominant theory when it emerged. This influence is especially visible in the preliminary analyses and the design phases. Preliminary analyses systematically include an epistemological component. In the design of tasks and situations, particular importance is attached to the search for situations which capture the epistemological essence of the mathematics to be learnt (in line with the concept of fundamental situation, Sect. 2.1.2.1); to the optimization of the potential of the milieu for students' autonomous learning (adidactic potential); to the management of devolution and institutionalization processes. Didactical engineering as a research methodology, however, has continuously developed since the early eighties. It has been used with the necessary adaptations in research at all educational levels from kindergarten to university, and also in research on teacher education. Allowing researchers to explore the potential of educational designs that cannot be observed in ordinary classrooms, it has played a particular role in the identification and study of the learning and teaching potential of digital environments. The development of Cabri-Géomètre (Laborde, 1995) has had, from this point of view, an emblematic role in the French community, and at an international level, articulating development of digital environment, development of mathematical tasks and didactical research.

In the last decade, ATD has developed its own design perspective based on what Chevallard calls the *Herbartian model*, in terms of study and research paths (SRP). In it, particular importance is given to the identification of generating questions with

strong mathematics potential, and also to the dialectics at stake between inquiry and the study and criticism of existing cultural answers. This is expressed through the idea of a *dialectic between medias and milieus*. A distinction is also made between *finalized SRP* whose praxeological aim is clear, and *non-finalized SRP* corresponding to more open forms of work such as interdisciplinary projects. More globally, the evolution of knowledge and perspectives in the field has promoted more flexible and collaborative forms of didactical engineering. A deep reflection on more than thirty years of didactical engineering took place at the 2009 *Summer school of didactics of mathematics* (Margolinas et al., 2011).

Design as a development activity has naturally taken place, especially in the action research activities developed in the IREMs and at the INRP (National Institute for Pedagogical Research, now IFé¹⁵ French Institute of Education). These have been more or less influenced by research products, which have also disseminated through textbooks and curricular documents, but with a great deal of reduction and distortion. However, up to recently, issues of development, dissemination or up-scaling have been the focus of only a few research projects. The project ACE (Arithmetic and understanding at primary school)¹⁶ supported by IFé, and based on the idea of *cooperative didactical engineering* (Joffredo-Le Brun, Morelato, Sensevy, & Quilio, 2017) is a notable exception. Another interesting evolution in this respect is the idea of *second-generation didactical engineering* developed by Perrin-Glorian (2011).

2.1.5 The Role of Empirical Research

Empirical research has ever played a major role in the French didactics. It takes a variety of forms. However, empirical research is mainly qualitative; large-scale studies are not so frequent, and the use of randomized samples even less. Statistical tools are used, but more for data analysis than statistical inferences. *Implicative analysis* (Gras, 1996) is an example of statistical method for data analysis which has been initiated by Gras, a French didactician, and whose users, in and beyond mathematics education, organize regular conferences.

Realizations in classrooms have always been given a strong importance in empirical research. This took place within the framework of didactical engineering during the first decades of research development as explained above. However, since the early nineties the distortion frequently observed in the dissemination of didactical engineering products led to increasing attention being paid to teachers' representations and practices. As a consequence, more importance was given to naturalistic observations in empirical methodologies. The enrolment of many didacticians in the IUFM (University institutes for teacher education) from their creation in 1991,¹⁷ and

¹⁵<http://ife.ens-lyon.fr/ife> (accessed 2018/01/08).

¹⁶<http://python.espe-bretagne.fr/ace/> (accessed 2018/01/08).

¹⁷IUFM became in 2013 ESPÉ (Higher schools for teacher professional development and for education, <http://www.reseau-espe.fr/>).

the subsequent move from the work with expert teachers in the IREMs to the work with pre-service and novice teachers, also contributed to this move.

The influence of the theory of didactic transposition and subsequently of ATD also expressed in empirical research. It led to the development of techniques for the analysis of transpositive processes, the identification of mathematics and didactic praxeologies through a diversity of sources (textbooks, curricular documents, educational material...). Empirical research is also influenced by technological evolution with the increasing use of videos, for instance in studies of teacher practices, and by increasing attention being paid to semiotic and linguistic processes requiring very detailed micro-analyses and appropriate tools (Forest, 2012). The involvement of teachers working with researchers in empirical research is one of the new issues that have been developed in French didactic research. It is central in the research studies considering collective work of groups of teachers and researchers, and working on the evolution of teacher practices; it is also central for the exploration of issues such as the documentary work of teachers mentioned above (Gueudet, Pepin, & Trouche, 2013).

The next two sections illustrate this general description through two case studies, regarding the research carried out on line symmetry-reflection and on algebra over several decades. These themes have been selected to show the importance of mathematics in the French didactic tradition, and also because they offer complementary perspectives on this tradition.

2.2 Research on Line Symmetry and Reflection in the French Didactic Tradition

Line symmetry and reflection¹⁸ appear as a fundamental subject, both in mathematics as a science and in mathematics education. In particular, as Longo (2012) points out,

We need to ground mathematical proofs also on geometric judgments which are no less solid than logical ones: “symmetry”, for example, is at least as fundamental as the logical “modus ponens”; it features heavily in mathematical constructions and proofs. (p. 53)

Moreover, several characteristics of this subject make it an important and interesting research topic for the didactics of mathematics: line symmetry is not only a mathematical object but also an everyday notion, familiar to students, and it is also involved in many professional activities; it is taught from primary school to university; among geometric transformations, it is a core object as a generator of the group of isometries of the Euclidean plane; it has played a central role in the (French) geometry curriculum since the New Math reform. This curriculum and the subsequent ones,

¹⁸In French, the expression «symétrie axiale» refers both to the property of a figure (line/reflection symmetry) and the geometric transformation (reflection). Therefore, these two aspects of the concept are probably more intertwined in French school and research than in English speaking countries. The word “réflexion” might be used in French but only when considering negative isometries in spaces of dimension greater than 2.

indeed, obeyed the logic of a progressive introduction of geometrical transformations, the other subjects being taught with reference to transformations (Tavignot, 1993)—even if this logic faded progressively since the eighties (Chesnais, 2012). For all these reasons, the learning and teaching of reflection and line symmetry has been the subject of a great deal of didactic research in France from the eighties up to the present.

From the first studies about line symmetry and reflection in the eighties, researchers have explored and modelled students' conceptions and teachers' decisions using the theory of conceptual fields (Vergnaud, 1991, 2009) and TDS (Brousseau, 1986, 1997), and later the *cKç model of knowledge*¹⁹ (Balacheff & Margolinas, 2005; Balacheff, 2013), in relation to questions about classroom design, in particular through the methodology of didactical engineering (see Sect. 2.1). Since the nineties and the increasing interest in understanding how ordinary classrooms work, however, new questions have arisen about teachers' practices. They were tackled in particular using the double didactic-ergonomic approach mentioned in Sect. 2.1. In current research, as the role of language in teaching and learning processes has become a crucial subject for part of the French research community, line symmetry and reflection once again appear as a particularly interesting subject to be investigated, in particular when using a logical analysis of language (Durand-Guerrier, 2013) or studying the relations between physical actions and verbal productions in mathematical activity.

This example is particularly relevant for a case study in the French didactic tradition, showing the crucial role of concepts and of curriculum developments in research on the one hand, and the progressive capitalization of research results, the evolution of research questions and the links between theoretical and empirical aspects on the other hand.

2.2.1 Students' Conceptions, Including Proof and Proving, and Classroom Design

The first Ph.D. thesis on the teaching and learning of line symmetry and reflection was defended by Grenier (1988). Her study connected TCF and the methodology of didactical engineering relying on TDS. She studied very precisely students' conceptions of line symmetry, using activities where students were asked to answer open questions such as (Fig. 2.1).

In addition, various tasks were proposed to students of constructing symmetrical drawings and figures that were crucial in identifying erroneous conceptions and

¹⁹cKç (conception, knowing, concept) was developed by Balacheff to build a bridge between mathematics education and research in educational technology. It proposes a model of learners' conceptions inspired by TDS and TCF. In it conceptions are defined as quadruplets (P, R, L, Σ) in which P is a set of problems, R a set of operators, L a representation system, and Σ a control structure. As pointed out in (Balacheff, 2013) the first three elements are almost directly borrowed from Vergnaud's model of conception as a triplet.

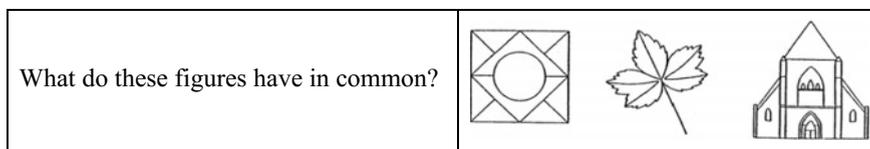


Fig. 2.1 Example of question to study students' conceptions (Grenier 1988, p. 105)

key didactical variables of tasks. This constituted essential results on which further research was based (see below). In her thesis, Grenier also designed and trialled a didactical engineering for the teaching of line symmetry and reflection in middle school (Grade 6). Beyond the fine grained analysis of students' conceptions and difficulties presented in the study, her research produced the important result that to communicate a teaching process to teachers, it is necessary to consider not only students' conceptions, but also teachers' representations about the mathematic knowledge at stake, the previous knowledge of the students, and the way students develop their knowledge.

Relying on the results of Grenier, Tahri (1993) developed a theoretical model of students' conceptions of line symmetry. She proposed a modelling of didactical interaction in a hybrid tutorial based on the micro-world Cabri-Géomètre in order to analyse teachers' decisions. Tahri's results served then as a starting point for the research of Lima (2006). Lima (2006) referred to the $cK\zeta$ (conception, knowing, concept) model of knowledge in order to identify a priori the controls that the students can mobilize when solving problems related to the construction and the recognition of symmetrical figures. In the $cK\zeta$ model, a conception is characterized by a set of problems (P), a set of operators (R), a representation system (L), and also a control structure (S), as explained above. Lima's study showed the relevance of the modelling of the structure of control in the case of line symmetry; in particular, it allowed the author to reconstruct some coherent reasoning in cases where students' answers seemed confused. This refinement of the identification of students' conceptions of line symmetry was then used to study teachers' decisions when designing tasks aimed at allowing students to reach adequate conceptions of line symmetry. Lima concluded that a next step was to identify problems favouring the transition from a given student's conception to a target conception, both being known.

$CK\zeta$ was also used in the Ph.D. of Miyakawa (2005) with a focus on validation. Considering with Balacheff that «learning proof is learning mathematics», Miyakawa studied the relationships between mathematical knowledge and proof in the case of line symmetry. His main research question concerned the gap between pragmatic validation, using pragmatic rules that cannot be proved in the theory (e.g. relying on perception, drawings or mental experience) and theoretical validation relying on rules that can be proved in the theory (Euclidian geometry). Miyakawa especially focused on rules that grade 9 students are supposed to mobilize when asked to solve either construction problems or proving problems. He showed that, while the rules at stake were apparently the same from a theoretical point of view, students able

to solve construction problems using explicitly the appropriate rules might not be able to use the corresponding rules in proving problems. The author concluded that although construction problems are playing an important role to overcome the gap between pragmatic and theoretical validation, they are not sufficient for this purpose.

A new thesis, articulating the TCF and the geometrical working spaces approach mentioned in Sect. 2.1 was then defended by Bulf (2008). She studied the effects of reflection on the conceptualization of other isometries and on the nature of geometrical work at secondary school. As part of her research, she was interested in the role of symmetry in the work of stone carvers and carpenters. Considering with Vergnaud that action plays a crucial role in conceptualization in mathematics, she tried to identify invariants through observations and interviews. The results of her study support the claim that the concept of symmetry organizes the action of these stone carvers and carpenters.

2.2.2 *The Study of Teachers' Practices and Their Effects on Students' Learning*

As Chevallard (1997) points out, it took a long time to problematize teaching practices:

Considered only in terms of his weaknesses, [...] or, on the other hand, as the didactician's double [...], the teacher largely remained a virtual object in our research field. As a result, didactical modelling of the teacher's position [...] is still in its infancy. (p. 24, translated by the authors)

Results of research in didactics in France—and, seemingly, in the rest of the world—progressively led researchers to problematize the question of the role of the teacher in the teaching and learning process. Questions arose in particular from the “transmission” of didactical engineering, as illustrated by Grenier's Ph.D. thesis. For example, teachers would not offer sufficient time for the students to explore problems. Instead they would give hints or answers quickly. It also appeared that when implementing a situation, teachers would not necessarily institutionalize with students what the researcher had planned. Along the same lines, Perrin-Glorian, when working with students in schools from disadvantaged areas (Perrin-Glorian, 1992), identified that the teachers' decisions are not necessarily coherent with the logic of the situation (and knowledge at stake). Some hypotheses emerged about the factors that might cause these “distortions”. Mainly, researchers explained them by the need for teachers to adjust to the reality of classrooms and students:

But it also appears that control of problem-situations cannot guarantee the reproducibility of the process, because some of these discrepancies result from decisions that the teacher takes to respond to the reality of the class. (Grenier, 1988, p. 7, translated by the authors)

A second important hypothesis was that the researchers and teachers had different conceptions of what learning and teaching mathematics is. Hence, in the eighties,

many researchers decided to investigate the role of the teacher in the teaching process and to study ordinary teachers' practices. This led to new developments within existing theories and to the emergence of a new approach. A *structured model of milieu* was developed in TDS (Brousseau, 1986; Margolinas, 2004); modelling of the teacher's position was initiated in ATD by Chevallard (1999). Robert and Rogalski developed the double approach of teachers' practices mentioned in 1.3 based on Activity Theory and the socio-historical theory of Vygostky, and connecting didactic and ergonomic points of view on teachers' practices. Research in this approach is driven by the investigation of regularities and variability of ordinary teaching practices depending on contexts, mathematical subjects, grades, teachers, etc. This investigation then allows the identification of causes and rationales underlying teachers' practices. Another focus is to investigate the effects of teaching practices on students' learning. Chesnais' Ph.D. thesis about the teaching and learning of reflection and line symmetry (Chesnais, 2009) is a good example of this evolution. Relying on previous research and using the didactic and ergonomic double approach, she compared the teaching and learning of these subjects in the 6th grade between a school situated in a socially disadvantaged area and an ordinary one. The research was based on both 'naturalistic observation' (during the first year) and an experiment (during the second year) which consisted of the transmission of a teaching scenario about reflection and line symmetry elaborated by one teacher from the ordinary school to another one from the socially disadvantaged school. The results showed that socially disadvantaged students could perform "as well as" ordinary ones provided that certain conditions are fulfilled. In particular, the following crucial conditions were identified: an important conceptual ambition of the teaching scenario, its coherence and "robustness" and the fact that the teacher receiving it is sufficiently aware of some specificities of the content and students' learning difficulties. Moreover, the research showed that multiple reasons drove the teachers' choices, and explained some differences identified between them: the fact that they taught to different audiences had probably a great influence but also their experience in teaching, determining their ability to identify the important issues about the teaching of line symmetry and reflection (in this case, it appeared that the experienced teacher of the ordinary school had a more coherent idea about the teaching of line symmetry and reflection because she had experienced teaching with older and more detailed teaching instructions). The research also suggested that collaborative work between teachers might be a good lever for professional development under certain conditions (in this work, the role of the researcher as an intermediary was crucial because the first teacher was not clearly conscious of what made her scenario efficient).

2.2.3 *Current Research*

A consistent part of recent French research heads toward a thorough investigation of the role of language in the teaching and learning of mathematics. Research globally considers language either as an object of learning (as part of concepts), as a medium

for learning (its role in the conceptualisation process) and for teaching, and/or as a methodological means for researchers to get access to students' and teachers' activity. Line symmetry and reflection represent once again an interesting subject with regard to the role of language. Indeed, a logical analysis (Vergnaud, 2009; Durand-Guerrier, 2013) shows that symmetry can be considered as a property of a given figure but also, via reflection, as a ternary relation involving two figures and an axis, or as a geometric transformation involving points, or even as a binary relation—when considering two figures and questioning the existence of a reflection transforming one into the other. Studying how these “variations of the meanings of words” may be expressed in the French language shows an incredible complexity, in particular because of the polysemy of the words “symétrie” and “symétrique”. This makes symmetry a good subject to study the relationships between action and language in mathematical activities, and how teachers and students deal with this complexity. For example Chesnais and Mathé (2015) showed that 5th grade²⁰ pupils' conceptualisation of the “flipping over property”²¹ of reflection results from the articulation of several types of interactions between students and milieu mediated by language, instruments like tracing paper, and by the teacher: for example, they showed that the manipulation of the tracing paper (flipping it to check if the initial figure and its image match) needs to be explicitly identified and that it is complementary to the use of an adequate vocabulary. It also appeared that teachers identified these issues differently.

Questions about teacher education and the development of teaching resources also play an important role in recent developments of research in the field. Here again, line symmetry and reflection were chosen as subjects for research. For instance, Perrin-Glorian elaborated the concept of second generation didactical engineering mentioned in Sect. 2.1 in the context of a long term and original research project regarding the teaching and learning of geometry, in which reflection and line symmetry play a crucial role (Perrin-Glorian, Mathé, & Leclercq, 2013). Searching for the construction of coherence from primary to secondary school, this project also led to a deepening of reflection on the role that the use of instruments can play in a progressive conceptualisation of geometrical objects.

We cannot enter into more details about this important set of research, but through this case study, we hope to have made clear an important feature of the development of didactic research in the French tradition: the intertwined progression of research questions, theoretical elaborations and empirical studies, coherently over long periods of time.

²⁰The 5th grade corresponds to the last grade of primary school in France.

²¹We refer here to the fact that reflection is a *negative* isometry.

2.3 Research on School Algebra in the French Didactic Tradition. From Didactic Transposition to Instrumental Genesis

Research on school algebra started at the very beginning of the development of the French didactic tradition in the 1980s with the first studies of the didactic transposition process (Chevallard, 1985a, b, 1994). Since then, and for more than 30 years, school algebra has been the touchstone of various approaches and research methodologies, which spread in the French, Spanish and Italian speaking communities, thanks to numerous collaborations and common research seminars like SFIDA (see Sect. 2.5).

Fortunately, all this work has been synthesized in two very good resources. The first one is a special issue of the journal *Recherches en Didactique des Mathématiques* (Coulange, Drouhard, Dorier, & Robert, 2012), which presents a summary of recent works in the subject, with studies covering a wide range of school mathematics, from the last years of primary school to the university. It includes sixteen papers grouped in two sections: a first one on teaching algebra practices and a second one presenting cross-perspectives with researchers from other traditions. The second resource is the survey presented by Chaachoua (2015), and by Coppé and Grugeon (2015) in two lectures given at the 17th *Summer school of didactics of mathematics* in 2013 (see Butlen et al., 2015). They focus on the effective and potential impacts between didactic research, the teaching profession and other instances intervening in instructional processes (curriculum developers, textbooks authors, teacher educators, policy makers, etc.). The discussion also deals with transfers that have not taken place and highlights various difficulties that seem to remain embedded in the school institution.

We cannot present this amount of work in a few pages. Instead of focusing on the results and the various contributions of each team, we have selected three core research questions that have guided these investigations in the field of secondary school mathematics. First, the analysis of the didactic transposition process and the associated questions about what algebra is and what kind of algebra is taught at school. Second, research based on didactical engineering proposals which address the question of what algebra could be taught and under what kind of conditions. Finally, both issues are approached focusing on ICT to show how computer-assisted tools can modify not only the way to teach algebra but also its own nature as a mathematical activity.

2.3.1 *What Algebra Is to Be Taught: Didactic Transposition Constraints*

To understand the role played by research on school algebra in the development of the field of didactics of mathematics, we should look back to the 1980s and Chevallard's attempt to approach secondary school mathematics from the new perspective

opened by the theory of didactic situations (TDS). At this period, TDS was mainly focused on pre-school and primary school mathematics. Its first enlargements to secondary school mathematics gave rise to the analyses in terms of didactic transposition (Chevallard, 1985a). Algebra was one of the case studies that received more attention. Contrary to the majority of investigations of this period, studies on the didactic transposition processes directly adopted an institutional perspective anchored in deep epistemological and historical analyses.

The first studies (Chevallard, 1985b) pointed out the nature and function of algebra in scholarly mathematics since the work of Vieta and Descartes, and its fading importance in secondary school curricula after the New Maths reform. What appears is a certain lack of definition of algebra as a school mathematical domain (in France as well as in many other countries), centred on the resolution of first and second degree equations. Many elements of what constitutes the driving force of the algebraic work (use of parameters and inter-play between parameters and unknowns, global modelling of arithmetic and geometrical systems, etc.) have disappeared from school curricula or only play a formal role. Part of this evolution can be explained by a cultural difficulty in accepting the primarily written nature of algebra, which strongly contrasts with the orality of the arithmetical world (Chevallard, 1989, 1990; see also Bosch, 2015).

A first attempt to describe the specificities of algebraic work was proposed by considering a broad notion of modelling that covers the modelling of extra-mathematical as well as mathematical systems (Chevallard, 1989). Analyses of algebra as a modelling process were later developed in terms of the new epistemological elements proposed by the Anthropological Theory of the Didactic based on the notion of praxeology. This extension brought about new research questions, such as the role of algebra and the students' difficulties in school institutional transitions (Gruegon, 1995) or the constraints appearing in the teaching of algebra when it is conceived as a process of algebraisation of mathematical praxeologies (Bolea, Bosch, & Gascón, 1999). In this context, algebra appears linked to the process of modelling and enlarging previously established praxeologies.

2.3.2 Teaching Algebra at Secondary School Level

While the aforementioned studies are more focused on what is considered as the external didactic transposition—the passage from scholarly knowledge to the knowledge to be taught—the question of “what algebra could be taught” is addressed by investigations approaching the second step of the didactic transposition process, the internal transposition, which transforms the knowledge to be taught into knowledge actually taught. This research addresses teaching and learning practices, either from a ‘naturalistic’ perspective considering what is effectively taught and learnt as algebra at secondary level—as well as what is not taught anymore—or in the design and implementation of new proposals following the methodology of didactical engineering. Research questions change from “What is (and what is not) algebra as knowledge

to be taught?” to “What can be taught as algebra in teaching institutions today?”. However, the answers elaborated to the first question remain crucial as methodological hypotheses. Experimental studies proposing new conditions to teach new kinds of algebraic activities rely on the previous elaboration of a priori epistemological models (what is considered as algebra) and didactic models (how is algebra taught and learnt).

The notion of calculation program (*programme de calcul*), used to rebuild the relationships between algebra and arithmetic in a modelling perspective, has been at the core of some of these instructional proposals, especially in the work carried out in Spain by the team led by Gascón and Bosch. Following Chevallard’s proposal to consider algebra as the “science of calculation programs”, a *reference epistemological model* is defined in terms of stages of the process of algebraisation (Bosch, 2015; Ruiz-Munzón, Bosch, & Gascón, 2007, 2013). This redefinition of school algebra appears to be an effective tool for the analysis of curricula and traditional teaching proposals. It shows that the algebraisation process in lower secondary school mathematical activities is very limited and contrasts with the “fully-algebraised” mathematics in higher secondary school or first year of university. It also provides grounds for new innovative instructional processes like those based on the notions of *study and research activities* and *study and research paths* (see Sect. 2.1.4) that cover the introduction of negative numbers in an algebraic context and the link of elementary algebra with functional modelling (Ruiz-Munzón, Matheron, Bosch, & Gascón, 2012).

2.3.3 Algebra and ICT

Finally, important investigations of school algebra address questions related to the integration of ICT in school mathematics. They look at the way this integration might influence not only students’ and teachers’ activities, but also the nature of algebraic work when paper and pencil work is enriched with new tools such as a spreadsheet, a CAS or a specific software specially designed to introduce and make sense of elementary algebraic manipulations. This research line starts from the hypothesis that ICT tools turn out to be operational when they become part of the students’ didactic milieu, thus focusing on the importance and nature of ICT tools feedback on the resolution of tasks. However, the integration of ICT tools in the didactic milieu cannot be taken for granted as research on instrumental genesis has shown. Many investigations have produced evidence on new teachers’ difficulties and problems of legitimacy to carry out such integration, pointing at didactic phenomena like the double reference (paper and pencil versus ICT tools) (Artigue, Assude, Grugeon, & Lenfant, 2001) or the emergence of new types of tasks raised by the new semiotic representations produced by ICT tools. In fact, the epistemological and didactic dimensions (what is algebra and how to teach it) appear so closely interrelated that a broad perspective is necessary to analyse the didactic transposition processes. The instrumental approach of technological integration emerged, especially in the

case of the use of CAS and technologies not initially thought for teaching. This approach was then extended to other technologies and has experienced significant internationalization (Drijvers, 2013).

In respect to technologies specifically designed for teaching, particular software devices have been designed on the basis of didactic investigations. They thus appear as a paradigm of connection between fundamental research, teaching development and empirical validation of ICT didactic tools (Chaachoua, 2015). The first one is *Aplusix*,²² a micro world especially designed for the practice of elementary algebra that remains very close to the students' paper and pencil manipulations, while providing feed-back based on a detailed epistemological and didactic analysis of potential learning processes (Trgalová & Chaachoua, 2009). *Aplusix* is based on a model of reasoning by equivalence between two algebraic expressions which is defined by the fact that they have the same denotation (Arzarello, Bazzini, & Chiappini, 2001). Therefore, *Aplusix* enables students to work on the relationship between sense and denotation, which is essential to effectuate and understand the transformation of algebraic expressions. The second device is *Pépité*, a computer-based environment providing diagnostics of students' competences in elementary algebra, as well as tools to manage the learning heterogeneity with the proposal of differentiating teaching paths (Pilet, Chenevotot, Grugeon, El Kechäï, & Delozanne, 2013). Both *Aplusix* and *Pépité* are the fruit of long term and continuous research work in close collaboration with computer scientists.

On the whole, all these investigations have contributed to a fundamental epistemological questioning on the nature of algebraic work and its components, sometimes enriched by linguistic or semiotic contributions. They all share the research aim of better identifying the universe of didactic possibilities offered by today's school systems and of determining the local conditions that would allow a renewed teaching of the field, directly aligned with mathematical work in primary school and connected with the world of functions and calculus for which it is preparatory

The longitudinal dimension of algebra and its predominance in higher secondary education and beyond show that one cannot approach school algebra without questioning the rationale of compulsory education as a whole and how it can supply the mathematical needs of citizens. We are thus led to the initial project of Brousseau and the fundamental problems that motivated the development of TDS: to reconstruct the compulsory mathematical curriculum based on democratic and effective principles that can be discussed in an objective and non-authoritarian way.

Let us finish this section with a tribute to Jean-Philippe Drouhard, one of the founders of the Franco-Italian Seminar in Didactics of Algebra (SFIDA, 1992–2012, see Sect. 2.5), who took special leadership in the development of the research on school algebra and devoted his research life in didactics to the study of the semio-linguistic complexity of elementary algebra (Drouhard, 2010).

After these two cases studies, we enter the second part of this chapter which is devoted to the influence of the French didactic tradition on other educational cultures, and the connections established, in Europe and beyond. These connections have a

²²<http://www.aplusix.com>.

long history as evidenced in Artigue (2016), and the IREMs have played an important role in their development, together with the many foreign students who have prepared their doctorate in France or been involved in co-supervision programs since the early eighties. The authors of the four following sections, from Germany, Italy, Mexico and Tunisia were such doctorate students who have now pursued their careers in their own country. These four case studies illustrate different ways through which a research tradition may diffuse, influencing individual researchers or communities, and the source of enrichment that these interactions are for the tradition itself.

2.4 View of the French Tradition Through the Lens of Validation and Proof

In this section I (Christine Knipping) present my view of the French didactic tradition. I take the position of a critical friend (the role given to me at ICME-13), and I consider this tradition through the lens of didactic research on validation and proof. This topic is at the core of my own research since my doctoral work, which I pursued in France and Germany. I will structure the section around three main strengths that I see in the French tradition: Cohesion, Interchange, and Dissemination. The first strength is *Cohesion* as the French community has a shared knowledge experience and theoretical frameworks that make it possible to speak of a French Didactique. The second strength is an open *Interchange* within the community and with others. The third strength allows the *dissemination* of ideas in the wider world of mathematics education. These three strengths, which have also influenced my own research, will be illustrated by examples from a personal perspective.

2.4.1 Cohesion

As a student, having just finished my Masters Degree in Mathematics, Philosophy and Education in the 1990s in Germany, I went to Paris, and enrolled as a student in the DEA-Programme in Didactics of Mathematics at University Paris 7 (see Sect. 2.1). Courses were well structured and introduced students to key ideas in the French didactic tradition: the theory of didactical situations (TDS) developed by Brousseau, the theory of conceptual fields due to Vergnaud, and the anthropological theory of the didactic that emerged from the theory of didactic transposition, conceptualised by Chevallard. The courses were taught by a wide range of colleagues from the research group DIDIREM of Paris 7, which is now the LDAR (*Laboratoire de Didactique André Revuz*). Among others Artigue, Douady, Perrin-Glorian, as well as colleagues from several teacher training institutes (IUFM) were involved. These colleagues not only introduced us to the theoretical pillars of the French tradition in mathematics education, but they also showed us how their own research and recent doctoral work

was based on these traditions. This made us aware of the power of French conceptual frameworks and demonstrated vividly how they could be applied. It also showed us some empirical results these frameworks had led to and how French research in mathematics education was expanding quickly at that time. The specific foci of research and research questions in the French community were an obvious strength, but phenomena that were not in these foci were not captured. A few of our professors reflected on this and made us aware that classroom interactions, issues of social justice and cultural contexts were more difficult to capture with the given theoretical frameworks. Also validation and proof, a topic I was very interested in, was hardly covered by our coursework in Paris, while in Grenoble there was a clear research focus on validation and proof since the 1980s (Balacheff, 1988), that probably was reflected in their DEA programme at the time.

The French National Seminar in Didactics of Mathematics, which we were invited to attend, was another experience of this phenomenon of cohesion. Many of the presentations at the National Seminar were based on the three pillars of French didactics and also French Ph.D.'s followed these lines. But there were topics and approaches beyond these frameworks that seemed to be important for the French community; validation and proof was apparently one of them. Presentations in this direction were regularly and vividly discussed at the National Seminar and quickly became public knowledge within the community. For example, Imre Lakatos' striking work *Proofs and Refutations* was not only translated by Balacheff and Laborde into French in 1984, but also presented to the French community at one of the first National Seminars. Looking through the *Actes du Séminaire National de Didactique des Mathématiques* shows that validation and proof is a consistent theme over decades. In this area of research French colleagues recognise and reference each other's work, but cohesion seems less strong in this context. Besides Balacheff's first school experiments with Lakatos' quasi-empirical approach (Balacheff, 1987), Legrand introduced the *scientific debate* as another way to establish processes of validation and proof in the mathematics class (Legrand, 2001). Coming from logic, Arsac and Durand-Guerrier approached the topic from a more traditional way, attempting to make proof accessible to students from this side (Arsac & Durand-Guerrier, 2000). Publications not only in the proceedings of the National Seminar, but also in the journal *Recherches en Didactique des Mathématiques* (RDM), made the diverse approaches in the field of validation and proof accessible. So in this context Interchange seemed the strength of the French community.

2.4.2 Interchange

Such interchange, within the community as described above but also with researchers from other countries, is in my view another striking strength. The *Colloque Franco-Allemand de Didactique des Mathématiques et de l'Informatique* is one example of this and valued by its publication in the book series associated with *Recherches en Didactique des Mathématiques* (Laborde, 1988). Many other on-going international

exchanges, as described for example in the following sections of this chapter, also illustrate this strength of the French community. I received a vivid impression of the passion for interchange at the tenth *Summer school of didactics of mathematics* held in Houlgate in 1999. The primary goal of such summer schools is to serve as a working site for researchers to study the work of their colleagues. Young researchers are welcome, but not the focus. Consistently colleagues from other countries are invited. Here Boero (Italy), Duval (France) and Herbst (USA) contributed to the topic “Validation, proof and formulation”, which was one of the four themes in 1999. Over the course of the summer school their different point of views—cognitive versus socio-cultural—became not only obvious, but were defended and challenged in many ways. These were inspiring debates. This interchange was also reflected in the *International Newsletter on the Teaching and Learning of Proof*,²³ established in 1997 whose first editor was Balacheff. Innovative articles and also the careful listing of recent publications made this Newsletter a rich resource for researchers in the area at the time. Colleagues from diverse countries published and also edited this online journal, which continues with Mariotti and Pedemonte as editors. Debates on argumentation and mathematical proof, their meaning and contextualization, as well as cultural differences are of interest for the Newsletter. Sekiguchi’s and Miyazaki’s (2000) publication on ‘Argumentation and Mathematical Proof in Japan’ for example already explicitly addressed this issue. One section in the Newsletter also explicitly highlights working groups and sections of international conferences that deal with the topic of validation and proof. References to authors and papers are listed and made accessible in this way. Interchange is therefore highly valued and also expressed by the fact that some articles are published in multiple languages (English, French, Spanish).

2.4.3 Dissemination

Interchange is in multiple ways related to dissemination, another strength of the French didactic tradition. Looking for vivid exchange also implies that perspectives and approaches from the community become more widely known and disseminated. The research work of Herbst is prominent in this way; French tradition had a visible impact on his own unique work, which then influenced not only his doctoral students but also other colleagues in the US. This is evident as well in the many bi-national theses (French-German; French-Italian) in mathematics education, including quite a few on the topic of validation and proof, my own Ph.D. thesis among them (Knipping, 2003). French universities were in general highly committed to this kind of double degree and had international graduate programs and inter-university coalitions with many countries. As Ph.D. students we were highly influenced by French research traditions and incorporated ideas from the French mathematics education community into our work. Pedemonte’s Ph.D. thesis (2002) entitled *Etude didactique et cogni-*

²³<http://www.lettredelapreuve.org> (accessed 2018/01/08).

tive des rapports de l'argumentation et de la démonstration dans l'apprentissage des mathématiques (Didactic and cognitive analyses of the relationship between argumentation and proof in mathematics learning) is an example of this. Her work and ideas were then further disseminated into the international validation and proof research community.

The working group on *argumentation and proof*, which has been meeting since the Third Conference of the European Society for Research in Mathematics Education (CERME 3, 2003) in Bellaria has been a vivid place not only for discussion and exchange, but also a site where ideas from the French community have continuously been prominent. French colleagues are always present, serving in guiding functions, and scholars like me who are familiar with ideas and approaches of the French tradition and have actively used them in our own work spread these ideas further into the international community. Interesting crossover work has also emerged throughout the years between different disciplines. Miyakawa, who also did his Ph.D. work in Grenoble with Balacheff (Miyakawa, 2005), is another interesting example of a colleague who stands for dissemination of French ideas and is interested in the kind of interdisciplinary work that I see as characteristic for French research in the field of validation and proof. He is now well established in Japan but he continues to collaborate with French colleagues and reaches out to other disciplines. Recently he presented at CERME 10 a paper with the title *Evolution of proof form in Japanese geometry textbooks* together with Cousin from the Lyon Institute of East Asian Studies (Cousin and Miyakawa, 2017). Collaboratively they use the anthropological theory of the didactic (ATD) to study the didactic transposition of proofs in the Japanese educational system and culture and to better understand proof taught/learned in this institutional context. Reflecting on the conditions and constraints specific to this institution Miyakawa and Cousin help us also to see in general the nature of difficulties for students in the context of proof-and-proving from a new perspective. French theoretical frameworks are again fruitful for this kind of inter-cultural-comparative work.

In closing, from my perspective as a critical friend working in the field of validation and proof, these three strengths, Cohesion, Interchange, and Dissemination, have contributed to the success of the French Didactique. As a *critical* friend, I should also observe that each of these strengths comes with costs. Theoretical cohesion can limit the research questions that can be addressed, and research groups strongly focused on one area inevitably neglect others. This is reflected in some limitations in interchange and dissemination. French voices are clearly heard in some contexts, but hardly at all in others, and interchanges are sometimes unbalanced. Overall, however, it is clear that France has been fortunate to have a strong community in didactics, supported by a range of institutions that foster interchange and dissemination, of which this set of presentations at ICME is yet another example.

2.5 Didactic Interactions Between France and Italy. A Personal Journey

Didactic interactions between Italy and France have a long history. For instance, Italian researchers participated in the French *Summer schools of didactics of mathematics* from the first. In this section, after pointing out some structures that have nurtured these interactions, I (Michela Maschietto) present and discuss them through the lenses of my personal experience, first as a doctoral student having both French and Italian supervisors, then as an Italian researcher regularly involved in collaborative projects with French researchers. I also approach them in a more general way, considering both the cultural and institutional conditions in which the research has developed in the two countries.

2.5.1 *Opportunities for Collaboration: SFIDA, Summer Schools and European Projects*

Among the different institutional structures that provided opportunities for collaboration between French and Italian researchers, SFIDA²⁴ (Séminaire Franco-Italien de Didactique de l'Algèbre) certainly played a crucial role. The idea of this seminar arose from the interest in teaching and learning algebra shared by the researchers of the Italian teams at the University of Genova and Turin (respectively, directed by Boero and Arzarello) and the French team at the University of Nice (directed by Drouhard). SFIDA sessions were organized twice per year from 1993 to 2012 (SFIDA-38 was the last edition), alternatively by the three research teams, and held in their respective universities. A unique feature of this seminar was that everyone spoke his/her own language, as the programs of each session show. This attitude to overcome language constraints fostered the participation of researchers from other universities, and also students, both Italian and French. SFIDA was not only a place for sharing projects or work in progress, but the seminar functioned also as a working group that allowed the emerging of new ideas in this field of research, as attested by the articles devoted to this seminar in the second part of the special issue of *Recherches en Didactique des Mathématiques* on didactic research in algebra (Coulange, Drouhard, Dorier, & Robert, 2012) already mentioned in Sect. 2.3.

Other scientific events allowed the two communities to meet each other and collaborate, like the French Summer school of didactics already mentioned and the conferences of the *Espace Mathématique Francophone*. The participation and contribution of Italian researchers and teacher-researchers to those events, the involvement of Italian researchers in their scientific and organizing committees strengthened the relationships between the communities. The team of the University of Palermo (directed by Spagnolo) was for itself especially involved in the regular scientific

²⁴<https://sites.google.com/site/seminairesfida/Home/> (accessed 2018/02/10).

meetings of the group on Implicative Analysis (Gras, 1996), and even organized two of them (in 2005 and 2010). Spagnolo, supervised by Brousseau was also one of the first Italian students to get his Ph.D. in a French university (Spagnolo, 1995), together with Polo supervised by Gras (Polo Capra, 1996). Furthermore, around the years 2000, several Italian students carried out their doctoral thesis in different French universities.

French and Italian research teams were also involved in several European projects on the use of technologies, on teacher training and on theoretical perspectives. For instance, the ReMATH project (Representing Mathematics with Digital Media) has focused on the analysis of the potentiality of semiotic representations offered by dynamic digital artefacts (Kynigos & Lagrange, 2014). Adopting a perspective of networking among theoretical frameworks, this project has fostered the development of specific methodologies for such networking like the idea of cross-experiment (Artigue & Mariotti, 2014). In recent times, other research teams collaborated within the FASMED project on formative assessment.²⁵

Despite those collaborations, relevant differences exist between the Italian and French traditions in mathematics education: they do not only have to deal with differences in theoretical frameworks, but also with cultural differences of the two communities in which research is carried out, as explained in Chap. 4. A critical perspective on them has been proposed by Boero, one of the promoters of SFIDA, who highlighted some difficulties to establish collaborations between French and Italian researchers since the beginning of SFIDA. For Boero (1994), they were due to:

- Italian researchers had been more interested in studying the relationships between innovative didactical proposals and their development in classes than modelling didactical phenomena, as in the French tradition;
- The experimental parts of the Italian research involving classes were not situations the researcher studied as an external observer, but they were an opportunity to make more precise and test the hypotheses about the Italian paradigm of “research for innovation”.
- The presence of several teacher-researchers in Italian research teams.

By a cultural analysis of the context in which researchers work Boero deepens his reflection in a more recent contribution: he claims that he is “now convinced that these difficulties do not derive only from researchers’ characteristics and personal positions, but also (and perhaps mainly) from ecological conditions under which research in mathematics education develops” (Arcavi et al., 2016, p. 26). Among these conditions, he especially points out: the features of the school systems (i.e., the Italian national guidelines for curricula are less prescriptive than French syllabuses and primary school teachers in Italy usually teach and follow the same students for five years); the economic constraints of research and the weight of the cultural environment (i.e., the cultural and social vision of mathematics, the spread of the

²⁵Improving Progress for Lower Achievers through Formative Assessment in Science and Mathematics Education, <https://research.ncl.ac.uk/fasmed/> (accessed 2018/02/10).

idea of mathematics laboratory, the development of mathematics research, and the influence of sociological studies that Boero considers weaker in Italy than in France).

I would like to give now another vision of the didactic relationships between Italy and France by reflecting on my personal scientific journey, from the perspective of boundary crossing (Akkerman & Bakker, 2011), as my transitions and interactions across different sites, and boundary objects, as artefacts, have had a bridging function for me.

2.5.2 A Personal Scientific Journey

This journey started at the University of Turin, where I obtained a one-year fellowship to study at a foreign university. The University of Bordeaux I, in particular the LADIST (*Laboratoire Aquitain de Didactique des Sciences et Techniques*) directed by Brousseau was my first destination, my first boundary crossing. At the LADIST I became more deeply involved in the Theory of Didactical Situations (Brousseau, 1997) that I had previously studied. A fundamental experience for me as a student in mathematics education was the observation of classes at the *École Michelet*, the primary school attached to the COREM (see Sect. 2.1). The activities of the COREM allowed me to compare the experimental reality with the factual components that Brousseau highlighted in his lessons to doctoral students at the LADIST. My first personal contact with French research was thus characterized by discussions, passion and enthusiasm for research and, of course, by the people I met. At the end of my fellowship, I moved to Paris to prepare a DEA.²⁶ There I met Artigue and other French colleagues, and I read Boero's paper (Boero, 1994) quoted above that encouraged me to become aware of the potential of my boundary crossing.

After the DEA, I continued my doctoral studies within an institutional agreement between the University of Paris 7 and the University of Turin. I had two supervisors (Artigue and Arzarello) from two didactic cultures who had not yet collaborated. Retrospectively, I can claim that the dialogue between these cultures was under my responsibility. Ante litteram, I looked for a kind of networking strategy (Prediger, Bikner-Ahsbahs, & Arzarello, 2008) appropriate to the topic of my research: the introduction of Calculus in high school with graphic and symbolic calculators (Maschietto, 2008). From the French culture I took the methodology of didactical engineering (Artigue, 2014) with its powerful a priori analysis, the idea of situation and a-didacticity, and the instrumental approach (see Sect. 2.1). From the Italian culture, I took the strong cognitive component following embodied cognition, a semiotic focus with the attention to gestures and metaphors, and the cognitive roots of concepts. The experimental part of my research was a didactical engineering carried out in some Italian classes. I had to negotiate the planned situations with the teachers of these classes who were members of the research team of the University of Turin. In that process, a relevant element was the a priori analysis of the planned situations.

²⁶Diplôme d'Etudes Approfondies.

It was a powerful tool for sharing the grounded idea of the didactical engineering with those teachers who did not belong to the French scientific culture, and became a boundary object.

At the end of my doctoral period, I moved to the University of Modena e Reggio Emilia with a research fellowship within the European project on mathematics exhibitions *Maths Alive*. Finally, I got a research position in that university some years later, and I currently work there. In Modena, I met the framework of the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti, 2008) (see Chap. 4), and the mathematical machines. These are tools mainly concerning geometry, which have been constructed by teacher-researchers (members of the team of that University directed by Bartolini Bussi, now Laboratorio delle Macchine Matematiche²⁷) following the Italian tradition of using material tools inspired by the work of Castelnuovo in mathematics laboratory. In my first designs of laboratory sessions, I tried to locally integrate the TDS and the Theory of Semiotic Mediation by alternating a moment of group work with a didactical feature and a moment of collective discussion with an institutional component, with the aim of mediating mathematical meanings. Another boundary crossing for me.

The collaboration with French colleagues was renewed when I moved to the INRP²⁸ in Lyon for two months as visiting researcher in the EducTice team. My new position as a researcher and not as a student, changed the conditions, making more evident the potential of boundary crossing. The work with the EducTice team was the source of new insights and productive exchanges about: the notion of mathematics laboratory and the use of tools in mathematics education (Maschietto & Trouche, 2010); the notion of resources and collaborative work among teachers from the perspective of the Documentational Approach (see Sect. 2.1); and the role of digital technologies in learning, teaching and teacher education. The second and third elements contributed to the MMLAB-ER project I was involved in Italy in terms of planning and analysis of the teachers' education program (Maschietto, 2015). The first and third elements have been essential in the development of the original idea of *duo of artefacts* based on using a material artefact and a digital one in an intertwined way in the same learning project (Maschietto & Soury-Lavergne, 2013). We jointly planned the design of the digital counterpart of a material artefact,²⁹ the tasks with the duo of artefacts, resources and training for primary school teachers, and the idea of duo of artefacts developed as a boundary object for all of us, bridging our respective practices.

The challenge of boundary crossing is to establish a continuity across the different sites and negotiated practices with other researchers and teachers. This story tells my personal experience of boundary crossing between the French and Italian didactic cultures. This is a particular story, but not an isolated case. It illustrates how

²⁷MMLab, www.mmlab.unimore.it (accessed 2018/02/10).

²⁸Institut National de Recherche Pédagogique, now Institut Français de l'Éducation (<http://ife.ens-lyon.fr/ife>) (accessed 2018/02/10).

²⁹In this case, the material artefact is the "pascaline Zero + 1", while the digital counterpart is the "e-pascaline" constructed in the new Cabri authoring environment.

the repeated boundary crossing of individual researchers has contributed and still contributes to the dissemination of the French didactic culture and, in return, to its enrichment by other didactic cultures.

2.6 Didactic Interactions Between France and Latin-America: The Case of Mexico

In Latin America, as in other regions of the world, Mathematics Education emerged in the 1970s, and its development was marked by a strong relationship with the French community of educational mathematicians. Initially, the Department of Mathematics Education at the Centro de Investigación y de Estudios Avanzados (DME-CINVESTAV, created in 1975), and the network of French IREMs have made exceptional contributions to this process by establishing relations between specialists in Mexico and France. Over time, other institutions emerged and the French theoretical approaches were adapted to respond to different educational needs of the continent, which resulted in a greater theoretical development: production of new tools, broadening of notions, questioning of scope, study of new issues, etc. Analyzing the case of Mexico, the focus of this section, we show other research communities how the relationships between institutions and research groups can generate scientific advances with social impact. Also, it leads to important reflections on current challenges of Latin America reality, especially that of achieving quality education for all, a key element in securing peace and social development.

2.6.1 The DME at CINVESTAV and Its Relation to “the” French School

CINVESTAV’s Department of Mathematics Education (DME) was created by Filloy and Imaz. Its founding reflected the urgent need to modernize study plans and programs for mathematics in harmony with an international movement which demanded that the mathematics being taught be brought more in line with current theories in the discipline. In that context, two French researchers, Brousseau and Pluvineau, were invited to Mexico City in 1979, thus initiating relations between Mexico and “the” French Didactic School.

Since its creation, the DME has maintained a well-defined scientific vocation. Because it is part of a centre devoted to research, its institutional conditions propitiate developing research projects, organizing and participating in congresses, colloquia and spaces for disseminating science, teacher-training courses, and Master’s and Doctoral programs, among other scientific and academic activities. Also, it has a copious production of materials, both research-based and didactic (designed for teachers and

students). Thus, it is an ideal space for establishing relations with research groups in other latitudes.

The first research projects developed at the DME centred on rational numbers, algebra, probability and functions, within theoretical frameworks that included epistemology, the cognitive sciences, and computation. Today, however, the fields explored have diversified broadly. Its methods for developing these activities and disseminating their results have been presented at international meetings (ICME, CIEAEM, PME) that led to forging contacts with research groups in France. The fact that two members of the DME—Hitt and Alarcón—did their doctoral studies in France, followed a few years later by the post-doctoral study periods of Cantoral and Farfán, have clearly influenced the development of Mathematics Education in Mexico through, for example, the creation of the *Escuela de Invierno en Matemática Educativa* (Winter School for Mathematics Education, EIME), reflecting the experience of French Summer School. A second key creation was the *Reunión Latinoamericana en Matemática Educativa* (Latin American Forum for Mathematics Education, RELME), which is held annually. The year 2017 has witnessed the 31st edition of this congress, which is opening up new channels for relations between France and Latin America, including a collection of studies framed in the Mathematical Working Space approach (see Sects. 2.1 and 2.2). Another event of this nature—propelled mainly by Brazilian researchers—is the *Simposio Latinoamericano de Didáctica de la Matemática* (Latin American Symposium on Mathematical Didactics, LADIMA³⁰), which was held in November, 2016, with a strong presence of researchers from France and Latin America. Also, researchers at the DME were founding members of the *Comité Latinoamericano de Matemática Educativa* (Latin American Committee for Mathematics Education, CLAME), whose achievements include founding the *Revista Latinoamericana de Matemática Educativa* (Latin American Journal of Mathematics Education, RELIME), which appears in such indexes as ISI Thomson Reuters. Indeed, the journal's Editorial Board includes French researchers and its list of authors reflects joint studies conducted by French and Latin American scholars.

Finally, academic life at the DME is characterized by the enrolment of students from Mexico and several other Latin American countries in its Master's and Doctoral programs. These graduate students enjoy the opportunity to contact "the" French School through specialized literature, seminars, congresses and seminars with French researchers.

2.6.2 The DIE-CINVESTAV: A Strong Influence on Basic Education Supported by TDS

Since its founding, the Department of Educational Research (DIE for its initials in Spanish) has maintained a close relationship with Mexico's Department of Pub-

³⁰<http://www.boineventos.com.br/ladima>, accessed January 2018.

lic Education that has allowed it to participate in producing textbooks, developing teaching manuals, and formulating study plans for math courses and processes of curriculum reform for basic education (students aged 6–15). The TDS has been the principal theoretical reference guiding these activities, as we show in the following section.

The 1980s brought the development of a project called “from six years” (*de los seis años*), one of the first TDS approaches, and one that would influence other programs, mainly in the curriculum reform for the area of basic education driven by the Department of Public Education in 1993. That reform introduced new textbooks, manuals and didactic activities strongly influenced by the French School, and especially TDS. Block, who would later complete his Doctorate (co-directed by Brousseau) participated actively in elaborating material for primary school, while Alarcón did the same for secondary school. That reform program was in place for 18 years (1993–2011), during a period that also saw the implementation of the Program for Actualizing Mathematics Professors (1995) and the introduction of several additional reforms that reflected the impact of TDS in Mexico’s Teachers Colleges (*Escuelas Normales*), the institutions entrusted with training elementary school teachers. In Block’s words, “after over 20 years of using those materials, it is likely that there are still traces of the contributions of TDS in the teaching culture”.

2.6.3 *The PROME at CICATA-IPN: A Professionalization Program for Teachers that Generates Relations Between France and Latin America*

In the year 2000, the CICATA at Mexico’s Instituto Politécnico Nacional created remote, online Master’s and Doctorate Programs in Mathematics Education (known as PROME) designed for mathematics teachers. This program has generated multiple academic interactions between France and Latin America because, while students come mostly from Latin American nations (Mexico, Argentina, Chile, Colombia and Uruguay) some instructors are French. Also, their teacher training programs include elements of TDS and ATD, exemplified by the Study and Research Periods for Teacher Training (REI-FP).

The Learning Units (LU) on which these programs are based have been designed with the increasingly clear objective of functioning as a bridge between research in Mathematics Education and teaching practice. To this end, the organizers encourage a broad perspective on research and its results by including LUs designed by instructors at PROME and from other areas of the world. French researchers such as Athanaze (National Institute of Applied Sciences of Lyon, INSA-Lyon), Georget (Caen University), and Hache (LDAR and IREM-Paris), have participated in designing and implementing LUs, while Kuzniak (LDAR and IREM-Paris) was active in developing a Doctoral-level seminar, and Castela (LDAR and Rouen University) has participated in online seminars, workshops, the Doctorate’s Colloquium,

and PROMÉ's inaugural Online Congress. These interactions allow the diffusion of research while recognizing teachers' professional knowledge and opening forums to answer urgent questions and identify demands that often develop into valuable research topics.

2.6.4 Theoretical Currents, Methodologies and Tools

Two theoretical currents that arose in Latin America—ethnomathematics and socioepistemology—have generated new ways of doing research and problematizing teaching and learning in the field of mathematics. The term ethnomathematics was coined by the Brazilian, D'Ambrosio, to label a perspective heavily influenced by Bishop's cultural perspective on mathematics education (Bishop, 1988). Socioepistemology, meanwhile, was first proposed by the Mexican scholar, Cantoral, and is now being developed by several Latin American researchers (see Cantoral, 2013). This approach that shares the importance attached to theoretical foundations in French research and the need for emancipation from the dominant traditions in this discipline, considers the epistemological role of social practices in the construction of mathematical knowledge; the mathematical object changes from being the focus of the didactical explanation to consider how it is used while certain normative practices take place. Since all kinds of knowledge matter—everyday and technical knowledge for example—Socioepistemology explains the permanent development of mathematical thinking considering not only the final mathematical production, but all those social circumstances—such as practices and uses—surrounding mathematical tasks.

Another theoretical development, though of narrower dimensions, is the extended praxeological model that emerged from Romo's doctoral dissertation (co-directed by Artigue and Castela), which has been widely disseminated in two publications (Romo-Vázquez, 2009; Castela & Romo, 2011). This approach makes it possible to analyse mathematical models in non-school contexts in order to transpose them to mathematics teaching by designing didactic activities shaped primarily for engineers and technicians; for example, the cases of electrical circuits and laminated materials (Siero & Romo-Vázquez, 2017), and blind source separation in engineering (Vázquez, Romo, Romo, & Trigueros, 2016). This model, accompanied by TDS tools, has also been applied in analyses of the practices of migrant child labourers in northern Mexico (Solares, 2012).

In terms of methodologies, it could be said that, as in the French School, research conducted in Mexico is largely qualitative in nature, though data-gathering and implicative analyses using the CHIC program developed by Gras are now being utilized. The methodology of didactical engineering (see Sect. 2.1) is still one of the most often employed, though rarely in its pure form. Socioepistemology is recognized by its use of Artigue (1990) and Farfán (1997).

2.6.5 *Areas of Opportunity and Perspectives*

The institutions introduced herein, and the brief historical profiles presented, reveal how interactions between France and Mexico have strengthened the development of Mathematics Education in Latin America. However, as Ávila (2016) points out, many challenges still need to be addressed, especially in terms of ensuring that all students have access to high-quality instruction in mathematics that will allow them to better understand and improve the world. This entails participating in specific types of education, including the following: in indigenous communities, with children of migrant workers, child labourers in cities, and children who lack access to technology, and in multi-grade schools, to mention only a few. In this regard, the research by Solares (2012) and Solares, Solares, and Padilla (2016) has shown how elements of TDS and ATD facilitate analysing and “valuing” the mathematical activity of child labourers in work contexts in Sonora and metropolitan Mexico City, and the design of didactic material that takes into account the knowledge and needs of these population sectors.

Finally, we consider that another important area of opportunity is generating didactic proposals for the Telesecundaria system (which serves mainly rural areas in Mexico) that, in the 2014 educational census escolar, represented 45.3% of all schools involved in teaching adolescents aged 12–15. This system requires a “tele-teacher” (classes are transmitted by television) and a teacher-monitor whose role is mainly to answer students’ questions and resolve their doubts; though they were recently granted more autonomy and are now expected to develop didactic material on “transversal” topics for various subjects (e.g. physics, history, mathematics and Spanish). In this educational setting, many scholars feel that designing SRPs (see Sect. 2.1 on ATD) may be an optimal approach, since SRPs are characterized by their “co-disciplinary” nature. Designing and implementing such materials will make it possible to better regulate teaching practice and the formation of citizens who are capable of questioning the world. It is further argued that there are more general areas, such as the role of multimedia tools in students’ autonomous work, large-scale evaluations, online math education, the study of cerebral activity associated with mathematical activities, and the nature of mathematical activities in technical and professional contexts, among other fields, that could lead to establishing new relations between France and Mexico to conduct research in these, so far, little-explored areas.

There is no doubt that both French and Mexican research have benefitted from this long term collaboration. Mexican research has enriched the perspectives of French didactics, especially with the development of socioepistemology and ethnomathematics, which have shown—beyond European logic—how autochthonous and native American communities produce mathematical knowledge, while highlighting the role of social practices in the social construction of knowledge. These approaches have produced studies in the ATD framework (Castela, 2009; Castela & Elguero, 2013). Likewise, studies of equality, and of education-at-a-distance, primarily for the professionalization of teachers, have developed very significantly in Latin Amer-

ica, and may bring about new relations that seek a greater impact on innovation in the teaching of mathematics in the classroom.

2.7 Didactic Interactions Between France and African Countries. The Case of Tunisia

The past decade has seen an important development of research in mathematics education across Francophone Africa, and the collaboration with French didacticians has played a decisive role in this development. Collaborations in mathematics education between Francophone African countries and France started in fact early after these countries got their independence, in the New Math period, with the support of the recently created IREM network and the INRP. Some IREMs were even created in African countries, for instance in Senegal as early as 1975 (Sokhna & Trouche, 2016). When doctorate programs in the didactics of mathematics opened in French universities in 1975 (see Sect. 2.1), African students entered these programs and prepared doctoral theses under the supervision of French researchers. The data collected for the preparation of the thematic afternoon at ICME-13 show that by 1985, eleven such theses had been already defended by students from four countries, and twenty more at the end of the 20th century, with nine countries represented. One can also observe the progressive development of co-supervision with African researchers, especially within the institutional system of co-tutoring doctorate. This is the case for sixteen of the twenty-five theses defended since 2000.

Research collaboration was also nurtured by the regular participation of African didacticians in the biannual Summer schools of didactics of mathematics (see Sect. 2.1) and, since the year 2000, by the tri-annual conferences of the Francophone Mathematical Space (EMF) created on the initiative of the French sub-commission of ICMI, the CFEM. One important aim of the creation of the EMF structure was indeed to favour the inclusion of Francophone researchers from non-affluent countries into the international community of mathematics education. EMF conferences alternate South and North locations, and three have already been held in Africa, in Tozeur (Tunisia) in 2003, Dakar (Senegal) in 2009, and Algier (Algeria) in 2015.

As evidenced by the four case studies regarding Benin, Mali, Senegal and Tunisia included in Artigue (2016), these collaborations enabled the creation of several master and doctorate programs in didactics of mathematics in Francophone African countries, and supported the emergence and progressive maturation of a community of didacticians of mathematics in the region. A clear sign of this maturation is the recent creation of ADiMA (Association of African didacticians of mathematics), the first conference of which was held at the ENS (Ecole Normale Supérieure) of Yaounde in Cameroon in August 2016; the second one is planned at the Institute of Mathematics and Physical Sciences of Dangbo in Benin in August 2018.

Tunisia is a perfect example of such fruitful interactions. In the next paragraphs, I (Faïza Chellougui) review them, from the first collaborations, in the seventies,

involving the Tunisian association of mathematical sciences (ATSM), the French association of mathematics teachers (APMEP) and the IREM network until today. I show how these collaborations have nurtured the progressive maturation of a Tunisian community of didacticians of mathematics, today structured in the Tunisian association of didactics of mathematics (ATDM), and enriched the research perspectives in both countries. More details can be found in (Chellougui & Durand-Guerrier, 2016).

2.7.1 The Emergence of Didactic Interactions Between France and Tunisia

As just mentioned above, didactic interactions between Tunisia and France in mathematics stem from the relationships between the APMEP and the ATSM created in 1968, the oldest association of mathematics teachers in the Arab world and in Africa. The two associations indeed have a long term tradition of cross-invitation and participation in their respective “National days” meetings and seminars, of regular exchange of publications, etc. As early as 1977, Brousseau was invited to the national days of the ATSM. He presented the didactics of mathematics, its questions, concepts and research methods to a large audience of teachers, illustrating these with the research work he was developing on the teaching of rational and decimal numbers.

The IREM network then played an important role, especially the IREM of Lyon, thanks to its director, Tisseron. Tisseron had taught mathematics to pre-service mathematics teachers at the ENS in Tunis in the seventies, and also didactics at the ENS in Bizerte in the eighties when a didactic course was introduced in the preparation of secondary mathematics teachers. The decisive step for the development of didactics of mathematics as a research field in Tunisia occurred in fact in 1998, with the accreditation of a graduate program in the didactics of mathematics (DEA) at the Institute for higher education and continuous training in Tunis (ISEFC).³¹ This accreditation had been prepared by Abdeljaouad, mathematician and historian of mathematics at the University of Tunis, who has played a crucial role in the emergence and development of didactic research in Tunisia, and Tisseron (see Abdeljaouad, 2009 for more details).

2.7.2 Development and Institutionalization

In order to set-up this program, a fruitful collaboration developed between the ISEFC and four research teams in French universities having doctorate programs in the didactics of mathematics: the LIRDHIST team (now S2HEP³²) in the University

³¹<http://www.isefc.rnu.tn/home.htm> (accessed 2018/01/08).

³²<https://s2hep.univ-lyon1.fr> (accessed 2018/01/08).

of Lyon 1 of which Tisseron was a member, the DIDIREM team (now LDAR³³) in the University Paris 7, the Leibniz team in the University Grenoble 1, and the LACES³⁴ at the University of Bordeaux 2. Initially the DEA courses were taught by researchers from these teams, but gradually Tunisian scholars took them partially in charge. In 2006, the DEA turned into a Master program in the framework of the LMD (License-Master-Doctorate) reform of university. This institutional change led to modification of the organization of the program, and from the fall of 2010, ISEFC was empowered to offer a Master of research in didactics of science and pedagogy. The overall objective is ensuring a high level of training taking into account the multiple components of careers in science education. Eventually, a new Master in didactics of mathematics was set-up in October 2015, aiming to be innovative and open to the international community.

Another important step for the institutionalization of the field was the creation of the Tunisian association of didactics of mathematics (ATDM) in 2007. This association provides an institutional status to the young community of didacticians of mathematics and supports the dissemination of its research activities and results. For instance, it organizes an annual seminar to which contribute both well-known international and Tunisian researchers, to allow the regular diffusion of new or on-going research and to promote exchanges and debates within the didactic community.

Since the establishment of the DEA, fifteen doctoral theses and more than forty DEA or Master dissertations have been defended, most of these under the co-supervision of French and Tunisian researchers. Research reported in these theses and dissertations mainly addresses the teaching and learning of specific mathematics concepts, from elementary grades up to university. Important attention is paid to the epistemology and history of the concepts and domains at stake. The existence of Tunisian researchers specialized in the history of mathematics especially Arabic mathematics, such as Abdeljaouad, certainly contributes to nurture and instrument this attention. The main theoretical frameworks used are those of the French didactics, and especially ATD, TDS and TCF (see Sect. 2.1). One specificity however, is the number of theses and research projects that concern higher education and the transition from secondary to tertiary mathematics education (see for instance the theses by Chellougui on the use of quantifiers in university teaching (Chellougui, 2004), by Ghedamsi on the teaching of Analysis in the first university year (Ghedamsi, 2008), and by Najar on the secondary-tertiary transition in the area of functions (Najar, 2010). In fact, these theses and more global collaboration between French and Tunisian researchers have substantially contributed to the development of research in the area of higher education in France and Francophone countries in the last decade. This is also the case for logical perspectives in mathematics education, as shown by the thesis of Ben Kilani, Chellougui and Kouki (Chellougui, 2004; Ben Kilani, 2005; Kouki, 2008). Tunisian researchers have also pushed new lines of research, such as those related to the teaching and learning of mathematics in multilingual contexts, poorly addressed by French didacticians. In that area, a

³³<https://www.ldar.website> (accessed 2018/01/08).

³⁴<http://www.laces.univ-bordeauxsegalen.fr> (accessed 2018/01/08).

pioneering work was the thesis of Ben Kilani who used logic to show the differences of functioning of the negation in the Arabic and French language, and to understand the difficulties induced by these differences in the transition between Arabic and French as language of instruction in grade 9.

2.7.3 Some Outcomes of the French-Tunisian Didactic Collaboration

This long term collaboration has enabled the emergence, progressive development and institutionalization of didactics of mathematics as a field of research and practice in Tunisia. The majority of Ph.D. graduates have found a position in Tunisian higher education; they constitute today a community with the capacity of taking in charge the didactic preparation of primary and secondary mathematics teachers, and most Master courses. In 2017 moreover, the two first habilitations for research supervision³⁵ have been delivered to Tunisian didacticians (Ghedamsi, 2017; Kouki, 2017), which is a promising step for this community.

The French-Tunisian collaboration has certainly played a role in the increasing regional and international visibility and recognition of Tunisian researchers in the didactics of mathematics observed in the last decade. International visibility and recognition expresses through contributions to Francophone international events in the field such as the Summer school of didactics of mathematics to which the Tunisian delegation is regularly the largest foreign delegation, the EMF and ADiMa conferences. Recently it has also expressed through contributions to CERME conferences organized by the European Society for Research in Mathematics Education or the recently created INDRUM network, federating mathematics education research at university level. International recognition also expresses through invited lectures at seminars and congresses, and diverse scientific responsibilities. My personal case is a good illustration. I had an invited lecture at ICME-13 in 2016, and am a member of the International program committee of ICME-14. I have been a member of the scientific committee of two Summer schools, in 2007 and 2017, of EMF 2015 and INDRUM 2016. I was co-chair of the Topic Study Group entitled “*Pluralités culturelles et universalité des mathématiques: enjeux et perspectives pour leur enseignement et apprentissage*” (Cultural diversity and universality of mathematics: stakes and perspectives) at EMF2015, and of the group “Logic, numbers and algebra” at INDRUM 2016.

There is no doubt that there exists today a Tunisian community of didacticians of mathematics, dynamic and mature, open to the world beyond the sole frontiers of the Francophone world. While maintaining privileged links with the French didactic community, which has supported its emergence and development, it is creating its

³⁵Habilitation for research supervision is a diploma compulsory to compete for full professor position at university in Tunisia as is the case in France.

own identity, and more and more offers challenging perspectives and contributions to this French community.

2.8 Epilogue

In this chapter reflecting the contributions at the ICME-13 thematic afternoon, we have tried to introduce the reader to the French didactic tradition, describing its emergence and historical development, highlighting some of its important characteristics, providing some examples of its achievements, and also paying particular attention to the ways this tradition has migrated outside the frontiers of the French hexagon and nurtured productive relationships with researchers in a diversity of countries, worldwide. This tradition has a long history, shaped as all histories by the conditions and constraints of the context where it has grown and matured. Seen from the outside, it may look especially homogeneous, leading to the term of French school of didactics often used to label it. The three theoretical pillars that have structured it from its origin and progressively developed with it, with their strong epistemological foundations, the permanent efforts of the community for maintaining unity and coherence, for capitalizing research knowledge, despite the divergent trends normally resulting from the development of the field, certainly contribute to this perception. We hope that this chapter shows that cultivating coherence and identity can go along with a vivid dynamics. The sources of this dynamics are to be found both in questions internal and external to the field itself and also, as we have tried to show, in the sources of inspiration and questions that French didacticians find in the rich connections and collaborations they have established and increasingly continue to establish both with close fields of research and with researchers living in other traditions and cultures. The first sections, for instance, have made clear the important role played by connections with psychology, cognitive ergonomics and computer sciences. The last four sections have illustrated the particular role played by foreign doctorate students, by the support offered to the creation of master and doctorate programs, and also by institutional structures such as the IREM network or the INRP, now IFÉ, since the early seventies; this is confirmed by the eight case studies presented in (Artigue, 2016). Looking more precisely at these international connections, there is no doubt that they are not equally distributed over the world. Beyond Europe, Francophone African countries, Vietnam, and Latin America are especially represented. For instance, among the 181 doctoral theses of foreign students supervised or co-supervised by French didacticians between 1979 and 2015, the distribution is the following: 56 from Francophone Africa, 54 from America, all but one from Latin America among with 28 for Brazil, 15 from East Asia among with 12 from Vietnam, 15 from Middle East among with 15 from Lebanon, and 36 from Europe.³⁶ For Francophone African countries, Vietnam and Lebanon, the educational links established

³⁶Data collected by Patrick Gibel (ESPE d'Aquitaine) with the support of Jérôme Barberon (IREM de Paris).

in the colonial era are an evident reason. For Latin America, the long term cultural connections with France and the place given to the French language in secondary education until recently, and also the long term collaboration between French and Latin American mathematics communities, have certainly played a major role. These connections are dynamic ones, and regularly new ones emerge. For instance, due to the will of the French Ecoles normales supérieures (ENS) and to the East China Normal University (ECNU, Shanghai) to develop their collaboration in various domains of research (philosophy, biology, history, ... including education), and to the presence on each side of researchers in mathematics education, new links have emerged since 2015 and, currently, 4 Ph.D. are in preparation, co-supervised by researchers from the two institutions.

All these connections and collaborations allow us to see our tradition from the outside, to better identify its strengths and weaknesses, as pointed out by Knipping in Sect. 2.4, and to envisage ways to jointly progress, at a time when the need of research in mathematics education is more important than ever.

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