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Linear Matrix Inequality Techniques in Optimal Control

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Abstract

LMI (linear matrix inequality) techniques offer more flexibility in the design of dynamic linear systems than techniques that minimize a scalar functional for optimization. For linear state space models, multiple goals (performance bounds) can be characterized in terms of LMIs, and these can serve as the basis for controller optimization via finite-dimensional convex feasibility problems. LMI formulations of various standard control problems are described in this article, including dynamic feedback stabilization, covariance control, LQR, H_∞ control, L_∞ control, and information architecture design.

Keywords Matrix inequalities • Control system design • Covariance control • LQR/LQG • H_∞ control • L_∞ control • Sensor/actuator design

Early Optimization History

Hamilton invented state space models of nonlinear dynamic systems with his generalized momenta work in the 1800s (Hamilton 1834, 1835), but at that time the lack of computational tools prevented broad acceptance of the first-order form of dynamic equations. With the rapid development of computers in the 1960s, state space models evoked a formal control theory for minimizing a scalar function of control and state, propelled by the calculus of variations and Pontryagin's maximum principle. Optimal control has been a pillar of control theory for the last 50 years. In fact, all of the problems discussed in this article can perhaps be solved by minimizing a scalar functional, but a search is required to find the right functional. Globally convergent algorithms are available to do just that for quadratic functionals, but more direct methods are now available.

Since the early 1990s, the focus for linear system design has been to pose control problems as feasibility problems, to satisfy multiple constraints. Since then, feasibility approaches have dominated design decisions, and such feasibility problems may be convex or not. If the problem can be reduced to a set of linear matrix inequalities (LMIs) to solve, then convexity is proven. However, failure to find such LMI formulations of the problem does not mean it is not convex, and computer-assisted methods for convex problems are available to avoid the search for LMIs (see Camino et al. 2003).

In the case of linear dynamic models of stochastic processes, optimization methods led to the popularization of linear quadratic Gaussian (LQG) optimal control, which had globally optimal solutions (see Skelton 1988). The first two moments of the stochastic process (the mean and the covariance) can be controlled with these methods, even if the distribution of the random variables involved is not Gaussian. Hence, LQG became just an acronym for the solution of quadratic functionals of control and state variables, even when the stochastic processes were not Gaussian. The label LQG was often used even for deterministic problems, where a time integral, rather than

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an expectation operator, was minimized, with given initial conditions or impulse excitations. These were formally called LQR (linear quadratic regulator) problems. Later the book (Skelton 1988) gave the formal conditions under which the LQG and the LQR answers were numerically identical, and this particular version of LQR was called the *deterministic LQG*.

It was always recognized that the quadratic form of the state and control in the LQG problem was an artificial goal. The real control goals usually involved prespecified performance bounds on *each* of the outputs and bounds on *each* channel of control. This leads to matrix inequalities (MIs) rather than scalar minimizations. While it was known early that *any* stabilizing linear controller could be obtained by some choice of weights in an LQG optimization problem (see Chap. 6 and references in Skelton 1988), it was not known until the 1980s *what* particular choice of weights in LQG would yield a solution to the matrix inequality (MI) problem. See early attempts in Skelton (1988), and see Zhu and Skelton (1992) and Zhu et al. (1997) for a globally convergent algorithm to find such LQG weights when the MI problem has a solution. Since then, rather than stating a minimization problem for a meaningless sum of outputs and inputs, linear control problems can now be stated simply in terms of norm bounds on *each* input vector and/or *each* output vector of the system (L_2 bounds, L_∞ bounds, or variance bounds and covariance bounds). These *feasibility* problems are convex for state feedback or full-order output feedback controllers (the focus of this elementary introduction), and these can be solved using linear matrix inequalities (LMIs), as illustrated in this article. However, the earliest approach to these MI problems was iterative LQG solutions (to find the correct weights to use in the quadratic penalty of the state), as in Skelton (1988), Zhu and Skelton (1992), and Zhu et al. (1997).

Matrix Inequalities

Let $\mathbf{Q}$ be any square matrix. The linear matrix inequality (LMI) "$\mathbf{Q} > \mathbf{0}$" is just a short-hand notation to represent a certain scalar inequality. That is, the matrix notation "$\mathbf{Q} > \mathbf{0}$" means "the scalar $\mathbf{x^T Q x}$ is positive for all values of $\mathbf{x}$, except $\mathbf{x} = \mathbf{0}$." Obviously this is a property of $\mathbf{Q}$, not $\mathbf{x}$, hence the abbreviated matrix notation $\mathbf{Q} > \mathbf{0}$. This is called a linear matrix inequality (LMI), since the matrix unknown $\mathbf{Q}$ appears linearly in the inequality $\mathbf{Q} > \mathbf{0}$. Note also that any square matrix $\mathbf{Q}$ can be written as the sum of a symmetric matrix $\mathbf{Q_s} = \frac{1}{2}(\mathbf{Q}+\mathbf{Q^T})$, and a skew-symmetric matrix $\mathbf{Q_k} = \frac{1}{2}(\mathbf{Q}-\mathbf{Q^T})$, but $\mathbf{x^T Q_k x} = \mathbf{0}$, so only the symmetric part of the matrix $\mathbf{Q}$ affects the scalar $\mathbf{x^T Q x}$. We assume hereafter without loss of generality that $\mathbf{Q}$ is symmetric. The notation "$\mathbf{Q} \geq \mathbf{0}$" means "the scalar $\mathbf{x^T Q x}$ cannot be negative for any $\mathbf{x}$."

Lyapunov proved that $\mathbf{x(t)}$ converges to zero if there exists a matrix $\mathbf{Q}$ such that, along the nonzero trajectory of a dynamic system (e.g., the system $\mathbf{\dot{x}} = \mathbf{Ax}$), two scalars have the property, $\mathbf{x(t)^T Q x(t)} > 0$ and $d/dt(\mathbf{x^T(t) Q x(t)}) < 0$. This proves that the following statements are all equivalent:

1. For any initial condition $\mathbf{x(0)}$ of the system $\mathbf{\dot{x}} = \mathbf{Ax}$, the state $\mathbf{x(t)}$ converges to zero.
2. All eigenvalues of $\mathbf{A}$ lie in the open left half plane.
3. There exists a matrix $\mathbf{Q}$ with the two properties $\mathbf{Q} > \mathbf{0}$ and $\mathbf{QA} + \mathbf{A^T Q} < \mathbf{0}$.
4. The set of all quadratic Lyapunov functions that can be used to prove the stability or instability of the null solution of $\mathbf{\dot{x}} = \mathbf{Ax}$ is given by $\mathbf{x^T Q^{-1} x}$, where $\mathbf{Q}$ is any square matrix with the two properties of item 3 above.

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LMIs are prevalent throughout the fundamental concepts of control theory, such as controllability and observability. For the linear system example $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$, $\mathbf{y} = \mathbf{Cx}$, the "Observability Gramian" is the infinite integral $\mathbf{Q} = \int \mathbf{e}^{\mathbf{A}^{\mathbf{T}}\mathbf{t}}\mathbf{C}^{\mathbf{T}}\mathbf{C}\mathbf{e}^{\mathbf{At}}\mathbf{dt}$. Furthermore $\mathbf{Q} > \mathbf{0}$ if and only if $(\mathbf{A}, \mathbf{C})$ is an observable pair, and $\mathbf{Q}$ is bounded only if the observable modes are asymptotically stable. When it exists, the solution of $\mathbf{QA} + \mathbf{A}^{\mathbf{T}}\mathbf{Q} + \mathbf{C}^{\mathbf{T}}\mathbf{C} = \mathbf{0}$ satisfies $\mathbf{Q} > \mathbf{0}$ if and only if the matrix pair $(\mathbf{A}, \mathbf{C})$ is observable.

Likewise the "Controllability Gramian" $\mathbf{X} = \int \mathbf{e}^{\mathbf{At}}\mathbf{B}\mathbf{B}^{\mathbf{T}}\mathbf{e}^{\mathbf{A}^{\mathbf{T}}\mathbf{t}}\mathbf{dt} > \mathbf{0}$ if and only if the pair $(\mathbf{A}, \mathbf{B})$ is controllable. If $\mathbf{X}$ exists, it satisfies $\mathbf{XA}^{\mathbf{T}} + \mathbf{AX} + \mathbf{BB}^{\mathbf{T}} = \mathbf{0}$, and $\mathbf{X} > \mathbf{0}$ if and only if $(\mathbf{A}, \mathbf{B})$ is a controllable pair. Note also that the matrix pair $(\mathbf{A}, \mathbf{B})$ is controllable for any $\mathbf{A}$ if $\mathbf{BB}^{\mathbf{T}} > \mathbf{0}$, and the matrix pair $(\mathbf{A}, \mathbf{C})$ is observable for any $\mathbf{A}$ if $\mathbf{C}^{\mathbf{T}}\mathbf{C} > \mathbf{0}$. Hence, the existence of $\mathbf{Q} > \mathbf{0}$ or $\mathbf{X} > \mathbf{0}$ satisfying either $(\mathbf{QA} + \mathbf{A}^{\mathbf{T}}\mathbf{Q} < \mathbf{0})$ or $(\mathbf{AX} + \mathbf{XA}^{\mathbf{T}} < \mathbf{0})$ is equivalent to the statement that "all eigenvalues of $\mathbf{A}$ lie in the open left half plane."

It should now be clear that the set of all stabilizing state feedback controllers, $\mathbf{u} = \mathbf{Gx}$, is parametrized by the inequalities $\mathbf{Q} > \mathbf{0}$, $\mathbf{Q}(\mathbf{A} + \mathbf{BG}) + (\mathbf{A} + \mathbf{BG})^{\mathbf{T}}\mathbf{Q} < \mathbf{0}$. The difficulty in this MI is the appearance of the product of the two unknowns $\mathbf{Q}$ and $\mathbf{G}$, so more work is required to show how to use LMIs to solve this problem.

In the sequel some techniques are borrowed from linear algebra, where a linear matrix equality (LME) $\mathbf{\Gamma G \Lambda} = \mathbf{\Theta}$ may or may not have a solution $\mathbf{G}$. For LMEs there are two separate questions to answer. The first question is "Does there exist a solution?" and the answer is "if and only if $\mathbf{\Gamma \Gamma^{+} \Theta \Lambda^{+} \Lambda} = \mathbf{\Theta}$." The second question is "What is the set of all solutions?" and the answer is "$\mathbf{G} = \mathbf{\Gamma^{+}\Theta\Lambda^{+}} + \mathbf{Z} - \mathbf{\Gamma^{+}\Gamma Z \Lambda \Lambda^{+}}$, where $\mathbf{Z}$ is arbitrary, and the $+$ symbol denotes Pseudo Inverse." LMI approaches employ the same two questions by formulating the necessary and sufficient conditions for the existence of an LMI solution and then to parametrize all solutions.

Perhaps the earliest book on LMI control methods was Boyd et al. (1994), but the results and notations used herein are taken from Skelton et al. (1998). Other important LMI papers and books can give the reader a broader background, including Iwasaki and Skelton (1994), Gahinet and Apkarian (1994), de Oliveira et al. (2002), Li et al. (2008), de Oliveira and Skelton (2001), Camino et al. (2001, 2003), Boyd and Vandenberghe (2004), Iwasaki et al. (2000), Khargonekar and Rotea (1991), Vandenberghe and Boyd (1996), Scherer (1995), Scherer et al. (1997), Balakrishnan et al. (1994), Gahinet et al. (1995), and Dullerud and Paganini (2000).

Control Design Using LMIs

Consider the feedback control system

$$\begin{bmatrix} \dot{\mathbf{x}}_{\mathbf{p}} \\ \mathbf{y} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{A_p} & \mathbf{D_p} & \mathbf{B_p} \\ \mathbf{C_p} & \mathbf{D_y} & \mathbf{B_y} \\ \mathbf{M_p} & \mathbf{D_z} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x_p} \\ \mathbf{w} \\ \mathbf{u} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{x}}_{\mathbf{c}} \end{bmatrix} = \begin{bmatrix} \mathbf{D_c} & \mathbf{C_c} \\ \mathbf{B_c} & \mathbf{A_c} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{x_c} \end{bmatrix} = \mathbf{G} \begin{bmatrix} \mathbf{z} \\ \mathbf{x_c} \end{bmatrix}, \tag{1}$$

where $\mathbf{z}$ is the measurement vector, $\mathbf{y}$ is the output to be controlled, $\mathbf{u}$ is the control vector, $\mathbf{x_p}$ is the plant state vector, $\mathbf{x_c}$ is the state of the controller, and $\mathbf{w}$ is the external disturbance (in some cases below we treat $\mathbf{w}$ as a zero-mean white noise). We seek to choose the control matrix $\mathbf{G}$ to satisfy the given upper bounds on the output covariance $E[\mathbf{yy}^{\mathbf{T}}] \leq \bar{\mathbf{Y}}$, where E represents the steady-state expectation operator in the stochastic case (i.e., when $\mathbf{w}$ is white noise), and in the deterministic case E represents the infinite integral of the matrix $[\mathbf{yy}^{\mathbf{T}}]$. The math is the same in each case, with

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appropriate interpretations of certain matrices. For a rigorous equivalence of the deterministic and stochastic interpretations, see Skelton (1988). By defining the matrices,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_p} \\ \mathbf{x_c} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{A_{cl}} & \mathbf{B_{cl}} \\ \mathbf{C_{cl}} & \mathbf{D_{cl}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{C} & \mathbf{F} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{H} \end{bmatrix} \mathbf{G} \begin{bmatrix} \mathbf{M} \; \mathbf{E} \end{bmatrix} \tag{2}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B_p} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} \mathbf{M_p} & \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} \mathbf{D_p} \\ \mathbf{0} \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} \mathbf{D_z} \\ \mathbf{0} \end{bmatrix} \tag{3}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C_p} \; \mathbf{0} \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{B_y} \; \mathbf{0} \end{bmatrix}, \quad \mathbf{F} = \mathbf{D_y}, \tag{4}$$

one can write the closed-loop system dynamics in the form

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A_{cl}} & \mathbf{B_{cl}} \\ \mathbf{C_{cl}} & \mathbf{D_{cl}} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix}. \tag{5}$$

Often it is of interest to characterize the set of all controllers that can satisfy performance bounds on both the outputs and inputs, $E[\mathbf{y}\mathbf{y}^\mathbf{T}] \leq \bar{\mathbf{Y}}$ and $E[\mathbf{u}\mathbf{u}^\mathbf{T}] \leq \bar{\mathbf{U}}$, and we call these *covariance* control problems. But without prespecified performance bounds $\bar{\mathbf{Y}}, \bar{\mathbf{U}}$, one can require stability only. Such examples are given below.

Many Control Problems Reduce to the Same LMI

Let the left (right) null spaces of any matrix $\mathbf{B}$ be defined by matrices $\mathbf{U_B}$ ($\mathbf{V_B}$), where $\mathbf{U_B^T B} = \mathbf{0}$, $\mathbf{U_B^T U_B} > \mathbf{0}$, ($\mathbf{B V_B} = \mathbf{0}$, $\mathbf{V_B^T V_B} > \mathbf{0}$). For any given matrices $\mathbf{\Gamma}, \mathbf{\Lambda}, \mathbf{\Theta}$, Chap. 9 of the book (Skelton et al. 1998) provides all $\mathbf{G}$ which solve

$$\mathbf{\Gamma G \Lambda} + (\mathbf{\Gamma G \Lambda})^\mathbf{T} + \mathbf{\Theta} < \mathbf{0}, \tag{6}$$

and proves that there exists such a matrix $\mathbf{G}$ if and only if the following two conditions hold:

$$\mathbf{U_\Gamma^T \Theta} U_\mathbf{\Gamma} < \mathbf{0}, \quad \textbf{or} \quad \mathbf{\Gamma \Gamma^T} > \mathbf{0}, \tag{7}$$

$$\mathbf{V_\Lambda^T \Theta} V_\mathbf{\Lambda} < \mathbf{0}, \quad \textbf{or} \quad \mathbf{\Lambda^T \Lambda} > \mathbf{0}. \tag{8}$$

If $\mathbf{G}$ exists, then one set of such $\mathbf{G}$ is given by

$$\mathbf{G} = -\rho \mathbf{\Gamma^T \Phi \Lambda^T} (\mathbf{\Lambda \Phi \Lambda^T})^{-1}, \qquad \mathbf{\Phi} = (\rho \mathbf{\Gamma \Gamma^T} - \mathbf{\Theta})^{-1}, \tag{9}$$

where $\rho > 0$ is an arbitrary scalar such that

$$\mathbf{\Phi} = (\rho \mathbf{\Gamma \Gamma^T} - \mathbf{\Theta})^{-1} > \mathbf{0}. \tag{10}$$

All $\mathbf{G}$ which solve the problem are given by Theorem 2.3.12 in Skelton et al. (1998). As elaborated in Chap. 9 of Skelton et al. (1998), 17 different control problems (using either state feedback or full-order dynamic controllers) all reduce to this same mathematical problem. That is, by

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defining the appropriate $\mathbf{\Theta}, \mathbf{\Lambda}, \mathbf{\Gamma}$, a very large number of different control problems, including the characterization of all stabilizing controllers, covariance control, H-infinity control, L-infinity control, LQG control, and H_2 control, can be reduced to the *same* matrix inequality (13). Several examples from Skelton et al. (1998) follow.

Stabilizing Control

There exists a controller $\mathbf{G}$ that stabilizes the system (1) if and only if (7) and (8) hold, where the matrices are defined by

$$\begin{bmatrix}\mathbf{\Gamma} & \mathbf{\Lambda}^{\mathbf{T}} & \mathbf{\Theta}\end{bmatrix} = \begin{bmatrix}\mathbf{B} & \mathbf{X}\mathbf{M}^{\mathbf{T}} & \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathbf{T}}\end{bmatrix}. \tag{11}$$

One can also write such results in another way, as in Corollary 6.2.1 of Skelton et al. (1998, p. 135): There exists a control of the form $\mathbf{u} = \mathbf{G}\mathbf{x}$ that can stabilize the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ if and only if there exists a matrix $\mathbf{X} > \mathbf{0}$ satisfying $\mathbf{B}^{\perp}(\mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathbf{T}})(\mathbf{B}^{\perp})^{\mathbf{T}} < \mathbf{0}$, where $\mathbf{B}^{\perp}$ denotes the left null space of $\mathbf{B}$. In this case all stabilizing controllers may be parametrized by $\mathbf{G} = -\mathbf{B}^{\mathbf{T}}\mathbf{P} + \mathbf{L}\mathbf{Q}^{1/2}$, for any $\mathbf{Q} > \mathbf{0}$ and a $\mathbf{P} > \mathbf{0}$ satisfying $\mathbf{P}\mathbf{A} + \mathbf{A}^{\mathbf{T}}\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{B}^{\mathbf{T}}\mathbf{P} + \mathbf{Q} = \mathbf{0}$. The matrix $\mathbf{L}$ is any matrix that satisfies the norm bound $\|\mathbf{L}\| < 1$. Youla et al. (1976) provided a parametrization of the set of all stabilizing controllers, but the parametrization was infinite dimensional (as it did not impose any restriction on the order or form of the controller). So for finite calculations one had to truncate the set to a finite number before optimization or stabilization started. As noted above, on the other hand, all stabilizing state feedback controllers $\mathbf{G}$ can be parametrized in terms of an arbitrary but finite-dimensional norm-bounded matrix $\mathbf{L}$. Similar results apply for the dynamic controllers of any fixed order (see Chap. 6 in Skelton et al. 1998).

Covariance Upper Bound Control

In the system (1), suppose that $\mathbf{D}_{\mathbf{y}} = \mathbf{0}$, $\mathbf{B}_{\mathbf{y}} = \mathbf{0}$ and that $\mathbf{w}$ is zero-mean white noise with intensity $\mathbf{I}$. Let a required upper bound $\overline{\mathbf{Y}} > \mathbf{0}$ on the steady-state output covariance $\mathbf{Y} = E[\mathbf{y}\mathbf{y}^{\mathbf{T}}]$ be given. The following statements are equivalent:

(i) There exists a controller $\mathbf{G}$ that solves the covariance upper bound control problem $\mathbf{Y} < \overline{\mathbf{Y}}$.
(ii) There exists a matrix $\mathbf{X} > \mathbf{0}$ such that $\mathbf{Y} = \mathbf{C}\mathbf{X}\mathbf{C}^{\mathbf{T}} < \overline{\mathbf{Y}}$ and (7) and (8) hold, where the matrices are defined by

$$\begin{bmatrix}\mathbf{\Gamma} & \mathbf{\Lambda}^{\mathbf{T}} & \mathbf{\Theta}\end{bmatrix} = \begin{bmatrix}\mathbf{B} & \mathbf{X}\mathbf{M}^{\mathbf{T}} & \mathbf{A}\mathbf{X} + \mathbf{X}\mathbf{A}^{\mathbf{T}} & \mathbf{D} \\ \mathbf{0} & \mathbf{E}^{\mathbf{T}} & \mathbf{D}^{\mathbf{T}} & -\mathbf{I}\end{bmatrix} \tag{12}$$

($\mathbf{\Theta}$ occupies the last two columns).

Proof is provided by Theorem 9.1.2 in Skelton et al. (1998).

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Linear Quadratic Regulator

Consider the linear time-invariant system (1). Suppose that $\mathbf{D_y} = \mathbf{0}, \mathbf{D_z} = \mathbf{0}$ and that $\mathbf{w}$ is the impulsive disturbance $\mathbf{w(t)} = \mathbf{w_0}\boldsymbol{\delta}\mathbf{(t)}$. Let a performance bound $\gamma > 0$ be given, where the required performance is to keep the integral squared output ($\|\mathbf{y}\|_{\mathbf{L2}}^2$) less than the prespecified value $\|\mathbf{y}\|_{\mathbf{L2}} < \gamma$ for any vector $\mathbf{w_0}$ such that $\mathbf{w_0^T w_0} \le \mathbf{1}$, and $\mathbf{x_0} = \mathbf{0}$. This problem is labeled linear quadratic regulator (LQR). The following statements are equivalent:

(i) There exists a controller $\mathbf{G}$ that solves the LQR problem.
(ii) There exists a matrix $\mathbf{Y} > \mathbf{0}$ such that $\|\mathbf{D^T Y D}\| < \gamma^2$ and (7) and (8) hold, where the matrices are defined by

$$\begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^{\mathbf{T}} & \boldsymbol{\Theta} \end{bmatrix} = \begin{bmatrix} \mathbf{YB} & \mathbf{M^T} & \mathbf{YA + A^T Y} & \mathbf{M^T} \\ \mathbf{H} & \mathbf{0} & \mathbf{M} & \mathbf{-I} \end{bmatrix}. \tag{13}$$

Proof is provided by Theorem 9.1.3 in Skelton et al. (1998).

H_∞ Control

LMI techniques provided the first papers to solve the general H_∞ problem, without any restrictions on the plant. See Iwasaki and Skelton (1994) and Gahinet and Apkarian (1994).

Let the closed-loop transfer matrix from $\mathbf{w}$ to $\mathbf{y}$ with the controller in (1) be denoted by $\mathbf{T(s)}$:

$$\mathbf{T(s)} = \mathbf{C_{cl}(sI - A_{cl})^{-1}B_{cl}} + \mathbf{D_{cl}}. \tag{14}$$

The H_∞ control problem can be defined as follows:

Let a performance bound $\gamma > 0$ be given. Determine whether or not there exists a controller $\mathbf{G}$ in (1) which asymptotically stabilizes the system and yields the closed-loop transfer matrix (14) such that the peak value of the frequency response is less than γ. That is, $\|\mathbf{T}\|_{\mathbf{H_\infty}} = \sup \|\mathbf{T}(j\omega)\| < \gamma$.

For the H_∞ control problem, we have the following result. Let a required H_∞ performance bound $\gamma > 0$ be given. The following statements are equivalent:

(i) A controller $\mathbf{G}$ solves the H_∞ control problem.
(ii) There exists a matrix $\mathbf{X} > \mathbf{0}$ such that (7) and (8) holds, where

$$\begin{bmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^{\mathbf{T}} & \boldsymbol{\Theta} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{XM^T} & \mathbf{AX + XA^T} & \mathbf{XC^T} & \mathbf{D} \\ \mathbf{H} & \mathbf{0} & \mathbf{CX} & -\gamma\mathbf{I} & \mathbf{F} \\ \mathbf{0} & \mathbf{E^T} & \mathbf{D^T} & \mathbf{F^T} & -\gamma\mathbf{I} \end{bmatrix} \tag{15}$$

($\boldsymbol{\Theta}$ occupies the last three columns).

Proof is provided by Theorem 9.1.5 in Skelton et al. (1998).

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L_∞ Control

The peak value of the frequency response is controlled by the above H_∞ controller. A similar theorem can be written to control the peak in the time domain.

Define $\sup \mathbf{y(t)^T y(t)} = \|\mathbf{y}\|^2_{L\infty}$, and let the statement $\|\mathbf{y}\|_{L\infty} < \gamma$ mean that the peak value of $\mathbf{y(t)^T y(t)}$ is less than γ^2. Suppose that $\mathbf{D_y} = \mathbf{0}$ and $\mathbf{B_y} = \mathbf{0}$. There exists a controller $\mathbf{G}$ which maintains $\|\mathbf{y}\|_{L\infty} < \gamma$ in the presence of any energy-bounded input $\mathbf{w(t)}$ (i.e., $\int_0^\infty \mathbf{w^T w dt} \le \mathbf{1}$) if and only if there exists a matrix $\mathbf{X} > \mathbf{0}$ such that $\mathbf{CXC^T} < \gamma^2 \mathbf{I}$ and (7) and (8) hold, where

$$\begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Lambda^T} & \mathbf{\Theta} \end{bmatrix} = \begin{bmatrix} \mathbf{B} & \mathbf{XM^T} & \mathbf{AX + XA^T} & \mathbf{D} \\ \mathbf{0} & \mathbf{E^T} & \mathbf{D^T} & -\gamma \mathbf{I} \end{bmatrix}. \tag{16}$$

Proof is provided by Theorem 9.1.4 in Skelton et al. (1998).

Information Architecture in Estimation and Control Problems

In the typical "control problem" that occupies most research literature, the sensors and actuators have already been selected. Yet the selection of sensors and actuators and their locations greatly affect the ability of the control system to do its job efficiently. Perhaps in one location a high-precision sensor is needed, and in another location high precision is not needed, and paying for high precision in that location would therefore be a waste of resources. These decisions must be influenced by the control dynamics which are yet to be designed. How does one know where to effectively spend money to improve the system? To answer this question, we must optimize the information architecture jointly with the control law.

Let us consider the problem of selecting the control law jointly with the selection of the precision (defined here as the inverse of the noise intensity) of each actuator/sensor, subject to the constraint of specified upper bounds on the covariance of output error and control signals, and specified upper bounds on the sensor/actuator cost. We assume the cost of these devices is proportional to their precision (i.e., the cost is equal to the *price per unit of precision*, times the precision). Traditionally, with full-order controllers, and *prespecified* sensor/actuator instruments (with specified precisions); this is a well-known solved convex problem (which means it can be converted to an LMI problem if desired), see Chap. 6 of Skelton et al. (1998). If we enlarge the domain of the freedom to include sensor/actuator precisions, it is not obvious whether the feasibility problem is convex or not. The following shows that this problem of including the sensor/actuator precisions within the control design problem is indeed convex and therefore completely solved. The proof is provided in Li et al. (2008).

Consider the linear control system (1)–(5). Assume that the cost of sensors and actuators is proportional to their precision, which we herein define to be the inverse of the noise intensity (or variance, in the discrete-time case). So if the price per unit of precision of the i-th sensor/actuator is P_{ii}, and if the variance (or intensity) of the noise associated with the i-th sensor/actuator is W_{ii}, then the total cost of all sensors and actuators is $\sum P_{ii} W_{ii}^{-1}$, or simply $\mathbf{tr(PW^{-1})}$, where $\mathbf{P} = \mathrm{diag}(P_{ii})$ and $\mathbf{W}^{-1} = \mathrm{diag}(W_{ii}^{-1})$.

Consider the control system (1). Suppose that $\mathbf{D_y} = \mathbf{0}$, $\mathbf{B_y} = \mathbf{0}$, $\mathbf{w} = \begin{bmatrix} \mathbf{w_s^T} & \mathbf{w_a^T} \end{bmatrix}^{\mathbf{T}}$ is the zero-mean sensor/actuator noise, $\mathbf{D_p} = [\mathbf{0} \;\; \mathbf{D_a}]$ and $\mathbf{D_z} = [\mathbf{D_s} \;\; \mathbf{0}]$. If the $\bar{\$}$ represents the allowed upper bound on sensor/actuator costs, there exists a dynamic controller $\mathbf{G}$ that satisfies the constraints

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$$E[\mathbf{uu^T}] < \bar{\mathbf{U}}, \quad E[\mathbf{yy^T}] < \bar{\mathbf{Y}}, \quad \mathbf{tr}(\mathbf{PW^{-1}}) < \bar{\$} \tag{17}$$

in the presence of sensor/actuator noise with intensity $\text{diag}(W_{ii}) = \mathbf{W}$ (which like $\mathbf{G}$ should be considered a design variable not fixed a priori) if and only if there exist matrices $\mathbf{L}, \mathbf{F}, \mathbf{Q}, \mathbf{X}, \mathbf{Z}, \mathbf{W^{-1}}$ such that

$$\mathbf{tr}(\mathbf{PW^{-1}}) < \bar{\$} \tag{18}$$

$$\begin{bmatrix} \bar{\mathbf{Y}} & \mathbf{C_pX} & \mathbf{C_p} \\ (\mathbf{C_pX})^\mathbf{T} & \mathbf{X} & \mathbf{I} \\ \mathbf{C_p}^\mathbf{T} & \mathbf{I} & \mathbf{Z} \end{bmatrix} > \mathbf{0}, \quad \begin{bmatrix} \bar{\mathbf{U}} & \mathbf{L} & \mathbf{0} \\ \mathbf{L^T} & \mathbf{X} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} & \mathbf{Z} \end{bmatrix} > \mathbf{0}, \quad \begin{bmatrix} \mathbf{\Phi}_{11} & \mathbf{\Phi}_{21}^\mathbf{T} \\ \mathbf{\Phi}_{21} & -\mathbf{W^{-1}} \end{bmatrix} < \mathbf{0}, \tag{19}$$

$$\mathbf{\Phi}_{21} = \begin{bmatrix} \mathbf{D_a} & \mathbf{0} \\ \mathbf{ZD_a} & \mathbf{FD_s} \end{bmatrix}, \quad \boldsymbol{\phi} = \begin{bmatrix} \mathbf{A_pX} + \mathbf{B_pL} & \mathbf{A_p} \\ \mathbf{Q} & \mathbf{ZA_p} + \mathbf{FM_p} \end{bmatrix}, \quad \mathbf{\Phi}_{11} = \boldsymbol{\phi} + \boldsymbol{\phi}^\mathbf{T}. \tag{20}$$

Note that the matrix inequalities (18)–(20) are LMIs in the collection of variables $(\mathbf{L}, \mathbf{F}, \mathbf{Q}, \mathbf{X}, \mathbf{Z}, \mathbf{W^{-1}})$, whereby joint control/sensor/actuator design is a convex problem.

Assume a solution $(\mathbf{L}, \mathbf{F}, \mathbf{Q}, \mathbf{X}, \mathbf{Z}, \mathbf{W})$ is found for the LMIs (18)–(20). Then the problem (17) is solved by the controller

$$\mathbf{G} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{V_l^{-1}} & -\mathbf{V_l^{-1}ZB_p} \end{bmatrix} \begin{bmatrix} \mathbf{Q} - \mathbf{ZA_pX} & \mathbf{F} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{V_r^{-1}} \\ \mathbf{I} & -\mathbf{M_pXV_r^{-1}} \end{bmatrix}, \tag{21}$$

where $\mathbf{V_l}$ and $\mathbf{V_r}$ are left and right factors of the matrix $\mathbf{I} - \mathbf{YX}$ (which can be found from the singular value decomposition $\mathbf{I} - \mathbf{YX} = \mathbf{U}\Sigma\mathbf{V^T} = (\mathbf{U}\Sigma^{1/2})(\mathbf{\Sigma}^{1/2}\mathbf{V^T}) = (\mathbf{V_l})(\mathbf{V_r})$).

To emphasize the theme of this article, to relate optimization to LMIs, we note that three optimization problems present themselves in the above problem with three constraints: control effort $\bar{\mathbf{U}}$, output performance $\bar{\mathbf{Y}}$, and instrument costs $\bar{\$}$. To solve optimization problems, one can fix any two of these prespecified upper bounds and iteratively reduce the level set value of the third "constraint" until feasibility is lost. This process minimizes the resource expressed by the third constraint, while enforcing the other two constraints.

As an example, if cost is not a concern, one can always set large limits for $\bar{\$}$ and discover the best assignment of sensor/actuator precisions for the specified performance requirements. These precisions produced by the algorithm are the values W_{ii}^{-1}, produced from the solution (18)–(20), where the observed rankings $W_{ii}^{-1} > W_{jj}^{-1} > W_{kk}^{-1} > \ldots$ indicate which sensors or actuators are most critical to the required performance goals $(\bar{\mathbf{U}}, \bar{\mathbf{Y}}, \bar{\$})$. If any precision W_{nn}^{-1} is essentially zero, compared to other required precisions, then the math is asserting that the information from this sensor (n) is not important for the control objectives specified, or the control signals through this actuator channel (n) are ineffective in controlling the system to these specifications. This information leads us to a technique for choosing the best sensor actuators and their location.

The previous discussion provides the precisions (W_{ii}^{-1}) required of each sensor and each actuator in the system. Our final application of this theory locates sensors and actuators in a large-scale system, by discarding the least effective ones. Suppose we solve any of the above feasibility problems, by starting with the entire admissible set of sensors and actuators (without regard to cost). For example, in a flexible structure control problem we might not know whether to place a rate sensor or displacement sensors at a given location, so we add both. We might not know whether

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to use torque or force actuators, so we add both. We fill up the system with all the possibilities we might want to consider, and let the above precision rankings (available after the above LMI problem is solved) reveal how much precision is needed at each location and at each sensor/actuator. If there is a large gap in the precisions required (say $W_{11}^{-1} > W_{22}^{-1} > W_{33}^{-1} >> \ldots W_{nn}^{-1}$), then delete the sensor/actuator n and repeat the LMI problem with one less sensor or actuator. Continue deleting sensors/actuators in this manner until feasibility of the problem is lost. Then this algorithm, stopping at the previous iteration, has selected the best distribution of sensors/actuators for solving the specific problem specified by the allowable bounds ($\bar{\$}$, $\bar{\mathbf{U}}$, $\bar{\mathbf{Y}}$). The most important contribution of the above algorithm has been to extend control theory to solve system design problems that involve more than just deigning control gains. This enlarges the set of solved linear control problems, from solutions of linear controllers with sensors/actuators prespecified to solutions which specify the sensor/actuator requirements jointly with the control solution.

Summary

LMI techniques provide more powerful tools for designing dynamic linear systems than techniques that minimize a scalar functional for optimization, since multiple goals (bounds) can be achieved for *each* of the outputs and inputs. Optimal control has been a pillar of control theory for the last 50 years. In fact, all of the problems discussed in this article can perhaps be solved by minimizing a scalar functional, but a search is required to find the right functional. Globally convergent algorithms are available to do just that for quadratic functionals. But more direct methods are now available (since the early 1990s) for satisfying multiple constraints. Since then, feasibility approaches have dominated design decisions (at least for linear systems), and such feasibility problems may be convex or not. If the problem can be reduced to a set of LMIs to solve, then convexity is proven. However, failure to find such LMI formulations of the problem does not mean it is not convex, and computer-assisted methods for convex problems are available to avoid the search for LMIs (see Camino et al. 2003). Optimization can also be achieved with LMI methods by reducing the level set for one of the bounds, while maintaining all the other bounds. This level set is reduced iteratively, between convex (LMI) solutions, until feasibility is lost. A most amazing fact is that most of the common linear control design problems all reduce to exactly the same matrix inequality problem (6). The set of such equivalent problems includes LQR, the set of all stabilizing controllers, the set of all H_∞ controllers, and the set of all L_∞ controllers. The discrete and robust versions of these problems are also included in this equivalent set; 17 control problems have been found to be equivalent to LMI problems.

LMI techniques extend the range of solvable system design problems beyond just control design. By integrating information architecture and control design, one can simultaneously choose the control gains and the precision required of all sensor/actuators to satisfy the closed-loop performance constraints. These techniques can be used to select the information (with precision requirements) required to solve a control or estimation problem, using the best economic solution (minimal precision). For a more complete discussion of LMI problems in control, read Dullerud and Paganini (2000), de Oliveira et al. (2002), Li et al. (2008), de Oliveira and Skelton (2001), Gahinet and Apkarian (1994), Iwasaki and Skelton (1994), Camino et al. (2001, 2003), Skelton et al. (1998), Boyd and Vandenberghe (2004), Boyd et al. (1994), Iwasaki et al. (2000), Khargonekar and Rotea (1991), Vandenberghe and Boyd (1996), Scherer (1995), Scherer et al. (1997), Balakrishnan et al. (1994), and Gahinet et al. (1995).

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Cross-References

- ▸ H_∞ Control, Keith Glover
- ▸ H_2 Optimal Control, Ben Chen
- ▸ Linear Quadratic Optimal Control, Harry Trentleman
- ▸ LMI Approach to Robust Control, Kang-Zhi Liu
- ▸ Stochastic Linear-Quadratic Control, Shanjian Tang

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