Chapter 2
Perturbation Methods

2.1 Regular Expansions

To introduce the ideas underlying perturbation methods and asymptotic approximations, we will begin with an algebraic equation. The problem we will consider is how to find an accurate approximation of the solution x of the quadratic equation

$$x^2 + 2\epsilon x - 1 = 0, \tag{2.1}$$

in the case of when ϵ is a small positive number. The examples that follow this one are more complex and, unlike this equation, we will not necessarily know at the start how many solutions the equation has. A method for determining the number of real-valued solutions involves sketching the terms in the equation. With this in mind, we rewrite the equation as $x^2 - 1 = -2\epsilon x$. The left- and right-hand sides of this equation are sketched in Figure 2.1. Based on the intersection points, it is seen that there are two solutions. One is a bit smaller than $x = 1$ and the other is just to the left of $x = -1$. Another observation is that the number of solutions does not change as $\epsilon \to 0$. The fact that the reduced problem, which is the one obtained when setting $\epsilon = 0$, has the same number of solutions as the original problem is a hallmark of what are called regular perturbation problems.

Our goal is to derive approximations of the solutions for small ϵ, and for this simple problem we have a couple of options on how to do this.

Method 1: Solve then Expand

It is an easy matter to find the solution using the quadratic formula. The result is

$$x = -\epsilon \pm \sqrt{1 + \epsilon^2}. \tag{2.2}$$

This completes the solve phase of the process. To obtain an approximation for the two solutions, for small ϵ, we first use the binomial expansion (see Table 2.1) to obtain

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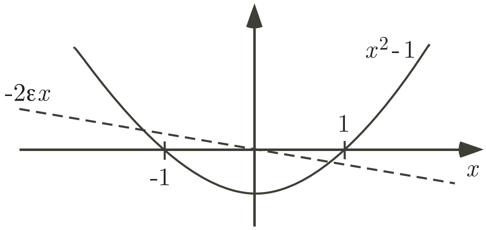


Figure 2.1 Sketch of the functions appearing in the quadratic equation in (2.1).

$$\sqrt{1+\epsilon^2} = 1 + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots . \tag{2.3}$$

A comment needs to be made here about the equal sign in this expression. The right hand side is an infinite series and by stating it is equal to $\sqrt{1+\epsilon^2}$ it is meant that given a value of ϵ that the series converges to $\sqrt{1+\epsilon^2}$. Said another way, given a value of ϵ, the more terms that are added together in the series the closer the sum gets to the value of $\sqrt{1+\epsilon^2}$. For this to be true it is necessary to require that $\epsilon^2 < 1$, but we are assuming ϵ is close to zero so this is not a restriction in this problem.

Substituting (2.3) into (2.2) yields

$$\begin{aligned} x &= -\epsilon \pm \left(1 + \frac{1}{2}\epsilon^2 - \frac{1}{8}\epsilon^4 + \cdots\right) \\ &= \pm 1 - \epsilon \pm \frac{1}{2}\epsilon^2 \mp \frac{1}{8}\epsilon^4 + \cdots . \end{aligned} \tag{2.4}$$

In the last step the terms are listed in order according to their power of ϵ. With this we can list various levels of approximation of the solutions, as follows

$$\begin{aligned} &x \approx \pm 1 && \text{1 term approximation} \\ &x \approx \pm 1 - \epsilon && \text{2 term approximation} \\ &x \approx \pm 1 - \epsilon \pm \tfrac{1}{2}\epsilon^2 && \text{3 term approximation.} \end{aligned}$$

So, we have accomplished what we set out to do, which is to derive an approximation of the solution for small ϵ. The procedure is straightforward but it has a major drawback because it requires us to be able to first solve the equation before constructing the approximation. For most problems this is simply impossible, so we need another approach.

Method 2: Expand then Solve

This approach requires us to first state what we consider to be the general form of the approximation for x. This requires a certain amount of experience and a reasonable place to start is with Taylor's theorem. We know that the solution depends on ϵ, we just don't know how. Emphasizing this dependence by writing $x(\epsilon)$, then using Taylor's theorem for ϵ near zero, we obtain

$$
\begin{aligned}
f(x) &= f(0) + xf'(0) + \frac{1}{2}x^2 f''(0) + \frac{1}{3!}x^3 f'''(0) + \cdots \\
(a+x)^\gamma &= a^\gamma + \gamma x a^{\gamma-1} + \frac{1}{2}\gamma(\gamma-1)x^2 a^{\gamma-2} + \frac{1}{3!}\gamma(\gamma-1)(\gamma-2)x^3 a^{\gamma-3} + \cdots \\
\frac{1}{1-x} &= 1 + x + x^2 + x^3 + \cdots \\
\frac{1}{(1-x)^2} &= 1 + 2x + 3x^2 + 4x^3 + \cdots \\
\frac{1}{\sqrt{1-x}} &= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \cdots \\
e^x &= 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots \\
a^x &= e^{x\ln(a)} = 1 + x\ln(a) + \frac{1}{2}(x\ln(a))^2 + \frac{1}{3!}(x\ln(a))^3 + \cdots \\
\sin(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \\
\cos(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots \\
\sin(a+x) &= \sin(a) + x\cos(a) - \frac{1}{2}x^2\sin(a) + \cdots \\
\cos(a+x) &= \cos(a) - x\sin(a) - \frac{1}{2}x^2\cos(a) + \cdots \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots \\
\ln(a+x) &= \ln(a) + \ln(1+x/a) = \ln(a) + \frac{x}{a} - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{3}\left(\frac{x}{a}\right)^3 + \cdots
\end{aligned}
$$

Table 2.1 Taylor series expansions, about $x = 0$, for some of the more commonly used functions.

$$x(\epsilon) = x(0) + \epsilon x'(0) + \frac{1}{2}\epsilon^2 x''(0) + \cdots . \tag{2.5}$$

This implies that x can be expanded using integer powers of ϵ. With nonlinear equations, there is no guarantee that the powers have to be integers. An example is the equation $x^2 - \epsilon = 0$. A reasonable assumption is that the solution can be expanded in powers of ϵ, although not necessarily in integer powers. For this reason we will assume that the general form of the expansion is

$$x \sim x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \cdots . \tag{2.6}$$

The values of $x_0, x_1, x_2, \ldots$ and $\alpha, \beta, \ldots$ will be determined when solving the equation. It is assumed that the terms are listed in order according to their power of ϵ, which means we assume $0 < \alpha < \beta < \cdots$. This requirement is known as a well-ordering assumption and we will make it every time we write down such an expression. It is also assumed that $x_0, x_1, x_2, \ldots$ do not depend on ϵ.

We will need to be able to identify the coefficients in an expansion and the big O notation will be used for this. So, in (2.6) the $O(\epsilon^\alpha)$ coefficient is x_1 and the $O(\epsilon^\beta)$ coefficient is x_2. For the same reason x_0 is the coefficient of the $O(1)$ term.

You might be wondering why we don't just assume at the start that $\alpha = 1$ and $\beta = 2$. After all, this is what we found earlier using the "Solve then Expand" approach. One reason for this has already been given, namely the solutions of nonlinear problems do not necessarily involve integer powers and we want a method that can handle such situations. A second reason is that we will learn something important by having the equation tell us that $\alpha = 1$. The expansion in (2.6) is really nothing more than an educated guess. It is quite possible that it is incorrect and it is important to see how the equation will tell us we have made an incorrect assumption.

We will substitute (2.6) into (2.1), but before doing so note

$$\begin{aligned} x^2 &\sim (x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \cdots)(x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \cdots) \\ &\sim x_0^2 + 2\epsilon^\alpha x_0 x_1 + 2\epsilon^\beta x_0 x_2 + \epsilon^{2\alpha} x_1^2 + \cdots . \end{aligned} \tag{2.7}$$

To start we will concentrate on finding the first two terms in the expansion for x. In this case, with (2.7), (2.1) takes the form

$$x_0^2 + 2\epsilon^\alpha x_0 x_1 + \cdots + 2\epsilon(x_0 + \epsilon^\alpha x_1 + \cdots) - 1 = 0. \tag{2.8}$$

We are constructing an approximation for small ϵ. In letting $\epsilon \to 0$ in the above equation we obtain the equation for the $O(1)$ term.

$O(1)$ $x_0^2 - 1 = 0$

The solutions are $x_0 = \pm 1$.

With this (2.8) reduces to

$$2\epsilon^\alpha x_0 x_1 + \cdots + 2\epsilon(x_0 + \epsilon^\alpha x_1 + \cdots) = 0. \tag{2.9}$$

Now, with the given values of x_0 we are left with a $2\epsilon x_0$ term in the above equation. There are no $O(\epsilon)$ terms on the right-hand side so there can be no $O(\epsilon)$ terms on the left-hand side. The only term available to cancel, or balance, out $2\epsilon x_0$ is $2\epsilon^\alpha x_0 x_1$, and for this to happen it is necessary that $\alpha = 1$. So, the equation has told us exactly what value this exponent must have. This gives us the following problem.

$O(\epsilon)$ $2x_0 x_1 + 2x_0 = 0$

The solution is $x_1 = -1$.

We have determined the first two terms in the expansion, but we could easily continue and find more. For example, to find the next term note that the equation in (2.8), using (2.7), reduces to

$$2\epsilon^{\beta}x_0x_2 + \epsilon^2x_1^2 + \cdots + 2\epsilon(\epsilon x_1 + \epsilon^{\beta}x_2 + \cdots) = 0. \qquad (2.10)$$

We now have $\epsilon^2x_1^2$ and $2\epsilon^2x_1$ on the left, with no $O(\epsilon^2)$ terms on the right. The only term available to eliminate them is $2\epsilon^{\beta}x_0x_2$ and, therefore, $\beta = 2$. With this we obtain the equation for the $O(\epsilon^2)$ terms in the equation.

$O(\epsilon^2)$ $2x_0x_2 + x_0^2 - 2x_1 = 0$

The solutions are $x_2 = \pm\frac{1}{2}$.

The above procedure can be used to find the successively higher order terms in the expansion. Rather than do that it is more worthwhile to consider what we have done to get to this point. Our conclusion is that one of the solutions is

$$x \sim 1 - \epsilon + \frac{1}{2}\epsilon^2, \qquad (2.11)$$

and the other is

$$x \sim -1 - \epsilon - \frac{1}{2}\epsilon^2. \qquad (2.12)$$

These approximations hold for small ϵ, and for this reason they are said to be asymptotic expansions of the solutions as $\epsilon \to 0$.

The formal definition of what it means to be an asymptotic expansion states that the difference between x and the expansion goes to zero faster than the last term included in the expansion. For (2.11) this means that

$$\lim_{\epsilon \to 0} \frac{x - (1 - \epsilon + \frac{1}{2}\epsilon^2)}{\epsilon^2} = 0.$$

For the same reason, $x \sim 1 - \epsilon$ is also an asymptotic expansion of the solution because

$$\lim_{\epsilon \to 0} \frac{x - (1 - \epsilon)}{\epsilon} = 0.$$

This is the basis for what is known as the limit-process definition of an asymptotic expansion. This is important for those interested in the theoretical foundations of the subject. For us, the critical point is that the asymptotic expansion is determined by how the function, or solution, behaves as $\epsilon \to 0$. The definition does not say anything about what happens when more terms are used in the expansion for a given value of ϵ. If we were to calculate every term in the expansion, and produce an infinite series in the process, the fact that it is an asymptotic expansion does not mean the series has to converge. In fact, some of the more interesting asymptotic expansions diverge. For this reason it is inappropriate to use an equal sign in (2.6) and why the symbol $\sim$ is used instead.

2.2 How to Find a Regular Expansion

The ideas used to construct asymptotic expansions of the solutions of a quadratic equation are easily extended to more complex problems. Exactly how one proceeds depends on how the problem is stated, and the following three situations are the most common.

2.2.1 Given a Specific Function

The expansion in (2.3) is an example of this situation. For these problems Taylor's theorem is most often used to construct the expansion, and it is not unusual to have to use it more than once. Typical examples are used below to illustrate how this is done.

Example 1

$$f(\epsilon) = \sin(e^{\epsilon}).$$

This is a compound function. To find, say, a three-term expansion of this for small ϵ one starts with the innermost function, which in this case is e^{ϵ}. To find a three-term expansion of this we can use the Taylor expansion of e^x given in Table 2.1, because ϵ small is equivalent in this case to x near zero. So, we have $e^{\epsilon} \sim 1 + \epsilon + \frac{1}{2}\epsilon^2 + \cdots$ and from this we conclude

$$\sin(e^{\epsilon}) \sim \sin\left(1 + \epsilon + \frac{1}{2}\epsilon^2 + \cdots\right).$$

The next observation to make is that the argument of the sine function on the right hand side has the form $\sin(1+y)$ where $y = \epsilon + \frac{1}{2}\epsilon^2 + \cdots$ is close to zero for small ϵ. This means the expansion given in Table 2.1 for $\sin(a+x)$ is applicable, where $a = 1$ and $x = y$. Using this fact we obtain

$$
\begin{aligned}
\sin(e^\epsilon) &\sim \sin(1+y) \\
&\sim \sin(1) + \cos(1)y - \frac{1}{2}\sin(1)y^2 + \cdots \\
&\sim \sin(1) + \cos(1)\left(\epsilon + \frac{1}{2}\epsilon^2 + \cdots\right) \\
&\qquad - \frac{1}{2}\sin(1)\left(\epsilon + \frac{1}{2}\epsilon^2 + \cdots\right)^2 + \cdots \\
&\sim \sin(1) + \epsilon\cos(1) + \frac{1}{2}\epsilon^2\left[\cos(1) - \sin(1)\right] + \cdots . \quad \blacksquare
\end{aligned} \tag{2.13}
$$

One might argue that the above calculation is not necessary because (2.13) can be obtained easily, and directly, from Taylor's theorem applied to $f(\epsilon) = \sin(e^\epsilon)$. This is correct, and it is worthwhile to have multiple methods available for constructing an expansion. However, the direct approach only works on certain functions. It is easy to find examples when the direct approach does not work, one is given below, and another is given in Exercise 2.2.

Example 2

$$
f(\epsilon) = \frac{1}{[1-\cos(\epsilon)]^3} .
$$

To find a two-term expansion of this for small ϵ, we start with the inner most function, which is $\cos(\epsilon)$. Using the Taylor expansion of $\cos(x)$ given in Table 2.1, we have

$$
\cos(\epsilon) \sim 1 - \frac{1}{2}\epsilon^2 + \frac{1}{24}\epsilon^4 + \cdots .
$$

With this

$$
\begin{aligned}
\frac{1}{1-\cos(\epsilon)} &\sim \frac{1}{\frac{1}{2}\epsilon^2 - \frac{1}{24}\epsilon^4 + \cdots} \\
&= \frac{2}{\epsilon^2}\frac{1}{1 - \frac{1}{12}\epsilon^2 + \cdots} .
\end{aligned}
$$

The last term can be expanded using the binomial expansion, which is the second entry in Table 2.1. In particular, with $a = 1$, $x = -\frac{1}{12}\epsilon^2 + \cdots$, and $\gamma = -3$,

$$
\begin{aligned}
\frac{1}{(1-\frac{1}{12}\epsilon^2+\cdots)^3} &= \left(1-\frac{1}{12}\epsilon^2+\cdots\right)^{-3} \\
&\sim 1-3\left(-\frac{1}{12}\epsilon^2+\cdots\right)+6\left(-\frac{1}{12}\epsilon^2+\cdots\right)^2+\cdots \\
&\sim 1+\frac{1}{4}\epsilon^2+\cdots .
\end{aligned}
$$

The resulting two-term expansion is

$$
\frac{1}{[1-\cos(\epsilon)]^3} = \frac{8}{\epsilon^6}\left(1+\frac{1}{4}\epsilon^2+\cdots\right) . \quad \blacksquare
$$

A comment is warranted about the expansions obtained in the last two examples. A general form of the expansion obtained in (2.13) is

$$
f \sim f_0 + \epsilon^\alpha f_1 + \epsilon^\beta f_2 + \cdots , \tag{2.14}
$$

where $0 < \alpha < \beta < \cdots$. In comparison, a general form of the expansion obtained in the second example is

$$
f \sim \epsilon^\alpha f_0 + \epsilon^\beta f_1 + \epsilon^\gamma f_2 + \cdots , \tag{2.15}
$$

where $\alpha < \beta < \gamma < \cdots$. The reason for pointing this out is that in the problems to follow it is necessary to guess, at the start, what form the expansion has. Our default assumption will be (2.14). The more general version given in (2.15) will only be used if (2.14) fails, or else there is some indication that such a general form is necessary.

Taylor's theorem is the most used method for expanding functions, but this should not be interpreted that one always ends up with a power series. An example is the function $e^{-1/\epsilon}$. Assuming it can be expanded using a power series, so $e^{-1/\epsilon} \sim x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots$, then the coefficients must satisfy the following limits

$$
\begin{aligned}
x_0 &= \lim_{\epsilon \to 0^+} e^{-1/\epsilon} , \\
x_1 &= \lim_{\epsilon \to 0^+} \frac{e^{-1/\epsilon} - x_0}{\epsilon} , \\
x_2 &= \lim_{\epsilon \to 0^+} \frac{e^{-1/\epsilon} - x_0 - \epsilon x_1}{\epsilon^2} .
\end{aligned}
$$

Using l'Hospital's rule one finds that each limit is zero, and so $x_0 = 0, x_1 = 0, x_2 = 0, \ldots$. In other words, as far as the functions $1, \epsilon, \epsilon^2, \ldots$ are concerned, $e^{-1/\epsilon}$ is just zero. This function certainly has rather small values but it is not identically zero. What is happening is that $e^{-1/\epsilon}$ goes to zero so quickly that the power functions are not able to describe it other than just conclude the function is zero. In this case $e^{-1/\epsilon}$ is said to be transcendentally small

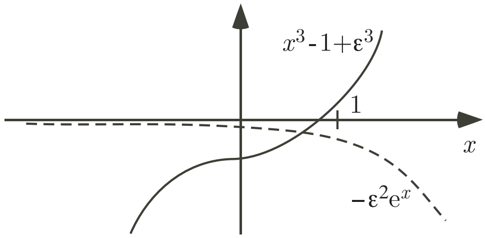


Figure 2.2 Sketch of the functions appearing in the transcendental equation in (2.16).

relative to the power functions. Even so, Taylor's theorem can be used with such functions and an example is $\sqrt{1+e^{-1/\epsilon}}$. Given the small value of $e^{-1/\epsilon}$ then we can think of the function as $\sqrt{1+y}$ where y is close to zero. Using Taylor's theorem, $\sqrt{1+y} \sim 1+\frac{1}{2}y-\frac{1}{8}y^2+\cdots$ and from this we conclude $\sqrt{1+e^{-1/\epsilon}} \sim 1+\frac{1}{2}e^{-1/\epsilon}-\frac{1}{8}e^{-2/\epsilon}+\cdots$. This result shows that the appropriate scale functions in this case are not power functions but the functions $1, e^{-1/\epsilon}, e^{-2/\epsilon}, e^{-3/\epsilon}, \ldots$.

2.2.2 Given an Algebraic or Transcendental Equation

The idea here is that we are given an algebraic or transcendental equation and we want to construct an approximation for the solution(s). This is exactly what we did for the quadratic equation example (2.1). To use the method on a slightly more difficult problem consider solving

$$x^3+\epsilon^2 e^x-1+\epsilon^3=0. \tag{2.16}$$

Our goal is to derive a two-term approximation of the solution. It is recommended that the first step in the construction is to assess how many solutions there are and, if possible, their approximate location. The reason for this is that we will have to guess the form of the expansion and any information we might have about the solution can be helpful. With this in mind the functions involved in this equation are sketched in Figure 2.2. There is one real-valued solution that is located slightly to the left of $x=1$ for small values of ϵ. In other words, the expansion for the solution should not start out as $x \sim \epsilon x_0+\cdots$ because this would be assuming that the solution goes to zero as $\epsilon \to 0$. Similarly, we should not assume $x \sim \frac{1}{\epsilon}x_0+\cdots$ as the solution does not become unbounded as $\epsilon \to 0$. For this reason we will assume that the appropriate expansion has the form

$$x \sim x_0+\epsilon^\alpha x_1+\cdots. \tag{2.17}$$

This is going to be substituted into (2.16) and this requires us to expand e^x. Using (2.17), and Table 2.1,

$$\begin{aligned} e^x &\sim e^{x_0+\epsilon^\alpha x_1+\cdots} \\ &= e^{x_0}e^{\epsilon^\alpha x_1+\cdots}. \end{aligned}$$

Setting $y = \epsilon^\alpha x_1 + \cdots$, and noting that y is close to zero for small ϵ, then

$$\begin{aligned} e^x &\sim e^{x_0}e^y \\ &= e^{x_0}\left(1 + y + \frac{1}{2}y^2 + \cdots\right) \\ &= e^{x_0}\left(1 + (\epsilon^\alpha x_1 + \cdots) + \frac{1}{2}(\epsilon^\alpha x_1 + \cdots)^2 + \cdots\right) \\ &= e^{x_0}\left(1 + \epsilon^\alpha x_1 + \cdots\right). \end{aligned}$$

Using the binomial expansion, given in Table 2.1, and (2.17) we also have that

$$x^3 \sim x_0^3 + 3\epsilon^\alpha x_0^2 x_1 + \cdots .$$

With this, the original equation given in (2.16) takes the form

$$x_0^3 + 3\epsilon^\alpha x_0^2 x_1 + \cdots + \epsilon^2 e^{x_0}\left(1 + \epsilon^\alpha x_1 + \cdots\right) - 1 + \epsilon^3 = 0. \tag{2.18}$$

The first problem to solve is obtained by simply setting $\epsilon = 0$, which gives us the following $O(1)$ problem.

$O(1)$ $x_0^3 - 1 = 0$

The real-valued solution is $x_0 = 1$. With this, the next term in (2.18) that must be considered is $\epsilon^2 e^{x_0}$. Given that there are no ϵ^2 terms on the right-hand side of the equation then one of the other terms on the left must balance with $\epsilon^2 e^{x_0}$. The only one available is $3\epsilon^\alpha x_0^2 x_1$, and for this to happen we get $\alpha = 2$. This gives us the following problem.

$O(\epsilon^2)$ $3x_0^2 x_1 + e^{x_0} = 0$

The solution is $x_1 = -\frac{1}{3}e$.

We have therefore found that a two-term expansion of the solution is

$$x \sim 1 - \frac{1}{3}\epsilon^2 e + \cdots . \tag{2.19}$$

This expansion is plotted in Figure 2.3 along with the numerical solution. The asymptotic nature of the approximation is evident as $\epsilon \to 0$.

The procedure used to find x_0 and x_1 can be continued without difficulty to find the higher-order terms x_2, x_3, etc. It is also very easy to extend

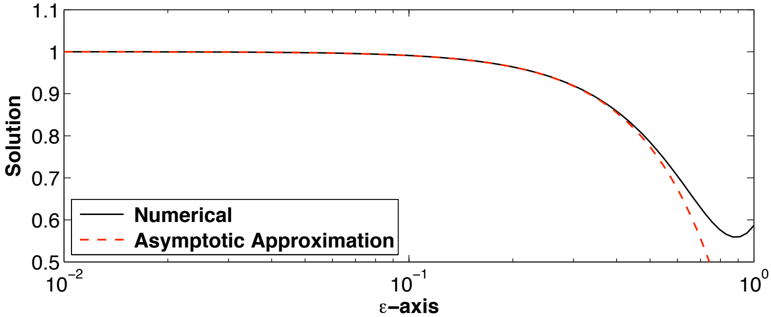


Figure 2.3 Comparison between the numerical solution of (2.16) and the asymptotic expansion (2.19).

the procedure to find the complex-valued solutions. What might not have been noticed is that not all of the terms in the original equation contribute to the approximation of the solution in (2.19). Namely, the ϵ^3 term does not contribute and the reason is that we have only computed the expansion through ϵ^2. In fact, if ϵ^3 were to be replaced with ϵ^4 or $\sin(\epsilon^3)$ the expansion in (2.19) still holds. It would not hold, however, if ϵ^3 were to be changed to ϵ or $\cos(\epsilon)$.

2.2.3 Given an Initial Value Problem

The next stage in the development is to apply regular expansions to problems involving differential equations. We will work out two examples, the first involves a single equation, and the second a system.

Example 1

The projectile problem furnishes an excellent example. Using (1.65) - (1.67) the problem to solve is

$$\frac{d^2x}{dt^2} = -\frac{1}{(1+\epsilon x)^2}\,, \quad \text{for} \quad 0 < t, \tag{2.20}$$

where

$$x(0) = 0, \tag{2.21}$$

$$\frac{dx}{dt}(0) = 1. \tag{2.22}$$

It is important to note that we are using the nondimensional problem and not the original given in (1.65) - (1.67). The use of an asymptotic expansion is predicated on one or more parameters taking on an extreme value. In the projectile problem the assumption is that the initial velocity v_0 is small. As we saw in the last chapter, the solution depends on the parameters through a combination of products, both dimensional and nondimensional. Consequently, the study of small v_0 is actually a study of what happens when the parameter group containing v_0 takes on an extreme value. Based on the scaling we used the specific limit is $\epsilon \to 0$.

The procedure for constructing the expansion will mimic what was done earlier. We start by stating what we believe to be the appropriate form for the expansion. Generalizing (2.6), our assumption is

$$x \sim x_0(t) + \epsilon^\alpha x_1(t) + \cdots . \tag{2.23}$$

The expansion is suppose to identify how the solution depends on ϵ. The terms in the expansion can, and almost inevitability will, depend on the other variables and parameters in the problem. For the projectile problem this means that each term in the expansion depends on time and this dependence is included in (2.23).

In preparation for substituting (2.23) into (2.20) note

$$\begin{aligned}
\frac{1}{(1+\epsilon x)^2} &= 1 - 2\epsilon x + 3\epsilon^2 x^2 + \cdots \\
&\sim 1 - 2\epsilon(x_0 + \epsilon^\alpha x_1 + \cdots) + 3\epsilon^2(x_0 + \cdots)^2 + \cdots \\
&= 1 - 2\epsilon x_0 + \cdots .
\end{aligned}$$

With this, the differential equation (2.20) becomes

$$x_0'' + \epsilon^\alpha x_1'' + \cdots = -1 + 2\epsilon x_0 + \cdots . \tag{2.24}$$

It is critical that the initial conditions are also included, and for these we have

$$x_0(0) + \epsilon^\alpha x_1(0) + \cdots = 0, \tag{2.25}$$

$$x_0'(0) + \epsilon^\alpha x_1'(0) + \cdots = 1. \tag{2.26}$$

As usual we break the above equations down into problems depending on the power of ϵ.

$O(1)$ $x_0'' = -1$
$x_0(0) = 0$, $x_0'(0) = 1$

The solution of this problem is $x_0 = t(1 - \frac{1}{2}t)$. With this the next highest term left in (2.24) is $2\epsilon x_0$. The term available to balance with this is $\epsilon^\alpha x_1''$, and from this we conclude $\alpha = 1$. This gives us the following problem.

$O(\epsilon)$ $x_1'' = 2x_0$
$x_1(0) = 0,\ x_1'(0) = 0$
The solution of this problem is $x_1 = \frac{1}{12}t^3(4-t)$.

We have therefore found that a two-term expansion of the solution is

$$x \sim t(1 - \frac{1}{2}t) + \frac{1}{12}\epsilon t^3(4-t) + \cdots . \tag{2.27}$$

This rather simple-looking expression is a two-term asymptotic expansion of the nonlinear projectile problem. Physically, the first term, $t(1-\frac{1}{2}t)$, gives the displacement of the projectile for a uniform gravitational field, and is the nondimensional version of (1.5). The second term, $\epsilon t^3(4-t)/12$, gives us the correction due to the nonlinear gravitational field.

To determine how well we have done in approximating the solution, a comparison is shown in Figure 2.4 for $\epsilon = 0.1$ and $\epsilon = 0.01$. It is seen that the one-term approximation, $x \sim t(1-\frac{1}{2}t)$, produces a reasonably accurate approximation for $\epsilon = 0.01$, but not when $\epsilon = 0.1$. In contrast, the two-term approximation (2.27) does very well for both values. To put this into perspective, if the object's initial velocity is the speed of sound then $\epsilon \approx 0.002$, while if it is equal to the Earth's escape velocity then $\epsilon \approx 2$. Figure 2.4 shows

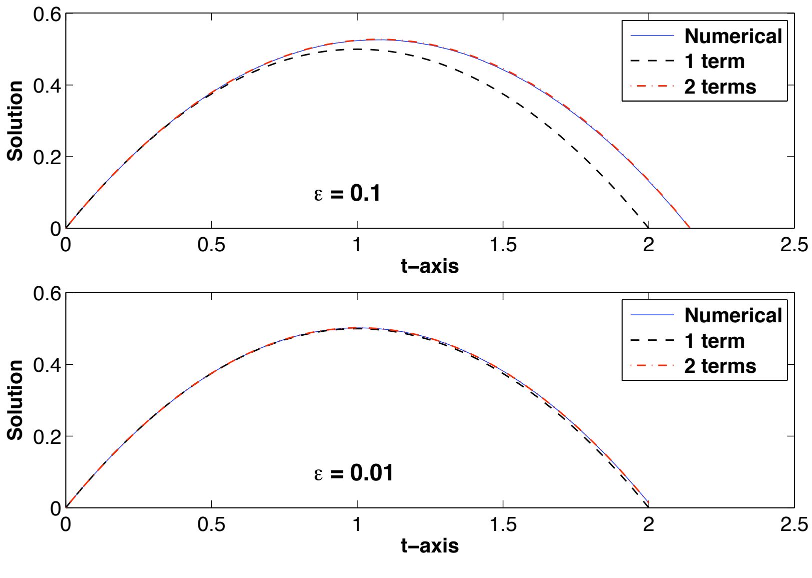


Figure 2.4 Comparison between the numerical solution of the projectile problem and the asymptotic expansion (2.27). In the upper graph $\epsilon = 0.1$, and in the lower graph $\epsilon = 0.01$. In both graphs the curves for the exact solution and two-term expansion are almost indistinguishable.

that at subsonic velocities the uniform gravitational field approximation is adequate. If the initial velocity is a bit larger, more than three times the speed of sound, then the nonlinear correction is needed. Finally, we expect the approximation to improve as ϵ gets closer to zero, and the graphs in Figure 2.4 are consistent with that expectation. ■

Example 2

The ideas used to find an approximation for a single equation are easily extended to systems. As an example, consider the thermokinetic model of Exercise 1.17. In nondimensional variables, the equations are

$$\frac{du}{dt} = 1 - ue^{\epsilon(q-1)}, \tag{2.28}$$

$$\frac{dq}{dt} = ue^{\epsilon(q-1)} - q. \tag{2.29}$$

The initial conditions are $u(0) = q(0) = 0$. We are assuming here that the nonlinearity is weak, which means that ϵ is small. Also, to simplify the problem, the other parameters that appear in the nondimensionalization have been set to one.

Generalizing (2.23), we expand both functions using our usual assumption, which is that

$$u \sim u_0(t) + \epsilon u_1(t) + \cdots ,$$
$$q \sim q_0(t) + \epsilon q_1(t) + \cdots .$$

In writing down the above expansions, it is assumed that the second terms in the expansions are $O(\epsilon)$, rather than $O(\epsilon^\alpha)$. This is done to simplify the calculations to follow.

Before substituting the expansions into the differential equations, note that

$$\begin{aligned} e^{\epsilon(q-1)} &\sim 1 + \epsilon(q-1) + \frac{1}{2}\epsilon^2(q-1)^2 + \cdots \\ &\sim 1 + \epsilon(q_0 + \epsilon q_1 + \cdots - 1) + \frac{1}{2}\epsilon^2(q_0 + \epsilon q_1 + \cdots - 1)^2 + \cdots \\ &\sim 1 + \epsilon(q_0 - 1) + \cdots , \end{aligned}$$

and

$$\begin{aligned} ue^{\epsilon(q-1)} &\sim (u_0 + \epsilon u_1 + \cdots)[1 + \epsilon(q_0 - 1) + \cdots] \\ &\sim u_0 + \epsilon\left[u_0(q_0 - 1) + u_1\right] + \cdots . \end{aligned}$$

With this, (2.28), (2.29) take the form

$$u_0' + \epsilon u_1' + \cdots = 1 - u_0 - \epsilon\left(u_0(q_0 - 1) + u_1\right) + \cdots ,$$
$$q_0' + \epsilon q_1' + \cdots = u_0 - q_0 + \epsilon\left(u_0(q_0 - 1) + u_1 - q_1\right) + \cdots .$$

As usual we break the above equations down into problems depending on the power of ϵ.

$O(1)$ $u_0' = 1 - u_0$
$q_0' = u_0 - q_0$
The solution of this problem that satisfies the initial conditions $u_0(0) = q_0(0) = 0$ is $u_0 = 1 - e^{-t}$, and $q_0 = 1 - (1+t)e^{-t}$.

$O(\epsilon)$ $u_1' = -u_1 - u_0(q_0 - 1)$
$q_1' = -q_1 + u_1 + u_0(q_0 - 1)$
The initial conditions are $u_1(0) = q_1(0) = 0$. The equation for u_1 is first order, and the solution can be found using an integrating factor. Once u_1 is determined then the q_1 equation can be solved using an integrating factor. Carrying out the calculation one finds that $u_1 = \frac{1}{2}(t^2 + 2t - 4)e^{-t} + (2+t)e^{-2t}$, $q_1 = \frac{1}{6}(t^3 - 18t + 30)e^{-t} - (2t+5)e^{-2t}$.

We have therefore found that a two-term expansion of the solution is

$$u(t) \sim 1 - e^{-t} + \epsilon\left(\frac{1}{2}(t^2 + 2t - 4)e^{-t} + (2+t)e^{-2t}\right), \tag{2.30}$$

$$q(t) \sim 1 - (1+t)e^{-t} + \epsilon\left(\frac{1}{6}(t^3 - 18t + 30)e^{-t} - (2t+5)e^{-2t}\right). \tag{2.31}$$

A comparison of the numerical solution for $q(t)$, and the above asymptotic approximation for $q(t)$ is shown in Figure 2.5 for $\epsilon = 0.1$. It is seen that even the one-term approximation, $q \sim 1 - (1+t)e^{-t}$, produces a reasonably accurate approximation, while the two-term approximation is indistinguish-

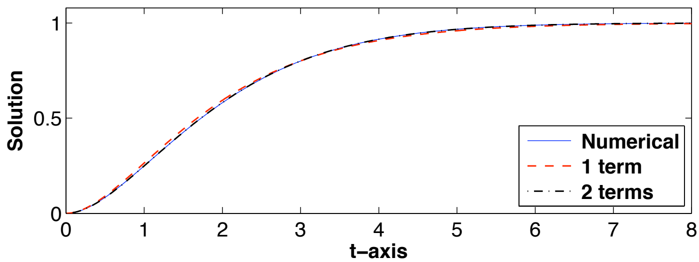


Figure 2.5 Comparison between the numerical solution for $q(t)$, and the asymptotic expansion (2.31). In the calculation, $\epsilon = 0.1$.

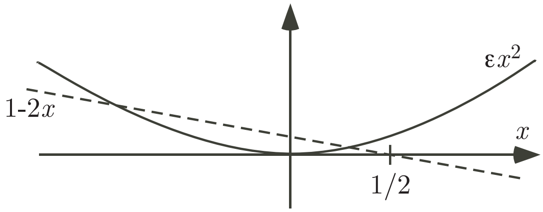


Figure 2.6 Sketch of the functions appearing in the quadratic equation in (2.32).

able from the numerical solution. The approximations for $u(t)$, which are not shown, are also as accurate. ■

2.3 Introduction to Singular Perturbations

All of the equations considered up to this point produce regular expansions. This means, roughly, that the expansions can be found without having to transform the problem. We now turn our attention to those that are not regular, what are known as singular perturbation problems. The first example considered is the quadratic equation

$$\epsilon x^2 + 2x - 1 = 0. \tag{2.32}$$

A tip-off that this is singular is that ϵ multiplies the highest-order term in the equation. Setting $\epsilon = 0$ drops the order down to linear, and this has dramatic effects on the number and types of solutions.

The best place to begin is to sketch the functions in the equation to get an idea of the number and location of the solutions. This is done in Figure 2.6, which shows that there are two real-valued solutions. One is close to $x = \frac{1}{2}$ and for this reason it is expected that an expansion of the form $x \sim x_0 + \epsilon^\alpha x_1 + \cdots$ will work. The second solution is far to the left on the negative x-axis, and the smaller ϵ the farther to the left it is located. Consequently, we should not be shocked later when we find that the expansion for this solution has the form $x \sim \frac{1}{\epsilon}x_0 + \cdots$, where x_0 is negative.

We start out as if this were a regular perturbation problem and assume the solutions can be expanded as

$$x \sim x_0 + \epsilon^\alpha x_1 + \cdots . \tag{2.33}$$

Substituting this into (2.32) we obtain

$$\epsilon(x_0^2 + 2\epsilon^\alpha x_0 x_1 + \cdots) + 2(x_0 + \epsilon^\alpha x_1 + \cdots) - 1 = 0. \tag{2.34}$$

Equating like powers of ϵ produces the following problems.

$O(1)$ $2x_0 - 1 = 0$

The solution is $x_0 = \frac{1}{2}$. With this, to balance the term $\epsilon^2 x_0^2$ we take $\alpha = 1$. This gives us the following problem.

$O(\epsilon^2)$ $x_0^2 + 2x_1 = 0$

The solution is $x_1 = -\frac{1}{8}$.

We therefore have

$$x \sim \frac{1}{2} - \frac{1}{8}\epsilon + \cdots . \tag{2.35}$$

This expansion is consistent with the conclusions we derived earlier from Figure 2.6 for one of the solutions. It is also apparent that no matter how many terms we calculate in the expansion (2.33) we will not obtain the second solution.

The failure of the regular expansion to find all of the solutions is typical of a singular perturbation problem. The method used to remedy the situation is to introduce a scaling transformation. Specifically, we will change variables and let

$$\bar{x} = \frac{x}{\epsilon^\gamma} . \tag{2.36}$$

With this, (2.32) takes the form

$$\underset{①}{\epsilon^{1+2\gamma}\bar{x}^2} + \underset{②}{2\epsilon^\gamma \bar{x}} - \underset{③}{1} = 0. \tag{2.37}$$

The reason for not finding two solutions earlier was that the quadratic term was lost when $\epsilon = 0$. Given the fact that this term is why there are two solutions in the first place we need to determine how to keep it in the equation as $\epsilon \to 0$. In other words, term ① in (2.37) must balance with one of the other terms and this must be the first problem solved as $\epsilon \to 0$. For example, suppose we assume the balance is between terms ① and ③, while term ② is of higher or equal order. For this to occur, we need $O(\epsilon^{1+2\gamma}) = O(1)$ and this would mean $\gamma = -\frac{1}{2}$. With this ①, ③ $= O(1)$ and ② $= O(\epsilon^{-1/2})$. This result is inconsistent with our original assumption that ② is higher order. Therefore, the balance must be with another term. This type of argument is central to singular problems and we will use a table format to present the steps used to determine the correct balance.

Balance	Condition on γ	Consistency Check	Conclusion
① ∼ ③ with ② higher order	$1 + 2\gamma = 0$ $\Rightarrow \gamma = -1/2$	①, ③ $= O(\epsilon)$ and ② $= O(\epsilon^{-1/2})$	Inconsistent with balance
① ∼ ② with ③ higher order	$1 + 2\gamma = \gamma$ $\Rightarrow \gamma = -1$	①, ② $= O(\epsilon^{-1})$ and ③ $= O(1)$	Consistent with balance

Based on the above analysis, $\gamma = -1$ and with this the equation takes the form

$$\bar{x}^2 + 2\bar{x} - \epsilon = 0. \tag{2.38}$$

With this we assume our usual expansion, which is

$$\bar{x} \sim \bar{x}_0 + \epsilon^\alpha \bar{x}_1 + \cdots . \tag{2.39}$$

The equation in this case becomes

$$\bar{x}_0^2 + 2\epsilon^\alpha \bar{x}_0 \bar{x}_1 + \cdots + 2(\bar{x}_0 + \epsilon^\alpha \bar{x}_1 + \cdots) - \epsilon = 0. \tag{2.40}$$

This gives us the following problems.

$O(1)$ $\bar{x}_0^2 + 2\bar{x}_0 = 0$

The solutions are $\bar{x}_0 = -2$ and $\bar{x}_0 = 0$. With this, to balance the $-\epsilon$ term in (2.40), we take $\alpha = 1$. This gives us the following problem.

$O(\epsilon)$ $2\bar{x}_0\bar{x}_1 + 2\bar{x}_1 - 1 = 0$

If $\bar{x}_0 = -2$ then $\bar{x}_1 = -\frac{1}{2}$, while if $\bar{x}_1 = 0$ then $\bar{x}_1 = \frac{1}{2}$.

It might appear that we have somehow produced three solutions, the one in (2.35) along with the two found above. However, it is not hard to show that the solution corresponding to $\bar{x}_0 = 0$ is the same one that was found earlier using a regular expansion. Consequently, the sought-after second solution is

$$x \sim \frac{1}{\epsilon}(-2 - \frac{1}{2}\epsilon + \cdots). \tag{2.41}$$

The procedure used to derive this result contained many of the ideas we used to find regular expansions. The most significant difference is the introduction of a scaled variable, (2.36), and the subsequent balancing used to determine how the highest-order term participates in the problem. As we will see shortly, these will play a critical role when analyzing similar problems involving differential equations.

2.4 Introduction to Boundary Layers

As our introductory example of a singular perturbation problem involving a differential equation we will consider solving

$$\epsilon y'' + 2y' + 2y = 0, \quad \text{for } \ 0 < x < 1, \tag{2.42}$$

where the boundary conditions are

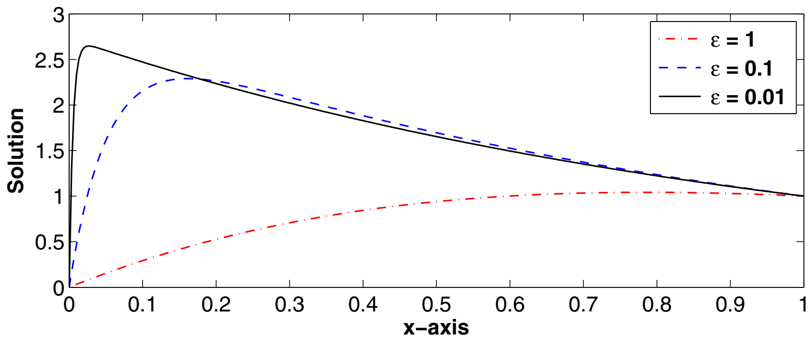


Figure 2.7 Graph of the exact solution of the boundary value problem (2.42)-(2.44), for various values of ϵ. Note the appearance of the boundary layer near $x = 0$ as ϵ decreases.

$$y(0) = 0, \tag{2.43}$$

and

$$y(1) = 1. \tag{2.44}$$

This is a boundary value problem and it has the telltale signs of a singular perturbation problem. Namely, ϵ is multiplying the highest derivative so setting $\epsilon = 0$ results in a lower order problem.

This problem has been selected as the introductory problem because it can be solved exactly, and we will be able to use this to evaluate the accuracy of our approximations. To find the exact solution one assumes $y = e^{rx}$ and from the differential equation concludes that $r_\pm = (-1 \pm \sqrt{1-2\epsilon})/\epsilon$. With this the general solution is $y = c_1 e^{r_+ x} + c_2 e^{r_- x}$. Imposing the two boundary conditions one finds that

$$y = \frac{e^{r_+ x} - e^{r_- x}}{e^{r_+} - e^{r_-}}. \tag{2.45}$$

This function is plotted in Figure 2.7, for various values of ϵ. It is seen that as ϵ deceases the solution starts to show a rapid transition in the region near $x = 0$. Also, if you look at the graphs you will notice that the rapid change takes place over a spatial interval that has a length about equal to the size of ϵ. The reason for making this observation is that our approximation will consist of two pieces, one for x near zero and the other that applies to the rest of the interval. The fact that we end up having to split the interval is not unexpected given what is occurring in Figure 2.7.

Step 1. Outer Solution

The first step is simply to use a regular expansion and see what results. Similar to what we did with the earlier projectile problem, it is assumed

$$y \sim y_0(x) + \epsilon y_1(x) + \cdots . \tag{2.46}$$

Introducing this into (2.42) we obtain

$$\epsilon(y_0'' + \epsilon y_1'' + \cdots) + 2(y_0' + \epsilon y_1' + \cdots) + 2(y_0 + \epsilon y_1 + \cdots) = 0, \tag{2.47}$$

where

$$y_0(0) + \epsilon y_1(0) + \cdots = 0, \tag{2.48}$$

and

$$y_0(1) + \epsilon y_1(1) + \cdots = 1. \tag{2.49}$$

Proceeding in the usual manner yields the following problems.

$O(1)$ $2y_0' + 2y_0 = 0$
$y_0(0) = 0$, $y_0(1) = 1$

The general solution of the differential equation is $y_0 = ae^{-x}$, where a is an arbitrary constant. This is where the singular nature of the problem starts to have an affect. We have one constant but there are two boundary conditions. Can we satisfy at least one of them? For some problems the answer is no, and we will consider such an example later. For this problem we can and we need to determine which one. To help with this decision, the solution is sketched in Figure 2.8 in the case of when $a > 0$ and when $a < 0$. The two boundary conditions are also shown in the figure. It is apparent that of the two possibilities, the $a > 0$ curve is the only one capable of satisfying one of the boundary conditions, and it is the one at $x = 1$. Assuming this is the case then $a = e$ and $y_0(x) = e^{1-x}$.

$O(\epsilon)$ $y_0'' + 2y_1' + 2y_1 = 0$
$y_1(1) = 0$

Note that only the boundary condition at $x = 1$ is listed here as this is the only one we believe this expansion is capable of satisfying. The general solution of the differential equation is $y_1 = (b - x/2)e^{1-x}$, where b is an arbitrary constant. With the given boundary condition we obtain $y_1(x) = (1 - x)e^{1-x}/2$.

Our regular expansion has yielded

$$y \sim e^{1-x} + \cdots . \tag{2.50}$$

Only the first term has been included here as this is all we are going to determine in this example. The second term was calculated earlier to demonstrate that it is easy to find, and also to show that including the second term does not help us satisfy the boundary condition at $x = 0$. It is this fact that will require us to scale the problem and this brings us to the next step.

Step 2. Inner, or Boundary Layer, Solution

We will now construct an approximation of the solution in the neighborhood of $x = 0$, which corresponds to the interval where the function undergoes a rapid increase as shown in Figure 2.7. Given its location, the approximation is called the boundary layer solution. It is also known as an inner solution, and correspondingly, the approximation in (2.50) is the outer solution. The width of this layer shrinks as $\epsilon \to 0$, so we must make a change of variables to account for this. With this in mind we introduce the boundary layer coordinate

$$\bar{x} = \frac{x}{\epsilon^\gamma}\,. \tag{2.51}$$

The exact value of γ will be determined shortly but we already have some inkling what it might be. We saw in Figure 2.7 that the rapid change in the solution near $x = 0$ takes place over an interval that has width of about ϵ. So, it should not be too surprising that we will find that $\gamma = 1$. In any case, using the chain rule

$$\frac{d}{dx} = \frac{d\bar{x}}{dx}\frac{d}{d\bar{x}} = \frac{1}{\epsilon^\gamma}\frac{d}{d\bar{x}}\,, \tag{2.52}$$

and

$$\frac{d^2}{dx^2} = \frac{1}{\epsilon^{2\gamma}}\frac{d^2}{d\bar{x}^2}\,. \tag{2.53}$$

We will designate the solution as $Y(\bar{x})$ when using $\bar{x}$ as the independent variable. With this the differential equation becomes

$$\underset{①}{\epsilon^{1-2\gamma}Y''} + \underset{②}{2\epsilon^{-\gamma}Y'} + \underset{③}{2Y} = 0. \tag{2.54}$$

We determine γ by balancing the terms in the above equation. Our goal is for the highest derivative to remain in the equation as $\epsilon \to 0$. This gives us the following two possibilities.

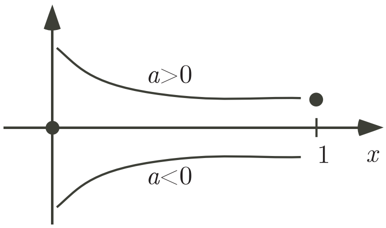


Figure 2.8 Sketch of the possible form of the outer solution $y_0 = ae^x$, depending on the sign of a. Also shown are the two given boundary conditions.

Balance	Condition on γ	Consistency Check	Conclusion
① ∼ ③ with ② higher-order	$1 - 2\gamma = 0$ $\Rightarrow \gamma = 1/2$	①, ③ $= O(1)$ and ② $= O(\epsilon^{-1/2})$	Inconsistent with balance
① ∼ ② with ③ higher-order	$1 - 2\gamma = -\gamma$ $\Rightarrow \gamma = 1$	①, ② $= O(\epsilon^{-1})$ and ③ $= O(1)$	Consistent with balance

Based on the above analysis we take $\gamma = 1$ and with this the differential equation takes the form

$$Y'' + 2Y' + 2\epsilon Y = 0. \tag{2.55}$$

Assuming $Y(\bar{x}) \sim Y_0(\bar{x}) + \cdots$ the differential equation becomes

$$(Y_0'' + \cdots) + 2(Y_0' + \cdots) + 2\epsilon(Y_0 + \cdots) = 0, \tag{2.56}$$

and from this we obtain the following problem.

$O(1)$ $Y_0'' + 2Y_0' = 0$
$Y_0(0) = 0$

Note that the boundary condition at $x = 0$ has been included here but not the one at $x = 1$. The reason is that we are building an approximation of the solution in the immediate vicinity of $x = 0$ and it is incorrect to assume it can satisfy the condition at the other end of the interval. Now, the general solution of the differential equation is $Y_0 = A + Be^{-2\bar{x}}$, where A, B are arbitrary constants. With the given boundary condition this reduces to $Y_0 = A(1 - e^{-2\bar{x}})$.

The approximation of the solution in the boundary layer is

$$Y(\bar{x}) \sim A(1 - e^{-2\bar{x}}) + \cdots . \tag{2.57}$$

We will determine A by connecting this result with the approximation we have for the outer region, and this brings us to the next step.

Step 3. Matching

We have made several assumptions about the solution and it is now time to prove that they are correct. To explain what this means, our approximation consists of two different expansions, and each applies to a different part of the interval. The situation we find ourselves in is sketched in Figure 2.9. This indicates that when coming out of the boundary layer the approximation in (2.57) approaches a constant value A. Similarly, the outer solution approaches a constant value, e, as it enters the boundary layer. There is a transition region, what is usually called an overlap domain, where the two

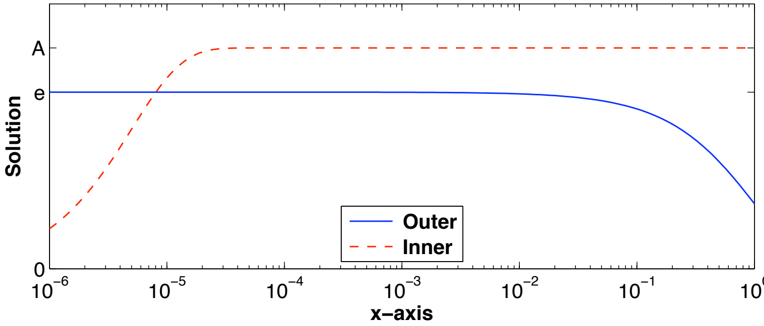


Figure 2.9 Graph of the inner approximation (2.57), and the outer approximation (2.50), before matching.

approximations are both constant. Given that they are approximations of the same function then we need to require that the inner and outer expansions are equal in this region. In more mathematical terms, the requirement we will impose on these two expansions is

$$\lim_{\bar{x}\to\infty} Y_0 = \lim_{x\to 0} y_0. \tag{2.58}$$

This is called the matching condition. With this we conclude $A = e$ and the resulting functions are plotted in Figure 2.10 for $\epsilon = 10^{-4}$. The overlap domain is clearly seen in this figure.

Step 4. Composite Expansion
The approximation of the solution we have comes in two pieces, one that applies near $x = 0$ and another that works everywhere else. Because neither can be used over the entire interval we say that they are not uniformly valid for $0 \leq x \leq 1$. The question we consider now is whether we can combine them

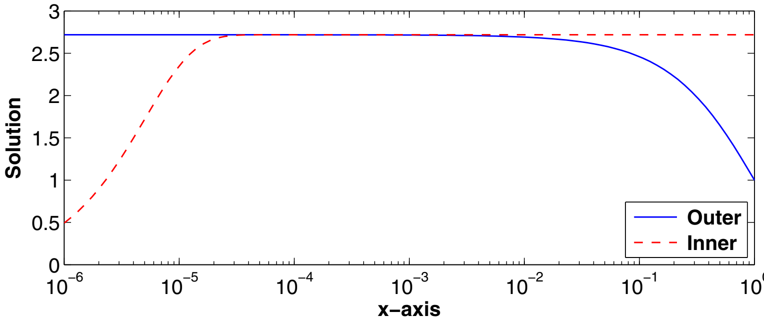


Figure 2.10 Graph of the inner approximation (2.57), and the outer approximation (2.50), after matching in the particular case of when $\epsilon = 10^{-4}$. Note the overlap region where the two approximations produce, approximately, the same result.

in some way to produce a uniform approximation, that is, one that works over the entire interval. The position we are in is summarized in Figure 2.11. The inner and outer solutions are constant outside the region where they are used to approximate the solution, and the constant is the same for both solutions. The value of the constant can be written as either $y_0(0)$ or as $Y_0(\infty)$, and the fact that they are equal is a consequence of the matching condition (2.58). This observation can be used to construct a uniform approximation. Namely, we just add the approximations together and then subtract the constant. The result is

$$\begin{aligned} y &\sim y_0(x) + Y_0(\bar{x}) - y_0(0) \\ &= e^{1-x} - e^{1-2x/\epsilon}. \end{aligned} \tag{2.59}$$

This function is known as a composite expansion and it is valid for $0 \leq x \leq 1$. To demonstrate its effectiveness it is plotted in Figure 2.12 along with the exact solution for $\epsilon = 10^{-1}$ and for $\epsilon = 10^{-2}$. It is evident from this figure that we have constructed a relatively simple expression that is a very good approximation of the solution over the entire interval.

2.4.1 Endnotes

One of the characteristics of a boundary layer is that its width goes to zero as $\epsilon \to 0$, yet the change in the solution across the layer does not go to zero. This type of behavior occurs in a wide variety of problems, although the terminology changes depending on the application and particular type of problem. For example, there are problems where the jump occurs in the interval $0 < x < 1$, a situation known as an interior layer. They are also not limited to BVPs and arise in IVPs, PDEs, etc. A example of this is shown in Figure 2.13. The boundary layer is the thin white region on the surface of the object. In this layer the air velocity changes rapidly, from zero on the object

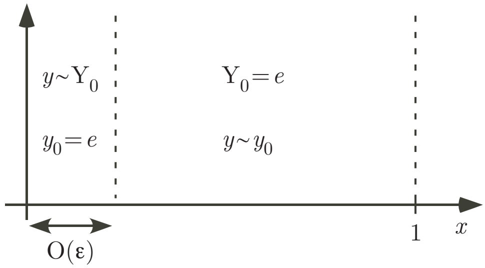


Figure 2.11 Sketch of the inner and outer regions and the values of the approximations in those regions.

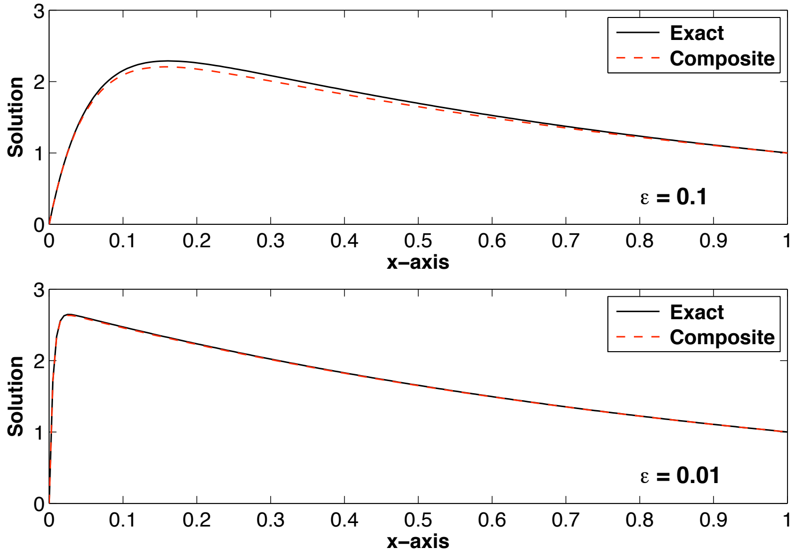
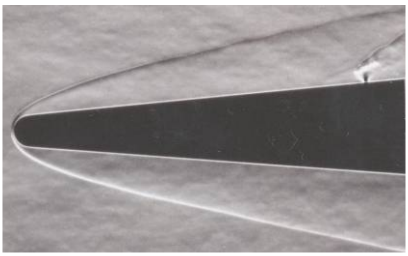


Figure 2.12 Graph of the exact solution (2.45) and composite approximation (2.59) for two values of ϵ.

to the large value in the outer flow. The parabolic curve that appears to be attached to the front of the object is a shock wave. The pressure undergoes a rapid change across the shock, and for this reason it is an example of an interior layer. The presence of a boundary layer is an issue when finding the numerical solution. As an example, Figure 2.14 shows the grid system used to solve the equations for the air flow over an object, in this case an airplane. The presence of a boundary layer necessitates the use of a large number of grid points near the surface, which greatly adds to the computational effort needed to solve the problem.

Another important comment to make concerns the existence of a boundary layer in the solution. In particular, an ϵ multiplying the highest derivative is not a guarantee of a boundary, or interior, layer. A simple example is $\epsilon y'' + y = 0$, for which the general solution is $y = a\sin(x/\sqrt{\epsilon}) + b\cos(x/\sqrt{\epsilon})$. In this case, instead of containing a rapidly decaying exponential function

Figure 2.13 Image of high speed flow, from left to right, over a fixed, wedge-shaped object. The thin white region on the surface of the object is the boundary layer. The parabolic curve is a shock wave, a topic which is studied in Chapter 5.



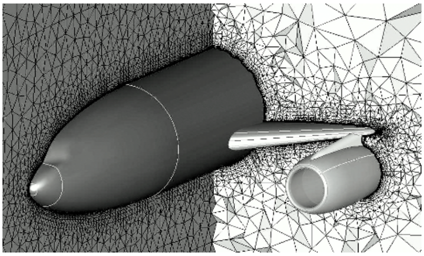


Figure 2.14 Grid refinement needed near the boundary to numerically calculate the air flow over an airplane (Steinbrenner and Abelanet [2007]).

that is characteristic of a boundary layer, the solution consists of rapidly varying oscillatory functions. The approximation method most often used in such situations is known as the WKB method. We will only scratch the surface of this subject, and a more extensive study of this can be found in Holmes [1995].

2.5 Multiple Boundary Layers

As a second boundary layer example we will consider the boundary value problem

$$\epsilon^2 y'' + \epsilon x y' - y = -e^x, \quad \text{for} \;\; 0 < x < 1, \tag{2.60}$$

where the boundary conditions are

$$y(0) = 2, \tag{2.61}$$

and

$$y(1) = 1. \tag{2.62}$$

When given a problem with a small parameter it is worthwhile to quickly check to see what might happen when $\epsilon = 0$. Setting $\epsilon = 0$ in (2.60), we lose all the derivative terms and simply end up with $y = e^x$. This function is incapable of satisfying either boundary condition, so we will find two boundary layers for this solution, one at each end of the interval. This is one of the reasons for considering this particular equation. Another is that it has variable coefficients and it is worth working out an example to see how to handle such situations. The procedure used to construct an asymptotic approximation of the solution will follow the steps we used in the last example, and for this reason there will be fewer explanations of what is being done.

Step 1. Outer Solution
Assuming $y \sim y_0(x) + \epsilon y_1(x) + \cdots$ one finds from the differential equation that $y_0 = e^x$. As stated above, this cannot satisfy either boundary condition, and this brings us to the next step.

STEPS 2 AND 3. Boundary Layer Solutions and Matching
Given that there is a layer at each end we need to split this step into two parts.

a) Layer at $x = 0$. In this region we will denote the solution as $Y(\bar{x})$. The boundary coordinate is the same as before. Setting $\bar{x} = x/\epsilon^{\gamma}$ and using the formulas in (2.52), (2.53) the differential equation (2.60) becomes

$$\underset{①}{\epsilon^{2-2\gamma}Y''} + \underset{②}{\epsilon\bar{x}Y'} - \underset{③}{Y} = \underset{④}{-e^{\epsilon^{\gamma}\bar{x}}}. \tag{2.63}$$

We will balance the terms in the usual manner but note that terms ③ and ④ are the same order. This is because $e^{\epsilon^{\gamma}\bar{x}} \sim 1 + \epsilon^{\gamma}\bar{x} + \cdots$. Consequently, when deciding on what balance we need in (2.63), term ④ will not be considered.

Balance	Condition on γ	Consistency Check	Conclusion
① ∼ ② with ③ higher-order	$2 - 2\gamma = 1$ $\Rightarrow \gamma = 1/2$	①, ② $= O(\epsilon)$ and ③ $= O(1)$	Inconsistent with balance
① ∼ ③ with ② higher-order	$2 - 2\gamma = 0$ $\Rightarrow \gamma = 1$	①, ③ $= O(1)$ and ② $= O(\epsilon)$	Consistent with balance

Consequently, with $\gamma = 1$, the differential equation becomes

$$Y'' + \epsilon\bar{x}Y' - Y = -e^{\epsilon\bar{x}}. \tag{2.64}$$

Assuming $Y(\bar{x}) \sim Y_0(\bar{x}) + \cdots$ we obtain the following problem to solve.

$O(1)$ $Y_0'' - Y_0 = -1$
$Y_0(0) = 2$

The general solution of the differential equation is $Y_0 = 1 + Ae^{\bar{x}} + Be^{-\bar{x}}$, where A, B are arbitrary constants. With the given boundary condition this reduces to $Y_0 = 1 + Ae^{\bar{x}} + (1 - A)e^{-\bar{x}}$.

As before, this boundary layer solution must match with the outer solution calculated earlier. The requirement is

$$\lim_{\bar{x}\to\infty} Y_0 = \lim_{x\to 0} y_0. \tag{2.65}$$

Given that $\lim_{\bar{x}\to\infty} e^{\bar{x}} = \infty$, for Y_0 to be able to match with the outer solution we must set $A = 0$. With this our first term approximation in this boundary layer is

$$Y_0(\bar{x}) = 1 + e^{-\bar{x}}. \tag{2.66}$$

b) Layer at $x = 1$. In this region we will denote the solution as $\widetilde{Y}(\widetilde{x})$. The boundary layer in this case is located at $x = 1$, and so the coordinate will be centered at this point. In particular, we let

$$\widetilde{x} = \frac{x-1}{\epsilon^{\gamma}} . \tag{2.67}$$

The differentiation formulas are similar to those in (2.52), (2.53). Also, we have that $x = 1 + \epsilon^{\gamma}\widetilde{x}$. With this the differential equation (2.60) becomes

$$\underset{①}{\epsilon^{2-2\gamma}\widetilde{Y}''} + \underset{②}{\epsilon^{1-\gamma}(1+\epsilon^{\gamma}\widetilde{x})\widetilde{Y}'} - \underset{③}{\widetilde{Y}} = \underset{④}{-e^{1+\epsilon^{\gamma}\widetilde{x}}}. \tag{2.68}$$

Similar to what happened earlier, terms ③ and ④ are the same order, so term ④ will not be considered in the balancing.

Balance	Condition on γ	Consistency Check	Conclusion
① ∼ ③ with ② higher-order	$2 - 2\gamma = 0$ $\Rightarrow \gamma = 1$	①, ③ $= O(1)$ and ② $= O(1)$	Consistent with balance

Consequently, with $\gamma = 1$, the differential equation becomes

$$\widetilde{Y}'' + (1+\epsilon\widetilde{x})\widetilde{Y}' - \widetilde{Y} = -e^{1+\epsilon\widetilde{x}}. \tag{2.69}$$

Assuming $\widetilde{Y}(\widetilde{x}) \sim \widetilde{Y}_0(\widetilde{x}) + \cdots$ we obtain the following problem to solve.

$O(1)$ $\widetilde{Y}_0'' + \widetilde{Y}_0' - \widetilde{Y}_0 = -e$
$\widetilde{Y}_0(0) = 1$

In the boundary condition, $\widetilde{Y}_0$ is evaluated at $\widetilde{x} = 0$ because $x = 1$ corresponds to $\widetilde{x} = 0$. The general solution of the differential equation is $\widetilde{Y}_0 = e + Ae^{r_+\widetilde{x}} + Be^{r_-\widetilde{x}}$, where $r_\pm = (-1 \pm \sqrt{5})/2$ and A, B are arbitrary constants. With the given boundary condition this reduces to $\widetilde{Y}_0 = e + Ae^{r_+\widetilde{x}} + (1 - e - A)e^{r_-\widetilde{x}}$.

This boundary layer solution must match with the outer solution calculated earlier. The requirement is

$$\lim_{\widetilde{x}\to-\infty} \widetilde{Y}_0 = \lim_{x\to 1} y_0. \tag{2.70}$$

This expression appears different from the one used earlier for the layer at $x = 0$. The reason is that the position of the layer has changed, but the matching principle is the same. Namely, for the approximations to match it is necessary that when you come out of the boundary layer into the outer

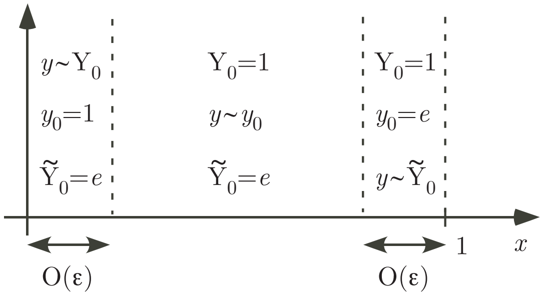


Figure 2.15 Sketch of the three regions and the values of the approximations in those regions.

region (i.e., $\widetilde{x} \to -\infty$) that you get the same value as when you enter the boundary layer from the outer region (i.e., $x \to 1$). Given that $r_+ > 0$ and $r_- < 0$ then $\lim_{\widetilde{x}\to-\infty} e^{r_-\widetilde{x}} = \infty$ and $\lim_{\widetilde{x}\to-\infty} e^{r_+\widetilde{x}} = 0$. For $\widetilde{Y}_0$ to be able to match with the outer solution we must set $1 - e - A = 0$. With this our first term approximation in this boundary layer is

$$\widetilde{Y}_0(\widetilde{x}) = e + (1-e)e^{r_+\widetilde{x}}. \tag{2.71}$$

Step 4. Composite
In a similar manner as in the last example, it is possible to combine the three approximations we have derived to produce a uniform approximation. The situation is shown schematically in Figure 2.15. It is seen that in each region the two approximations not associated with that region add to $1 + e$. This means we simply add the three approximations together and subtract $1 + e$. In other words,

$$\begin{aligned} y &\sim y_0(x) + Y_0(\bar{x}) + \widetilde{Y}_0(\widetilde{x}) - y_0(0) - y_0(1) \\ &= e^x + e^{-x/\epsilon} + (1-e)e^{r_+(x-1)/\epsilon}. \end{aligned} \tag{2.72}$$

This function is a composite expansion of the solution and it is valid for $0 \le x \le 1$. To demonstrate its effectiveness the composite approximation is plotted in Figure 2.16 along with the numerical solution for $\epsilon = 10^{-1}$ and for $\epsilon = 10^{-2}$. The approximations are not very accurate for $\epsilon = 10^{-1}$, but this is not unexpected given that ϵ is not particularly small. In contrast, for $\epsilon = 10^{-2}$ the composite approximation is quite good over the entire interval, and it is expected to get even better for smaller values of ϵ.

2.6 Multiple Scales and Two-Timing

As the last two examples have demonstrated, the presence of a boundary layer limits the region over which an approximation can be used. Said another way, the inner and outer approximations are not uniformly valid over the entire interval. The tell-tale sign that this is going to happen is that when $\epsilon = 0$ the highest derivative in the problem is lost. However, the lack of uniformity can occur in other ways and one investigated here relates to changes in the solution as a function of time. It is easier to explain what happens by working out a typical example. For this we use the pendulum problem. Letting $\theta(t)$ be the angular deflection made by the pendulum, as shown in Figure 2.17, the problem is

$$\theta'' + \sin(\theta) = 0, \tag{2.73}$$

where

$$\theta(0) = \epsilon, \tag{2.74}$$

and

$$\theta'(0) = 0. \tag{2.75}$$

The equation of motion (2.73) comes from Newton's second law, $F = ma$, where the external forcing F is gravity. It is assumed the initial angle is small,

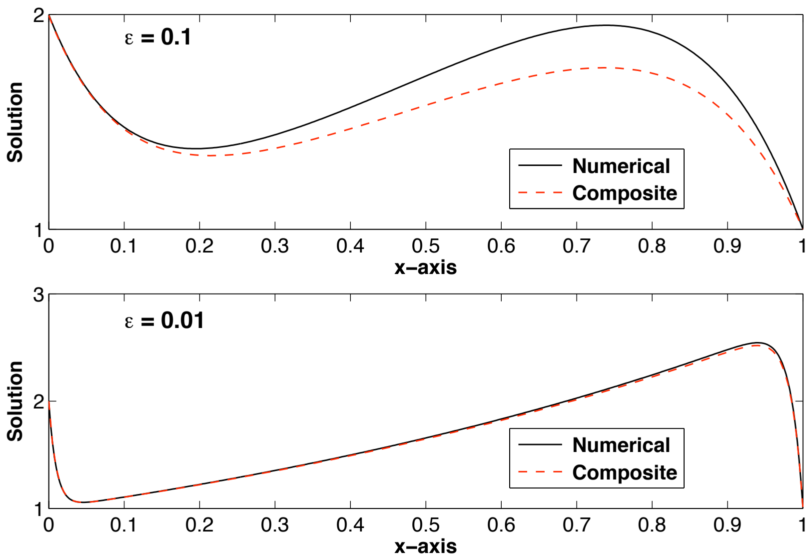


Figure 2.16 Graph of the numerical solution of the boundary value problem (2.60)-(2.62) and the composite approximation of the solution (2.72). In the upper plot $\epsilon = 10^{-1}$ and in the lower plot $\epsilon = 10^{-2}$.

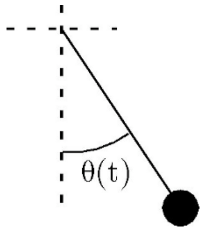


Figure 2.17 Pendulum example.

and this is the reason for the initial condition (2.74). It is also assumed that the pendulum starts from rest, so the initial velocity (2.75) is zero.

Although the problem is difficult to solve we have at least some idea of what the solution looks like because of everyday experience with a pendulum (e.g., watching a grandfather clock or using a swing). Starting out with the given initial conditions, the pendulum should simply oscillate back and forth. A real pendulum will eventually stop due to damping, but we have not included this in the model so our pendulum should go forever.

The fact that the small parameter is in the initial condition, and not in the differential equation, is a bit different from what we had in the last two examples but we are still able to use our usual approximation methods. The appropriate expansion in this case is

$$\theta(t) \sim \epsilon(\theta_0(t) + \epsilon^{\alpha}\theta_1(t) + \cdots). \tag{2.76}$$

The ϵ multiplying the series is there because of the initial condition. If we did not have it, and tried $\theta = \theta_0 + \epsilon^{\alpha}\theta_1 + \cdots$, we would find that $\theta_0 = 0$ and $\alpha = 1$. The assumption in (2.76) is made simply to avoid all the work to find that the first term in the expansion is just zero. Before substituting (2.76) into the problem recall $\sin(x) = x - \frac{1}{6}x^3 + \cdots$ when x is close to zero. This means, because the θ in (2.76) is close to zero,

$$\begin{aligned} \sin(\theta) &\sim \sin(\epsilon(\theta_0 + \epsilon^{\alpha}\theta_1 + \cdots)) \\ &\sim (\epsilon\theta_0 + \epsilon^{\alpha+1}\theta_1 + \cdots) - \frac{1}{6}(\epsilon\theta_0 + \ \cdots)^3 + \cdots \\ &\sim \epsilon\theta_0 + \epsilon^{\alpha+1}\theta_1 - \frac{1}{6}\epsilon^3\theta_0^3 + \cdots . \end{aligned} \tag{2.77}$$

With this the equation of motion (2.73) becomes

$$\epsilon\theta_0'' + \epsilon^{\alpha+1}\theta_1'' + \cdots + \epsilon\theta_0 + \epsilon^{\alpha+1}\theta_1 - \frac{1}{6}\epsilon^3\theta_0^3 + \cdots = 0, \tag{2.78}$$

and the initial conditions are

$$\epsilon\theta_0(0) + \epsilon^{\alpha+1}\theta_1(0) + \cdots = \epsilon, \tag{2.79}$$

and

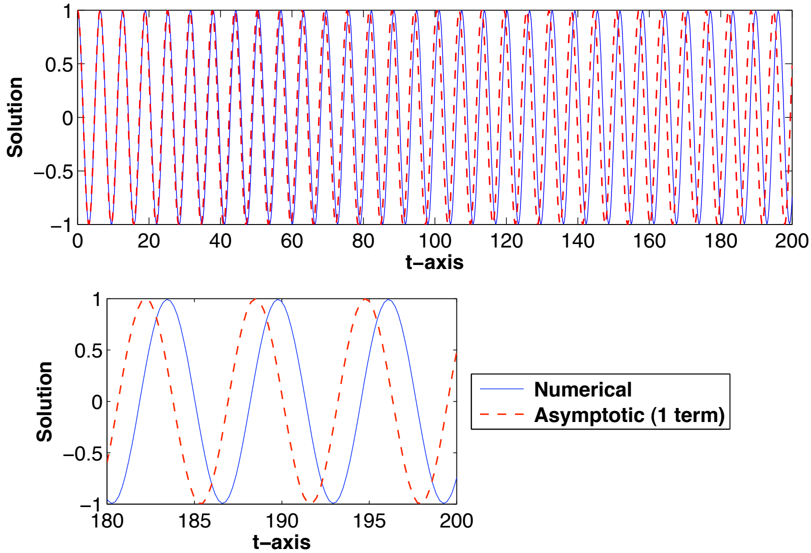


Figure 2.18 Graph of the numerical solution of the pendulum problem (2.60)-(2.62) and the first term in the regular perturbation approximation (2.76). Shown are the solutions over the entire time interval, as well as a close up of the solutions near $t = 200$. In the calculation $\epsilon = \frac{1}{3}$ and both solutions have been divided by $\epsilon = \frac{1}{3}$.

$$\epsilon\theta_0'(0) + \epsilon^{\alpha+1}\theta_1'(0) + \cdots = 0. \tag{2.80}$$

Proceeding in the usual manner yields the following problem.

$O(\epsilon)$ $\theta_0'' + \theta_0 = 0$
$\theta_0(0) = 1,\ \theta_0'(0) = 0$

The general solution of the differential equation is $\theta_0 = a\cos(t) + b\sin(t)$, where a, b are arbitrary constants. It is possible to write this solution in the more compact form $\theta_0 = A\cos(t + B)$, where A, B are arbitrary constants. As will be explained later, there is a reason for why the latter form is preferred in this problem. With this, and the initial conditions, it is found that $\theta_0 = \cos(t)$.

The plot of the one-term approximation, $\theta \sim \epsilon\cos(t)$, and the numerical solution are shown in Figure 2.18. The asymptotic approximation describes the solution accurately at the start, and reproduces the amplitude very well over the entire time interval. What it has trouble with is matching the phase and this is evident in the lower plot in Figure 2.18. One additional comment to make is the value for ϵ used in Figure 2.18 is not particularly small, so getting an approximation that is not very accurate is no surprise. However, if a smaller value is used the same difficulty arises. The difference is that the first term approximation works over a longer time interval but eventually the phase error seen in Figure 2.18 occurs.

In looking to correct the approximation to reduce the phase error we calculate the second term in the expansion. With the given θ_0 there is an $\epsilon^3\theta_0^3$ term in (2.78). To balance this we use the θ_1 term in the expansion and this requires $\alpha = 2$. With this we have the following problem to solve.

$O(\epsilon^3)$ $\theta_1'' + \theta_1 = \frac{1}{6}\theta_0^3$
$\theta_1(0) = 0$, $\theta_1'(0) = 0$

The method of undetermined coefficients can be used to find a particular solution of this equation. This requires the identity $\cos^3(t) = \frac{1}{4}(3\cos(t) + 3\cos(3t))$, in which case the differential equation becomes

$$\theta_1'' + \theta_1 = \frac{1}{24}(3\cos(t) + 3\cos(3t)). \tag{2.81}$$

With this the general solution is found to be $\theta_1 = a\cos(t) + b\sin(t) - \frac{1}{16}t\sin(t)$, where a, b are arbitrary constants. From the initial conditions this reduces to $\theta_1 = -\frac{1}{16}t\sin(t)$.

The plot of the two term approximation,

$$\theta \sim \epsilon\cos(t) - \frac{1}{16}\epsilon^3 t\sin(t), \tag{2.82}$$

and the numerical solution is shown in Figure 2.19. It is clear from this that we have been less than successful in improving the approximation. The culprit here is the $t\sin(t)$ term. As time increases its contribution grows, and it eventually gets as large as the first term in the expansion. Because of this it is called a secular term, and it causes the expansion not to be uniformly valid for $0 \le t < \infty$. This problem would not occur if time were limited to a finite

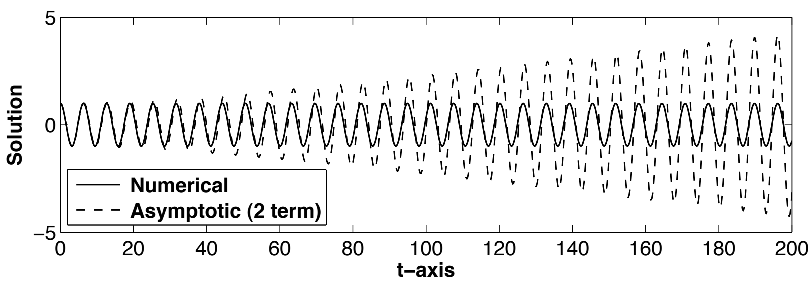


Figure 2.19 Graph of the numerical solution of the pendulum problem (2.60)-(2.62) and the regular perturbation approximation (2.72). In the calculation $\epsilon = \frac{1}{3}$ and the solution has been divided by $\epsilon = \frac{1}{3}$.

interval, as happened in the projectile problem. However, for the pendulum there is no limit on time and this means the expansion is restricted to when it can be used. One last comment to make concerns how this term ended up in the expansion in the first place. In the differential equation for θ_1, given in (2.81), the right hand side contains $\cos(t)$ and this is a solution of the associated homogeneous equation. It is this term that produces the $t\sin(t)$ in the expansion and it is this term we would like to prevent from appearing in the problem.

What is happening is that there is a slow change in the solution that the first term approximation is unable to describe. In effect there are two time scales acting in this problem. One is the basic period of oscillation, as seen in Figure 2.18, and the other is a slow time scale over which the phase changes. Our approximation method will be based on this observation. We will explicitly assume there are two concurrent time scales, given as

$$t_1 = t, \tag{2.83}$$

$$t_2 = \epsilon^\gamma t. \tag{2.84}$$

The value of γ is not known yet, and we will let the problem tell us the value as we construct the approximation. Based on this assumption it is not surprising that the method is called two-timing, or the method of multiple scales.

To illustrate the idea underlying two-timing, consider the function

$$u = e^{-3\epsilon t}\sin(5t).$$

This consists of an oscillatory function, with a slowly decaying amplitude. This can be written using the two-timing variables as

$$u = e^{-3t_2}\sin(5t_1),$$

where $\gamma = 1$.

The change of variables in (2.83), (2.84) is reminiscent of the boundary layer problems in the previous section. The difference here is that we are not separating the time axis into separate regions but, rather, using two time scales together. As we will see, this has a profound effect on how we construct the approximation.

To determine how the change of variables affects the time derivative, we have, using the chain rule,

$$\begin{aligned}\frac{d}{dt} &= \frac{dt_1}{dt}\frac{\partial}{\partial t_1} + \frac{dt_2}{dt}\frac{\partial}{\partial t_2}\\ &= \frac{\partial}{\partial t_1} + \epsilon^\gamma\frac{\partial}{\partial t_2}\,. \end{aligned}\tag{2.85}$$

The second derivative is

$$\frac{d^2}{dt^2} = \left(\frac{\partial}{\partial t_1} + \epsilon^\gamma \frac{\partial}{\partial t_2}\right)\left(\frac{\partial}{\partial t_1} + \epsilon^\gamma \frac{\partial}{\partial t_2}\right)$$
$$= \frac{\partial^2}{\partial t_1^2} + 2\epsilon^\gamma \frac{\partial^2}{\partial t_1 \partial t_2} + \epsilon^{2\gamma} \frac{\partial^2}{\partial t_2^2} . \tag{2.86}$$

The steps used to construct an asymptotic approximation of the solution will closely follow what we did earlier. It should be kept in mind during the derivation that the sole reason for introducing t_2 is to prevent a secular term from appearing in the second term.

With the introduction of a second time variable, the expansion is assumed to have the form

$$\theta \sim \epsilon(\theta_0(t_1, t_2) + \epsilon^\alpha \theta_1(t_1, t_2) + \cdots). \tag{2.87}$$

The only difference between this and the regular expansion (2.76) used earlier is that the terms are allowed to depend on both time variables. When this is substituted into the differential equation we obtain an expression similar to (2.78), except the time derivatives are given in (2.85) and (2.86). Specifically, we get

$$\epsilon \frac{\partial^2}{\partial t_1^2}\theta_0 + \epsilon^{\alpha+1} \frac{\partial^2}{\partial t_1^2}\theta_1 + 2\epsilon^{\gamma+1} \frac{\partial^2}{\partial t_1 \partial t_2}\theta_0 + \cdots + \epsilon\theta_0 + \epsilon^{\alpha+1}\theta_1 - \frac{1}{6}\epsilon^3\theta_0^3 + \cdots = 0, \tag{2.88}$$

and the initial conditions are

$$\epsilon\theta_0(0,0) + \epsilon^{\alpha+1}\theta_1(0,0) + \cdots = \epsilon, \tag{2.89}$$

and

$$\epsilon \frac{\partial}{\partial t_1}\theta_0(0,0) + \epsilon^{\alpha+1} \frac{\partial}{\partial t_1}\theta_1(0,0) + \epsilon^{\gamma+1} \frac{\partial}{\partial t_2}\theta_0(0,0) + \cdots = 0. \tag{2.90}$$

Proceeding in the usual manner yields the following problem.

$O(\epsilon)$ $\frac{\partial^2}{\partial t_1^2}\theta_0 + \theta_0 = 0$
$\theta_0(0,0) = 1,\ \frac{\partial}{\partial t_1}\theta_0(0,0) = 0$

The general solution of the differential equation is $\theta_0 = A(t_2)\cos(t_1 + B(t_2))$, where A, B are arbitrary functions of t_2. The effects of the second time variable are seen in this solution, because the coefficients are now functions of the second time variable. To satisfy the initial conditions we need $A(0)\cos(B(0)) = 1$ and $A(0)\sin(B(0)) = 0$. From this we have that $A(0) = 1$ and $B(0) = 0$.

In the differential equation (2.88), with the $O(\epsilon)$ terms out of the way, the next term to consider is $\epsilon^3\theta_0^3$. The only terms we have available to balance with this have order $\epsilon^{\alpha+1}$ and $\epsilon^{\gamma+1}$. To determine which terms to use we can use a balance argument, similar to what was done for boundary layers. It is found that both terms are needed and this means $\alpha+1 = \gamma+1 = 3$. This is an example of a distinguished balance. A somewhat different way to say this is, the more components of the equation you can include in the approximation the better. In any case, our conclusion is that $\alpha = \gamma = 2$ and this yields the next problem to solve.

$O(\epsilon^3)$ $\frac{\partial^2}{\partial t_1^2}\theta_1 + \theta_1 + 2\frac{\partial^2}{\partial t_1 \partial t_2}\theta_0 = \frac{1}{6}\theta_0^3$
$\theta_1(0,0) = 0,\ \frac{\partial}{\partial t_1}\theta_1(0,0) + \frac{\partial}{\partial t_2}\theta_0(0,0) = 0$

The method of undetermined coefficients can be used to find a particular solution. To be able to do this we first substitute the solution for θ_0 into the differential equation and then use the identity $\cos^3(t) = \frac{1}{4}(3\cos(t) + 3\cos(3t))$. The result is

$$\begin{aligned}\theta_1'' + \theta_1 = \frac{1}{24}&[3\cos(t_1 + B) + 3\cos(3(t_1 + B))] \\ &+2A'\sin(t_1 + B) + 2AB'\cos(t_1 + B).\end{aligned} \tag{2.91}$$

We are at a similar point to what occurred using a regular expansion, as given in (2.81). As before, the right-hand side of the differential equation contains functions that are solutions of the associated homogeneous equation, namely, $\cos(t_1 + B)$ and $\sin(t_1 + B)$. If they are allowed to remain they will produce a solution containing either $t_1\cos(t_1 + B)$ or $t_1\sin(t_1 + B)$. Either one will cause a secular term in the expansion and for this reason we will select A and B to prevent this from happening. To lose $\sin(t_1 + B)$ we take $A' = 0$ and to eliminate $\cos(t_1 + B)$ we take $2AB' = -\frac{1}{8}$. With the earlier determined initial conditions $A(0) = 1$ and $B(0) = 0$, we conclude that $A = 1$ and $B = -t_2/16$.

In the above analysis we never actually determined θ_1. It is enough to know that the problem for θ_1 will not result in a secular term in the expansion. We did find A and B, and with them the expansion is

$$\theta \sim \epsilon\cos(t - \epsilon^2 t/16) + \cdots . \tag{2.92}$$

To investigate the accuracy of this approximation, it is plotted in Figure 2.20 using the same values as for Figure 2.19. Clearly we have improved the first term approximation, and now do well with both amplitude and phase.

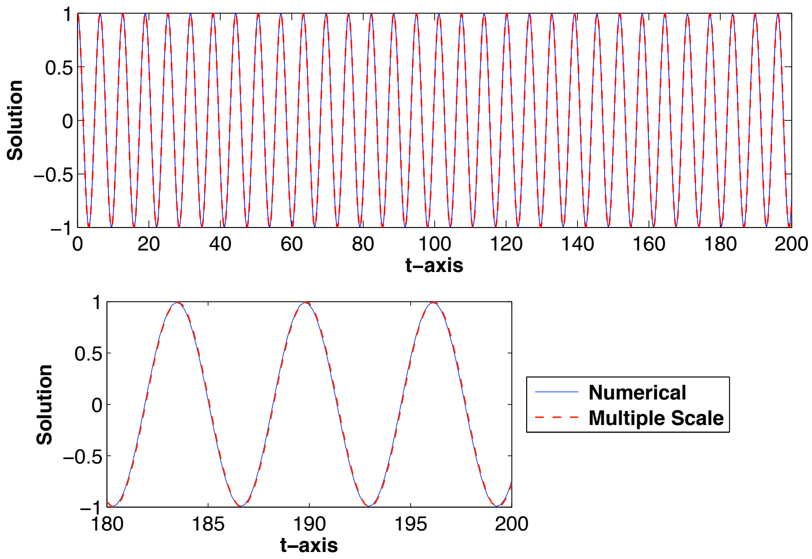


Figure 2.20 Graph of the numerical solution of the pendulum problem (2.60)-(2.62) and the multiple scale approximation (2.92). Shown are the solutions over the entire time interval, as well as a close up of the solutions near $t = 200$. In the calculation $\epsilon = \frac{1}{3}$ and the solution has been divided by $\epsilon = \frac{1}{3}$.

Exercises

2.1. Assuming $f \sim a_1\epsilon^\alpha + a_2\epsilon^\beta + \cdots$ find α, β (with $\alpha < \beta$), and nonzero a_1, a_2, for the following:
(a) $f = e^{\sin(\epsilon)}$.
(b) $f = \sqrt{1 + \cos(\epsilon)}$.
(c) $f = 1/\sqrt{\sin(\epsilon)}$.
(d) $f = 1/(1 - e^\epsilon)$.
(e) $f = \sin(\sqrt{1 + \epsilon x})$, for $0 \le x \le 1$.
(f) $f = \epsilon \exp(\sqrt{\epsilon} + \epsilon x)$, for $0 \le x \le 1$.

2.2. Let $f(\epsilon) = \sin(e^\epsilon)$.
(a) According to Taylor's theorem, $f(\epsilon) = f(0) + \epsilon f'(0) + \frac{1}{2}\epsilon^2 f''(0) + \cdots$. Show that this gives (2.13).
(b) Explain why the formula used in part (a) can not be used to find an expansion of $f(\epsilon) = \sin(e^{\sqrt{\epsilon}})$. Also, show that the method used to derive (2.13) still works, and derive the expansion.

2.3. Consider the equation

$$x^2 + (1 - 4\epsilon)x - \sqrt{1 + 4\epsilon} = 0.$$

(a) Sketch the functions in this equation and then use this to explain why there two real-valued solutions.
(b) Find a two-term asymptotic expansion, for small ϵ, of each solution.

2.4. Consider the equation
$$\ln(x) = \epsilon x.$$
(a) Sketch the functions in this equation and then use this to explain why there are two real-valued solutions.
(b) Find a two-term asymptotic expansion, for small ϵ, of the solution near $x = 1$.

2.5. Consider the equation
$$xe^x = \epsilon.$$
(a) Sketch the functions in this equation and then use this to explain why there is one real-valued solution.
(b) Find a two-term asymptotic expansion, for small ϵ, of the solution.

2.6. Consider the equation
$$x^3 = \epsilon e^{-x}.$$
(a) Sketch the functions in this equation and then use this to explain why there is one real-valued solution.
(b) Find a two-term asymptotic expansion, for small ϵ, of the solution.

2.7. Consider the equation
$$\sin(x + \epsilon) = x.$$
(a) Sketch the functions in this equation and then use this to explain why there is one real-valued solution.
(b) Find a two-term asymptotic expansion, for small ϵ, of the solution.

2.8. Consider the equation
$$\frac{x^3}{1 + x} = \epsilon.$$
(a) Sketch the functions in this equation and then use this to explain why there is only one real-valued solution and describe where it is located for small values of ϵ. Use this to explain why you might want to use an expansion of the form $x \sim \epsilon^\alpha x_0 + \epsilon^\beta x_1 + \cdots$ rather than the one in (2.17).
(b) Find a two-term asymptotic expansion, for small ϵ, of each solution.

2.9. Consider the equation
$$x(x + 2) = \epsilon(x - 1).$$

(a) Sketch the functions in this equation and then use this to explain why there are two real-valued solutions and describe where they are located for small values of ϵ.
(b) Find a two-term asymptotic expansion, for small ϵ, of each solution.

2.10. Consider the equation

$$x(x-1)(x+2) = \epsilon e^{x}.$$

(a) Sketch the functions in this equation. Use this to explain why there are three real-valued solutions, and describe where they are located for small values of ϵ. Use this to explain why you might want to use an expansion of the form $x \sim \epsilon^{\alpha} x_0 + \epsilon^{\beta} x_1 + \cdots$ for one of the solutions, while (2.17) should work for the other two.
(b) Find a two-term asymptotic expansion, for small ϵ, of each solution.

2.11. Consider the equation $\epsilon x^4 - x - 1 = 0$.
(a) Sketch the functions in this equation and then use this to explain why there are only two real-valued solutions to this equation and describe where they are located for small values of ϵ.
(b) Find a two-term asymptotic expansion, for small ϵ, of each solution.

2.12. Consider the equation

$$\frac{1}{1+x^2} = \epsilon x^2.$$

(a) Sketch the functions in this equation and then use this to explain why there are two real-valued solutions and describe where they are located for small values of ϵ. Use this to explain why an expansion of the form given in (2.17) is not a good idea.
(b) Find a two-term asymptotic expansion, for small ϵ, of the solutions.

2.13. Find a two-term expansion of the solution of

$$\frac{dv}{dt} + \epsilon v^2 + v = 0, \quad \text{for } \ 0 < t,$$

where $v(0) = 1$.

2.14. The projectile problem that includes air resistance is

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = -\frac{1}{(1+\epsilon x)^2},$$

where $x(0) = 0$, and $\frac{dx}{dt}(0) = 1$. For small ϵ, find a two-term expansion of the solution.

2.15. Air resistance is known to depend nonlinearly on velocity, and the dependence is often assumed to be quadratic. Assuming gravity is constant, the equations of motion are

$$\frac{d^2y}{dt^2} = -\epsilon\frac{dy}{dt}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2},$$
$$\frac{d^2x}{dt^2} = -1 - \epsilon\frac{dx}{dt}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

Here x is the vertical elevation of the object, and y is its horizontal location. The initial conditions are $x(0) = y(0) = 0$, and $\frac{dx}{dt}(0) = \frac{dy}{dt}(0) = 1$. The assumption is that air resistance is weak, and so ϵ is small and positive.
(a) For small ϵ, find the first terms in the expansions for x and y.
(b) Find the second terms in the expansions for x and y.

2.16. Consider the nonlinear boundary value problem

$$\frac{d}{dx}\left(\frac{y_x}{1+\epsilon y_x^2}\right) - y = 0, \quad \text{for } \ 0 < x < 1,$$

where $y(0) = 1$ and $y(1) = e^{-1}$. This type of nonlinearity arises in elasticity, a topic taken up in Chapter 6.
(a) Explain why a boundary layer is not expected in the problem, and a regular expansion should work.
(b) For small ϵ, find a two-term expansion of the solution.

2.17. The Friedrichs' (1942) model problem for a boundary layer in a viscous fluid is

$$\epsilon y'' = a - y', \quad \text{for } \ 0 < x < 1,$$

where $y(0) = 0$ and $y(1) = 1$ and a is a given positive constant.
(a) After finding a first term of the inner and outer expansions, derive a composite expansion of the solution.
(b) Taking $a = 1$, plot the exact and composite solutions, on the same axes, for $\epsilon = 10^{-1}$. Do the same thing for $\epsilon = 10^{-2}$ and for $\epsilon = 10^{-3}$. Comment on the effectiveness, or non-effectiveness, of the expansion in approximating the solution.
(c) Suppose you assume the boundary layer is at the other end of the interval. Show that the resulting first term approximations from the inner and outer regions do not match.

2.18. Given that the boundary layer is at $x = 0$, find a composite expansion of

$$\epsilon y'' + 3y' - y^4 = 0, \quad \text{for } \ 0 < x < 1,$$

where $y(0) = 1$ and $y(1) = 1$.

2.19. Given that the boundary layer is at $x = 0$, find a composite expansion of

$$\epsilon^2 y'' + y' + \epsilon y = x, \quad \text{for } 0 < x < 1,$$

where $y(0) = 1$ and $y(1) = 1$.

2.20. Find a composite expansion of

$$\epsilon y'' + y' - y = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = -1$.

2.21. Find a composite expansion of

$$\epsilon y'' + 2y' - y^3 = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$ and $y(1) = 1$.

2.22. Find a composite expansion of

$$\epsilon y'' + y' + \epsilon y = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = 1$, $y(1) = 2$.

2.23. Given that the boundary layer is at $x = 1$, find a composite expansion of

$$\epsilon y'' - 3y' - y^4 = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = 1$ and $y(1) = 1$.

2.24. Find a composite expansion of

$$\epsilon y'' - \frac{1}{2}y' - xy = 0, \quad \text{for } 0 < x < 1,$$

where $y(0) = 1$ and $y(1) = 1$.

2.25. Find a composite expansion of

$$\epsilon y'' - (2 - x^2)y = -1, \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 2$.

2.26. Find a composite expansion of

$$\epsilon y'' - (1 + x)y = 2, \quad \text{for } 0 < x < 1,$$

where $y(0) = 0$, $y(1) = 0$.

2.27. As found in Exercise 1.13, the equation for a weakly damped oscillator is

$$y'' + \epsilon y' + y = 0, \quad \text{for } 0 < t,$$

where $y(0) = 1$ and $y(0) = 0$.

(a) For small ϵ, find a two-term regular expansion of the solution.
(b) Explain why the expansion in (a) is not well-ordered for $0 \le t < \infty$. What requirement is needed on t so it is well-ordered?
(c) Use two-timing to construct a better approximation to the solution.

2.28. The weakly nonlinear Duffing equation is

$$y'' + y' + \epsilon y^3 = 0, \quad \text{for } \ 0 < t,$$

where $y(0) = 0$ and $y'(0) = 1$.
(a) For small ϵ, find a two-term regular expansion of the solution.
(b) Explain why the expansion in (a) is not well-ordered for $0 \le t < \infty$. What requirement is needed on t so it is well-ordered?
(c) Use two-timing to construct a better approximation to the solution.

2.29. This problem derives additional information from the projectile problem.
(a) Let t_M be the time at which the projectile reaches its maximum height. Given that the solution depends on ϵ, it follows that t_M depends on ϵ. Use (2.27) to find a two-term expansion of t_M for small ϵ. What is the resulting two-term expansion for the maximum height x_M?
(b) Let t_E be the time at which the projectile hits the ground. Given that the solution depends on ϵ, it follows that t_E depends on ϵ. Use (2.27) to find a two-term expansion of t_E for small ϵ.
(c) Based on your results from parts (a) and (b), describe the effects of the nonlinear gravitational field on the motion of the projectile.

2.30. In the study of reactions of chemical mixtures one comes across the following problem

$$\frac{d^2y}{dx^2} = -\epsilon e^y, \quad \text{for } \ 0 < x < 1,$$

where $y(0) = y(1) = 0$. This is known as Bratu's equation, and it illustrates some of the difficulties one faces when solving nonlinear equations.
(a) Explain why a boundary layer is not expected in the problem and find the first two terms in a regular expansion of the solution.
(b) The function

$$y = -2 \ln\left[\frac{\cosh(\beta(1-2x))}{\cosh(\beta)}\right],$$

where β satisfies

$$\cosh(\beta) = 2\beta\sqrt{\frac{2}{\epsilon}}, \tag{2.93}$$

is a solution of the Bratu problem. By sketching the functions in (2.93), as functions of β, explain why there is an ϵ_0 where if $0 < \epsilon < \epsilon_0$ then there are exactly two solutions, while if $\epsilon_0 < \epsilon$ then there are no solutions.

(c) Comment on the conclusion drawn from part (b) and your result in part (a). Explain why the regular expansion does not fail in a manner found in a boundary layer problem but that it is still not adequate for this problem.