SHAPE OPTIMIZATION FOR DYNAMIC CONTACT PROBLEMS WITH FRICTION

A. Myśliński
System Research Institute
Polish Academy of Sciences
01 - 447 Warsaw, ul. Newelska 6, Poland
myslinsk@ibspan.waw.pl

Abstract The paper deals with shape optimization of dynamic contact problem with Coulomb friction for viscoelastic bodies. The mass nonpenetrability condition is formulated in velocities. The friction coefficient is assumed to be bounded. Using material derivative method as well as the results concerning the regularity of solution to dynamic variational inequality the directional derivative of the cost functional is calculated and necessary optimality condition is formulated.

Keywords: Dynamic unilateral problem, shape optimization, sensitivity analysis, necessary optimality condition

1. Introduction

This paper deals with formulation of a necessary optimality condition for a shape optimization problem of a viscoelastic body in unilateral dynamic contact with a rigid foundation. It is assumed that the contact with given friction, described by Coulomb law [2], occurs at a portion of the boundary of the body. The contact condition is described in velocities. This first order approximation seems to be physically realistic for the case of small distance between the body and the obstacle and for small time intervals. The friction coefficient is assumed to be bounded. The equilibrium state of this contact problem is described by an hyperbolic variational inequality of the second order [2, 3, 5, 7, 17].

The shape optimization problem for the elastic body in contact consists in finding, in a contact region, such shape of the boundary of the domain occupied by the body that the normal contact stress is minimized. It is assumed that the volume of the body is constant.

The original version of this chapter was revised: The copyright line was incorrect. This has been corrected. The Erratum to this chapter is available at DOI: 10.1007/978-0-387-35699-0_19

E. W. Sachs et al. (eds.), *System Modeling and Optimization XX*

Shape optimization of static contact problems was considered, among others, in [3, 8, 9, 10, 11, 16]. In [3, 8] the existence of optimal solutions and convergence of finite–dimensional approximation was shown. In [9, 10, 11, 16] necessary optimality conditions were formulated using the material derivative approach (see [16]). Numerical results are reported in [3, 11].

In this paper we shall study this shape optimization problem for a viscoelastic body in unilateral dynamical contact. The essential difficulty to deal with the shape optimization problem for dynamic contact problem is regularity of solutions to the state system. Assuming small friction coefficient and suitable regularity of data it can be shown [6, 7] that the solution to dynamic contact problem is enough regular to differentiate it with respect to parameter. Using the material derivative method [16] as well as the results of regularity of solutions to the dynamic variational inequality [6, 7] we calculate the directional derivative of the cost functional and we formulate necessary optimality condition for this problem. The present paper extends the authors' results contained in [12].

We shall use the following notation : $\Omega \in R^2$ will denote the bounded domain with Lipschitz continuous boundary Γ. The time variable will be denoted by t and the time interval $I = (0, \mathcal{T}), \mathcal{T} > 0$. By $H^k(\Omega)$, $k \in (0, \infty)$ we will denote the Sobolev space of functions having derivatives in all directions of the order k belonging to $L^2(\Omega)$ [1]. For an interval I and a Banach space B $L^p(I; B)$, $p \in (1, \infty)$ denotes the usual Bochner space [2]. $u_t = du/dt$ and $u_{tt} = d^2u/dt^2$ will denote first and second order derivatives, respectively, with respect to t of function u. u_{tN} and u_{tT} will denote normal and tangential components, respectively, of function u_t. $Q = I \times \Omega$, $\gamma_i = I \times \Gamma_i$, $i = 1, 2, 3$ where Γ_i are pieces of the boundary Γ.

2. Contact problem formulation

Consider deformations of an elastic body occupying domain $\Omega \in R^2$. The boundary Γ of domain Ω is Lipschitz continuous. The body is subjected to body forces $f = (f_1, f_2)$. Moreover surface tractions $p = (p_1, p_2)$ are applied to a portion Γ_1 of the boundary Γ. We assume that the body is clamped along the portion Γ_0 of the boundary Γ and that the contact conditions are prescribed on the portion Γ_2 of the boundary Γ. Moreover $\Gamma_i \cap \Gamma_j = \emptyset$, $i \neq j, i, j = 0, 1, 2$, $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$.

We denote by $u = (u_1, u_2)$, $u = u(t, x)$, $x \in \Omega$, $t \in [0, \mathcal{T}]$, $\mathcal{T} > 0$ the displacement of the body and by $\sigma = \{\sigma_{ij}(u(t, x))\}, i, j = 1, 2$, the stress field in the body. We shall consider elastic bodies obeying Hooke's law

[2, 3, 5, 17] :

$$\sigma_{ij}(u) = c^0_{ijkl}(x)e_{kl}(u) + c^1_{ijkl}(x)e_{kl}(u_t) \;\; x \in \Omega, \quad e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \quad (1)$$

$i,j,k,l = 1,2$, $u_{k,l} = \partial u_k/\partial x_l$. We use here the summation convention over repeated indices [2]. $c^0_{ijkl}(x)$ and $c^1_{ijkl}(x)$, $i,j,k,l = 1,2$ are components of Hooke's tensor. It is assumed that elasticity coefficients c^0_{ijkl} and c^1_{ijkl} satisfy usual symmetry, boundedness and ellipticity conditions [2, 3, 5]. In an equilibrium state a stress field σ satisfies the system [2, 3, 6, 7] :

$$u_{tti} - \sigma_{ij}(x)_{,j} = f_i(x), \; (t,x) \in (0,\mathcal{T}) \times \Omega \;\; i,j = 1,2 \qquad (2)$$

where $\sigma_{ij}(x)_{,j} = \partial\sigma_{ij}(x)/\partial x_j$, $i,j = 1,2$. There are given the following boundary conditions :

$$u_i(x) = 0 \;\text{ on } (0,\mathcal{T}) \times \Gamma_0 \quad i = 1,2,$$
$$\sigma_{ij}(x)n_j = p_i \text{ on } (0,\mathcal{T}) \times \Gamma_1 \quad i,j = 2; \qquad (3)$$
$$u_{tN} \leq 0, \; \sigma_N \leq 0, \; u_{tN}\sigma_N = 0, \qquad \text{on } (0,\mathcal{T}) \times \Gamma_2; \qquad (4)$$
$$u_{tT} = 0 \quad \Rightarrow \; |\, \sigma_T \,| \leq \mathcal{F} \,|\, \sigma_N \,|;$$
$$u_{tT} \neq 0 \quad \Rightarrow \; \sigma_T = -\mathcal{F} \,|\, \sigma_N \,|\, \frac{u_{tT}}{|\, u_{tT} \,|}. \qquad (5)$$

Here we denote : $u_N = u_i n_i$, $\sigma_N = \sigma_{ij} n_i n_j$, $(u_T)_i = u_i - u_N n_i$, $(\sigma_T)_i = \sigma_{ij} n_j - \sigma_N n_i$ $i,j = 1,2$, $n = (n_1, n_2)$ is the unit outward vector to the boundary Γ. There are given the following initial conditions:

$$u_i(0,x) = u_0 \quad u_{ti}(0,x) = u_1, \quad i = 1,2, \quad x \in \Omega. \qquad (6)$$

We shall consider problem (2)–(6) in the variational form. Let us assume,

$$f \in H^{1/4}(I;(H^1(\Omega;R^2))^*) \cap L^2(Q;R^2),$$
$$p \in L^2(I;(H^{1/2}(\Gamma_1;R^2))^*),$$
$$u_0 \in H^{3/2}(\Omega;R^2) \quad u_1 \in H^{3/2}(\Omega;R^2), \quad u_{1|\Gamma_2} = 0, \qquad (7)$$
$$\mathcal{F} \in L^\infty(\Gamma_2;R^2) \quad \mathcal{F}(.,x) \text{ is continuous for a.e. } x \in \Gamma_2$$

be given. The space $L^2(Q;R^2)$ and the Sobolev spaces $H^{1/4}(I;(H^1(\Omega;R^2))^*)$ as well as $(H^{1/2}(\Gamma_1);R^2)$ are defined in [1, 2]. Let us introduce :

$$F = \{z \in L^2(I;H^1(\Omega;R^2)) \; : \; z_i = 0 \;\text{ on } (0,\mathcal{T}) \times \Gamma_0 \;, i = 1,2\} \qquad (8)$$

$$K = \{z \in F \; : \; z_{tN} \leq 0 \quad \text{on } (0, \mathcal{T}) \times \Gamma_2 \}. \tag{9}$$

The problem (1) - (6) is equivalent to the following variational problem [6, 7]: find $u \in L^\infty(I; H^1(\Omega; R^2)) \cap H^{1/2}(I; L^2(\Omega; R^2)) \cap K$ such that $u_t \in L^\infty(I; L^2(\Omega; R^2)) \cap H^{1/2}(I; L^2(\Omega; R^2)) \cap K$ and $u_{tt} \in L^\infty(I; H^{-1}(\Omega; R^2))$ $\cap (H^{1/2}(I; L^2(\Omega; R^2)))^*$ satisfying the following inequality [6, 7],

$$\begin{gathered}\int_Q u_{tti} dx d\tau + \int_Q \sigma_{ij}(u) e_{ij}(v_i - u_{ti}) dx d\tau + \\ \int_{\gamma_2} \mathcal{F} \mid \sigma_N(u) \mid (\mid v_T \mid - \mid u_{tT} \mid) dx d\tau \geq \int_Q f_i(v_i - u_{ti}) dx d\tau + \\ \int_{\gamma_1} p_i(v_i - u_{ti}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega; R^2)) \cap K.\end{gathered} \tag{10}$$

Note, that from (2) as well as from Imbedding Theorem of Sobolev spaces [1] it follows that u_0 and u_1 in (6) are continuous on the boundary of cylinder Q. The existence of solutions to system (1) - (6) was shown in [6, 7]:

Theorem 2.1 *Assume : (i) The data are smooth enough, i.e. (2) is satisfied. (ii) Γ_2 is of class $C^{1,1}$. (iii) The friction coefficient is small enough. Then there exists a unique weak solution to the problem (1) - (6).*

Proof. The proof is based on penalization of the inequality (10), friction regularization and employment of localization and shifting technique due to Lions and Magenes. For details of the proof see [7].

□

For the sake of brevity we shall consider the contact problem with prescribed friction, i.e., we shall assume

$$\mathcal{F} \mid \sigma_N \mid = \sigma_T \leq 1. \tag{11}$$

The condition (4) is replaced by the following one,

$$u_{tT}\sigma_T + \mid u_{tT} \mid = 0, \; \mid \sigma_T \mid \leq 1 \quad \text{on } I \times \Gamma_2. \tag{12}$$

Let us introduce the space

$$\Lambda = \{\lambda \in L^2(I; L^\infty(\Gamma_2)) \; : \mid \lambda \mid \leq 1 \quad \text{on } I \times \Gamma_2\}. \tag{13}$$

Taking into account (12) the system (10) takes the form : Find $u \in K$ and $\lambda \in \Lambda$ such that

$$\int_Q u_{tti} dx d\tau + \int_Q \sigma_{ij}(u) e_{ij}(v_i - u_{ti}) dx d\tau - \int_{\gamma_2} \lambda_T (v_T - u_{tT}) dx d\tau$$

$$\geq \int_Q f_i(v_i - u_{ti})dxd\tau + \int_{\gamma_1} p_i(v_i - u_{ti})dxd\tau \tag{14}$$

$$\forall v \in H^{1/2}(I; H^1(\Omega; R^2)) \cap K$$

$$\int_{\gamma_2} \sigma_T u_{tT} dsd\tau \leq \int_{\gamma_2} \lambda_T u_{tT} dsd\tau \quad \forall \lambda_T \in \Lambda. \tag{15}$$

3. Formulation of the shape optimization problem

We consider a family $\{\Omega_s\}$ of the domains Ω_s depending on parameter s. For each Ω_s we formulate a variational problem corresponding to (10). In this way we obtain a family of the variational problems depending on s and for this family we shall study a shape optimization problem , i.e., we minimize with respect to s a cost functional associated with the solutions to (10).

The domain Ω_s we shall consider as an image of a reference domain Ω under a smooth mapping $\mathbf{T}_s$. To describe the transformation $\mathbf{T}_s$ we shall use the speed method [16]. Let us denote by $V(s,x)$ an enough regular vector field depending on parameter $s \in [0,\vartheta), \vartheta > 0$:

$$V(.,.) \,:\, [0,\vartheta) \times R^2 \to R^2$$

$$V(s,.) \in C^2(R^2,R^2) \;\; \forall s \in [0,\vartheta), \;\; V(.,x) \in C([0,\vartheta),R^2) \;\; \forall x \in R^2. \tag{16}$$

Let $\mathbf{T}_s(V)$ denotes the family of mappings : $\mathbf{T}_s(V) \,:\, R^2 \ni X \to x(t,X) \in R^2$ where the vector function $x(.,X) = x(.)$ satisfies the systems of ordinary differential equations :

$$\frac{d}{d\tau}x(\tau,X) = V(\tau,x(\tau,X)), \tau \in [0,\vartheta), \quad x(0,X) = X \in R. \tag{17}$$

We denote by $D\mathbf{T}_s$ the Jacobian of the mapping $\mathbf{T}_s(V)$ at a point $X \in R^2$. We denote by $D\mathbf{T}_s^{-1}$ and ${}^{\star}D\mathbf{T}_s^{-1}$ the inverse and the transposed inverse of the Jacobian $D\mathbf{T}_s$, respectively. $J_s = \det D\mathbf{T}_s$ will denote the determinant of the Jacobian $D\mathbf{T}_s$. The family of domains $\{\Omega_s\}$ depending on parameter $s \in [0,\vartheta), \vartheta > 0$, is defined as follows : $\Omega_0 = \Omega$

$$\Omega_s = \mathbf{T}_s(\Omega)(V) = \{x \in R^2 \,:\, \exists X \in R^2 \text{ s. th. } x = x(s,X), \\ \text{where the function } x(.,X) \text{ satisfies (17) for } 0 \leq \tau \leq s\}. \tag{18}$$

Let us consider problem (14) - (15) in the domain Ω_s. Let F_s, K_s, Λ_s be defined, respectively, by (8, (9), (13) with Ω_s instead of Ω. We shall write $u_s = u(\Omega_s)$, $\sigma_s = \sigma(\Omega_s)$. The problem (14) - (15) in the domain Ω_s takes the form : find $u_s \in K_s$ and $\lambda_s \in \Lambda_s$ such that,

$$\int_{Q_s} u_{ttsi} v_i dx d\tau + \int_{Q_s} \sigma_{ij}(u_s) e_{ij}(v_i - u_{tsi}) dx d\tau -$$
$$\int_{\gamma_{s2}} \lambda_{sT}(v_T - u_{tsT}) dx d\tau \geq \int_{Q_s} f_i(v_i - u_{tsi}) dx d\tau + \qquad (19)$$
$$\int_{\gamma_{s1}} p_i(v_i - u_{tsi}) dx d\tau \quad \forall v \in H^{1/2}(I; H^1(\Omega_s; R^2)) \cap K$$

$$\int_{\gamma_{s2}} \sigma_{sT} u_{tsT} ds d\tau \leq \int_{\gamma_{s2}} \lambda_{sT} u_{tsT} ds d\tau \quad \forall \lambda_{sT} \in \Lambda_s. \qquad (20)$$

We are ready to formulate the optimization problem. By $\hat{\Omega} \subset R^2$ we denote a domain such that $\Omega_s \subset \hat{\Omega}$ for all $s \in [0, \vartheta), \vartheta > 0$. Let $\phi \in M$ be a given function. The set M is determined by :

$$M = \{\phi \in L^\infty(I; H_0^2(\hat{\Omega}; R^2) \ : \phi \leq 0 \text{ on } I \times \hat{\Omega}, \ \| \phi \|_{L^\infty(I; H_0^2(\hat{\Omega}; R^2)} \leq 1\} \qquad (21)$$

Let us introduce, for given $\phi \in M$, the following cost functional :

$$J_\phi(u_s) = \int_{\gamma_{s2}} \sigma_{sN} \phi_{tNs} dz d\tau, \qquad (22)$$

where ϕ_{tNs} and σ_{sN} are normal components of ϕ_{ts} and σ_s, respectively, depending on parameter s. Note, that the cost functional (22) approximates the normal contact stress [3, 8, 11]. We shall consider such a family of domains $\{\Omega_s\}$ that every Ω_s, $s \in [0, \vartheta)$, $\vartheta > 0$, has constant volume $c > 0$, i.e. : every Ω_s belongs to the constraint set U given by :

$$U = \{\Omega_s \ : \ \int_{\Omega_s} dx = c\}. \qquad (23)$$

We shall consider the following **shape optimization problem** :

For given $\phi \in M$, find the boundary Γ_{2s}
of the domain Ω_s occupied by the body, (24)
minimizing the cost functional (22) subject to $\Omega_s \in U$.

The set U given by (23) is assumed to be nonempty. $(u_s, \lambda_s) \in K_s \times \Lambda_s$ satisfy (19) - (20). Note, that the goal of the shape optimization problem (24) is to find such boundary Γ_2 of the domain Ω occupied by the body that the normal contact stress is minimized. Remark, that the cost functional (22) can be written in the following form [3, 17] :

$$\int_{\gamma_{2s}} \sigma_{sN}\phi_{tN} ds d\tau = \int_{Q_s} u_{tts}\phi_{ts} dx d\tau + \int_{Q_s} \sigma_{sij}(u_s) e_{kl}(\phi_{ts}) dx d\tau - \qquad (25)$$
$$\int_{Q_s} f\phi_{ts} dx d\tau - \int_{\gamma_{1s}} p_s \phi_{ts} ds d\tau - \int_{\gamma_{2s}} \sigma_{sT}\phi_{tTs} ds d\tau.$$

We shall assume there exists at least one solution to the optimization problem (24). It implies a compactness assumption of the set (23) in suitable topology. For detailed discussion concerning the conditions assuring the existence of optimal solutions see [3, 16].

4. Shape derivatives of contact problem solution

In order to calculate the Euler derivative (44) of the cost functional (22) we have to determine shape derivatives $(u', \lambda') \in F \times \Lambda$ of a solution $(u_s, \lambda_s) \in K_s \times \Lambda_s$ of the system (19)–(20). Let us recall from [16] :

Definition 4.1 *The shape derivative $u' \in F$ of the function $u_s \in F_s$ is determined by :*

$$(\tilde{u}_s)_{|\Omega} = u + su' + o(s), \qquad (26)$$

where $\| o(s) \|_F / s \to 0$ for $s \to 0$, $u = u_0 \in F$, $\tilde{u}_s \in F(R^2)$ is an extension of the function $u_s \in F_s$ into the space $F(R^2)$. $F(R^2)$ is defined by (8) with R^2 instead of Ω.

In order to calculate shape derivatives $(u', \lambda') \in F \times \Lambda$ of a solution $(u_s, \lambda_s) \in K_s \times \Lambda_s$ of the system (19),(20) first we calculate material derivatives $(\dot{u}, \dot{\lambda}) \in F \times \Lambda$ of the solution $(u_s, \lambda_s) \in K_s \times \Lambda_s$ to the system (19),(20). Let us recall the notion of the material derivative [16]:

Definition 4.2 *The material derivative $\dot{u} \in F$ of the function $u_s \in K_s$ at a point $X \in \Omega$ is determined by :*

$$\lim_{s \to 0} \| [(u_s \circ \mathbf{T}_s) - \sigma]/s - \dot{u} \|_F = 0, \qquad (27)$$

where $u \in K$, $u_s \circ \mathbf{T}_s \in K$ is an image of function $u_s \in K_s$ in the space F under the mapping $\mathbf{T}_s$.

Taking into account Definition 4.2 we can calculate material derivatives of a solution to the system (19),(20) :

Lemma 4.1 *The material derivatives $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$ of a solution $(u_s, \lambda_s) \in K_s \times \Lambda_s$ to the system (19)–(20) are determined as a unique* *solution to the following system :*

$$\int_Q \{(\dot{u}_{tt}\eta + u_{tt}\dot{\eta} + u_{tt}\eta divV(0)(DV(0)u)_{tt}\eta + u_{tt}(DV(0)\eta)$$

$$-\dot{f}\eta - f\dot{\eta} + (\sigma_{ij}(u)e_{kl}(\eta) - f\eta)divV(0)\}dxd\tau - \tag{28}$$

$$\int_{\gamma_1}(\dot{p}\eta + p\dot{\eta} + p\eta D)dxd\tau - \int_{\gamma_2}\{(\dot{\lambda}\eta_{tT} + \lambda\dot{\eta}_{tT} +$$

$$\lambda\nabla\eta_T V(0)n + \lambda\eta_{tT}D\}dxd\tau \geq 0 \quad \forall\eta \in K_1,$$

$$\int_{\gamma_2}(\dot{\lambda} - \mu)u_{tT} + (\lambda - \dot{\mu})u_{tT} + (\lambda - \mu)\dot{u}_{tT} + \lambda u_{tT}D\}dxd\tau \quad \forall\mu \in L_1, \tag{29}$$

where V(0) = V(0,X), DV(0) denotes the Jacobian matrix of the matrix V(0). Moreover :

$$K_1 = \{\xi \in F \;:\; \xi = u - DVu \;\; on\ \gamma_0,\;\; \xi n \geq nDV(0)u \;\; on\ A_1,$$
$$\xi n = nDV(0)u \;\; on\ A_2\ \}, \tag{30}$$

$$A_0 = \{x \in \gamma_2 \;:\; u_{tN} = 0\},\;\; A_1 = \{x \in B \;:\; \sigma_N = 0\},$$
$$A_2 = \{x \in B \;:\; \sigma_N < 0\}, \tag{31}$$

$$\begin{aligned} B_0 &= \{x \in \gamma_2 \;:\; \lambda_T = 1,\;\; u_{tT} \neq 0\}, \\ B_1 &= \{x \in \gamma_2 \;:\; \lambda_T = -1,\;\; u_{tT} = 0\}, \\ B_2 &= \{x \in \gamma_2 \;:\; \lambda_T = 1, \;:\; u_{tT} = 0\}. \end{aligned} \tag{32}$$

$$L_1 = \{\xi \in \Lambda \;:\; \xi \geq 0 \; on\ B_2,\;\; \xi \leq 0 \; on\ B_1,\; \xi = 0 \; on\ B_0\ \} \tag{33}$$

and D is given by

$$D = div\, V(0) - (DV(0)n, n). \tag{34}$$

Proof: It is based on approach proposed in [16]. First we transport the system (19)–(20) to the fixed domain Ω. Let $u^s = u_s \circ \mathbf{T}_s \in F$, $u = u_0 \in F$, $\lambda^s = \lambda_s \circ \mathbf{T}_s \in \Lambda$, $\lambda = \lambda_0 \in \Lambda$. Since in general $u^s \notin K(\Omega)$ we introduce a new variable $z^s = D\mathbf{T}_s^{-1}u^s \in K$. Moreover $\dot{z} = \dot{u} - DV(0)u$ [7, 15]. Using this new variable z^s as well as the formulae for transformation of the function and its gradient into reference domain Ω [15, 16] we write the system (19)–(20) in the reference domain Ω. Using the estimates on time derivative of function u [7] the Lipschitz continuity of u and λ satisfying (19) - (20) with respect to s can be proved. Applying to this system the result concerning the differentiability of solutions to variational inequality [15, 16] we obtain that the material derivative $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$ satisfies the system (28)-(29). Moreover from the ellipticity condition of the elasticity coefficients by a standard argument [15] it follows that $(\dot{u}, \dot{\lambda}) \in K_1 \times \Lambda$ is a unique solution to the system (28)-(29).

□

Recall [16], that if the shape derivative $u' \in F$ of the function $u_s \in F_s$ exists, then the following condition holds :

$$u' = \dot{u} - \nabla u V(0), \tag{35}$$

where $\dot{u} \in F$ is material derivative of the function $u_s \in F_s$.

From regularity result in [7] it follows that :

$$\nabla u V(0) \in F, \quad \nabla \lambda_T V(0) \in \Lambda, \tag{36}$$

where the spaces F and Λ are determined by (8) and (13) respectively.

Integrating by parts system (28),(29) and taking into account (35),(36) we obtain the similar system to (28),(29) determining the shape derivative $(u', \lambda'_T) \in F \times L$ of the solution $(u_s, \lambda_{sT}) \in K_s \times L_s$ of the system (19) - (20) :

$$\begin{aligned}\int_Q u'_{tt}\eta + u_t t\eta' + (DV(0) +^* DV(0))u_{tt}\eta dx d\tau + \int_\gamma u_{tt}\eta V(0)n \\ \int_Q \sigma_{ij}(u')e_k l\eta - \int_{\gamma_2} \lambda'\eta_{tT} + \lambda\eta'_{tT}\} dx d\tau + \\ I_1(u_t, \eta) + I_2(\lambda, u, \eta) \geq 0 \quad \forall \eta \in N_1,\end{aligned} \tag{37}$$

$$\int_{\gamma_2} [u'_{tT}(\mu - \lambda) - u_{tT}\lambda'] dx d\tau + I_3(u, \mu - \lambda) \geq 0 \quad \forall \mu \in L_1, \tag{38}$$

$$N_1 = \{\eta \in F \ : \ \eta = \lambda - DuV(0), \ \ \lambda \in K_1\}, \tag{39}$$

$$\begin{aligned}I_1(\varphi, \phi) = \int_\gamma \{\sigma_{ij}(\varphi)e_k l\phi - f\phi - \\ ((\nabla pn)\phi + (p\nabla\phi)n + p\phi H)V(0)n\} dx d\tau,\end{aligned} \tag{40}$$

$$\begin{aligned}I_2(\mu, \varphi, \phi) = \int_{\gamma_2} \{(\nabla\mu)n\nabla\phi + \mu(\nabla(\nabla\varphi n))\varphi + \\ \mu\nabla\varphi_{tT}H + \mu\nabla\varphi n\}V(0)n dx d\tau,\end{aligned} \tag{41}$$

$$\begin{aligned}I_3(\varphi, \mu - \lambda) = \int_{\gamma_2} (\varphi n)(\mu - \lambda) + \varphi(\nabla\mu n) - \varphi(\nabla\lambda n) + \\ \varphi(\mu - \lambda)H]V(0)n dx d\tau,\end{aligned} \tag{42}$$

where H denotes a mean curvature of the boundary Γ [16].

5. Necessary optimality condition

Our goal is to calculate the directional derivative of the cost functional (22) with respect to the parameter s. We will use this derivative to formulate necessary optimality condition for the optimization problem (24). First, let us recall from [16] the notion of Euler derivative of the cost functional depending on domain Ω :

Definition 5.1 . *Euler derivative $dJ(\Omega; V)$ of the cost functional J at a point Ω in the direction of the vector field V is given by :*

$$dJ(\Omega; V) = \limsup_{s \to 0} [J(\Omega_s) - J(\Omega)]/s. \tag{43}$$

The form of the directional derivative $dJ_\phi(u; V)$ of the cost functional (22) is given in :

Lemma 5.1 *The directional derivative $dJ_\phi(u; V)$ of the cost functional (22), for $\phi \in M$ given, at a point $u \in K$ in the direction of vector field V is determined by :*

$$\begin{aligned} dJ_\phi(u; V) = \int_Q [u'_{tt}\eta + u_{tt}\eta' + (DV(0) +^* DV(0))u_{tt}\eta] dx d\tau + \\ \int_\gamma u_{tt}\eta V(0)n + \int_Q (\sigma'_{ij} e_{kl}(\phi) dx + \\ \int_\Gamma (\sigma_{ij} e_{kl}(\phi) - f\phi) V(0) n ds - \int_{\Gamma_1} (\nabla p \phi V(0) + \\ p \nabla \phi V(0) + p\phi D) ds - \int_{\Gamma_2} \sigma'_T \phi_T ds + I_1(u, \phi) - I_2(\lambda, u, \phi), \end{aligned} \tag{44}$$

where σ' is a shape derivative of the function σ_s with respect to s. This derivative is defined by (26). ∇p is a gradient of function p with respect to x. Moreover V(0) = V(0,X), ϕ_T and σ_T are tangent components of functions ϕ and σ, respectively, as well as D is given by (34). DV(0) denotes the Jacobian matrix of the matrix V(0) and div denotes divergence operator.

Proof : Taking into account (22),(25) as well as formulae for transformation of the gradient of the function defined on domain Ω_s into the reference domain Ω [16] and using the mapping (16)– (17) we can express the cost functional (22) defined on domain Ω_s in the form of the functional $J_\phi(u^s)$ defined on domain Ω, determined by :

$$J_\phi(u^s) = \int_Q (D\mathbf{T}_s u^s)_{tt} D\mathbf{T}_s \phi_t^s det D\mathbf{T}_s dx d\tau +$$

$$\int_Q [\sigma_{ij} D\mathbf{T}_s e_{kl}(D\mathbf{T}_s \phi_t^s) - f^s D\mathbf{T}_s \phi) \det D\mathbf{T}_s dx -$$
$$\int_{\gamma_1} p^s D\mathbf{T}_s \phi \parallel \det D\mathbf{T}_s{}^\star D\mathbf{T}_s^{-1} n \parallel ds - \tag{45}$$
$$\int_{\gamma_2} \lambda_{sT} D\mathbf{T}_s \phi_T \parallel \det D\mathbf{T}_s{}^\star D\mathbf{T}_s^{-1} n \parallel ds,$$

where $u^s = u_s \circ \mathbf{T}_s \in F$, $u = u_0 \in F$ and $\lambda = \lambda_0 \in \Lambda$. By (43) we have :

$$dJ_\phi(u; V) = \limsup_{t \to 0} [J_\phi(u^s) - J_\phi(u)]/s. \tag{46}$$

Remark, it follows by standard arguments [3] that the pair $(\sigma_s, u_s) \in Q_s \times K_s$, $s \in [0, \vartheta)$, $\vartheta > 0$, satisfying the system (19)–(20) is Lipschitz continuous with respect to the parameter s. Passing to the limit with $s \to 0$ in (46) as well as taking into account the formulae for derivatives of $D\mathbf{T}_s^{-1}$ and $\det D\mathbf{T}_s$ with respect to the parameter s [16] and (26) we obtain (44).

□

In order to eliminate the shape derivative (u', λ') from (44) we introduce an adjoint state $(r, q) \in K_2 \times L_2$ defined as follows :

$$\int_Q r_{tt} \zeta dx d\tau + \int_Q \sigma_{ij}(\zeta) e_{kl}(\phi + r) dx d\tau +$$
$$\int_{\gamma_2} \zeta_{tT}(q - \lambda) \zeta dx d\tau = 0 \quad \forall \zeta \in K_2, \tag{47}$$

with

$$r(\mathcal{T}, x) = 0, \quad r_t(\mathcal{T}, x) = 0,$$

$$\int_{\gamma_2} (r_{tT} + \phi_{tT} - u_{tT}) \delta dx d\tau = 0 \;\; \forall \delta \in L_2, \tag{48}$$

$$K_2 = \{\zeta \in K_1 \; : \; \zeta n = 0 \text{ on } A_0\}, \tag{49}$$

$$L_2 = \{\delta \in \Lambda \; : \; \delta = 0 \;\text{ on } A_0 \cap B_0 \;\}. \tag{50}$$

Since $\phi \in M$ is a given element, then by the same arguments as used to show the existence of solution $(u, \lambda) \in K \times L$ to the system (19)–(20) we can show the existence of the solution $(r, q) \in K_2 \times L_2$ to the system (47),(48).

From (44),(37),(38), (47),(48) we obtain :

$$dJ_\phi(u; V) = I_1(u, \phi + r) + I_2(\lambda, u, \phi + r) + I_3(u, q - \lambda). \tag{51}$$

The necessary optimality condition has a standard form :

Theorem 5.1 *There exists a Lagrange multiplier $\mu \in R$ such that for all vector fields V determined by (16),(17) the following condition holds :*

$$dJ_\phi(u;V) + \mu \int_\Gamma V(0)nds \geq 0, \tag{52}$$

where $dJ_\phi(u;V)$ is given by (51).

Proof : It is given in [3, 4, 5, 16, 17].

6. Conclusions

In the paper the necessary optimality condition for the shape optimization problem for the dynamical contact problem was formulated. Preliminary numerical results can be found in [13] where the continuous optimization problem was discretized by piecewise linear and piecewise constant functions on each finite element. The discretized problem was numerically solved by an Augmented Lagrangian Algorithm combined with active set strategy and updating of the dual variables.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] G.Duvaut and J.L. Lions, *Les inequations en mecanique et en physique*, Dunod, Paris, 1972.

[3] J. Haslinger, Neittaanmaki P., *Finite Element Approximation for Optimal Shape Design. Theory and Application.*, John Wiley& Sons, 1988.

[4] E.J. Haug, K.K Choi, V. Komkov, *Design Sensitivity Analysis of Structural Systems*, Academic Press, 1986.

[5] I. Hlavacek, J. Haslinger, J. Necas, J.Lovisek, *Solving of Variational Inequalities in Mechanics*(in Russian), Mir, Moscow, 1986.

[6] J. Jarusek and C. Eck, Dynamic Contact Problems with Small Coulomb Friction for Viscoelastic Bodies. Existence of Solutions, *Mathematical Models and Methods in Applied Sciences*, **9**, pp. 11 - 34, 1999.

[7] J. Jarusek, Dynamical Contact Problem with Given Friction for Viscoelastic Bodies, *Czech. Math. Journal* , **46**, pp. 475 - 487, 1996.

[8] A. Klabring, J. Haslinger, On almost Constant Contact Stress Distributions by Shape Optimization, *Structural Optimization*, **5**, pp. 213-216, 1993.

[9] A. Myśliński, Mixed Variational Approach for Shape Optimization of Contact Problem with Prescribed Friction, in : Numerical Methods for Free Boundary Problems, P.Neittaanmaki ed., *International Series of Numerical Mathematics*, Birkhäuser, Basel,**99**, pp. 286-296, 1991.

[10] A. Myśliński, Shape Optimization of Contact Problems Using Mixed Variational Formulation, *Lecture Notes in Control and Information Sciences*, Springer,Berlin, **160**, pp. 414 - 423, 1992.

[11] A. Myśliński, Mixed Finite Element Approximation of a Shape Optimization Problem for Systems Described by Elliptic Variational Inequalities, *Archives of Control Sciences*, **3**, No 3-4, pp. 243 - 257, 1994.

[12] A. Myśliński, Shape Optimization for Dynamic Contact Problems, *Discussiones Mathematicae, Differential Inclusions, Control and Optimization*, **20**, pp. 79 - 91, 2000.

[13] A. Myśliński, Augmented Lagrangian Techniques for Shape Optimal Design of Dynamic Contact Problems, *Preprint*, System Research Institute, Warsaw, 2001 - to be published in Proceedings of WCSMO4 Conference, 2001.

[14] J. Necas, *Les Methodes Directes en Theorie des Equations Elliptiques*, Masson, Paris, 1967.

[15] J. Sokolowski and J.P. Zolesio, Shape sensitivity analysis of contact problem with prescribed friction, *Nonlinear Analysis, Theory, Methods and Applications* , **12**, pp. 1399 - 1411, 1988.

[16] J. Sokolowski and J.P. Zolesio, *Introduction to Shape Optimization. Shape Sensitivity Analysis.* Springer, Berlin, 1992.

[17] J. Telega, Variational Methods in Contact Problems of Mechanics(in Russian), *Advances in Mechanics*, **10**, pp. 3-95, 1987.