DETERMINING FUNCTIONALS FOR A CLASS OF SECOND ORDER IN TIME EVOLUTION EQUATIONS WITH APPLICATIONS TO VON KARMAN EQUATIONS

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Abstract In this paper we present a general approach to construction of determining functionals for second order in time evolution equations with nonlinear damping. As an example we consider von Karman evolution equations which describe nonlinear oscillations of an elastic plate.

Keywords: Long-time dynamics, determining parameters, von Karman equations.

Introduction

The question of the number of parameters that are necessary for the description of the long-time behaviour of solutions to nonlinear partial differential equations was first discussed by Foias and Prodi [7] and Ladyzhenskaya [11] for the 2D Navier-Stokes equations. They proved that

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the asymptotic behaviour of the solutions is completely determined by dynamics of the first N Fourier modes, if N is sufficiently large. Later a general approach to the problem of the existence of a finite number of determining functionals (parameters) for dissipative evolution PDE has been developed (see [3] and [4, Chap.5] for a survey).

In this paper we present an approach to construction of determining functionals for second order in time evolution equations with *nonlinear* damping. Nonlinear dissipation does not allow to apply general methods developed in [3, 4]. As an example von Karman evolution equations which describe nonlinear plate oscillations is considered.

As in [2, 3, 4] we involve the concept of the completeness defect for a description of sets of determining functionals. Assume that X and Y are Banach spaces and X continuously and densely embedded into Y. Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a finite set of linearly independent continuous functionals on X. We define the completeness defect $\epsilon_{\mathcal{L}}(X, Y)$ of the set $\mathcal{L}$ with respect to the pair of the spaces X and Y by the formula

$$\epsilon_{\mathcal{L}}(X,Y) = \sup\{\|w\|_Y \ : w \in X, \ l_j(w) = 0, \ l_j \in \mathcal{L}, \ \|w\|_X \leq 1\}.$$

The value $\epsilon_{\mathcal{L}}$ is proved to be very useful for characterization of sets of determining functionals (see, e.g., [3, 4] and the references therein). One can show that the completeness defect $\epsilon_{\mathcal{L}}(X, Y)$ is the best possible global error of approximation in Y of elements $u \in X$ by elements of the form $u_{\mathcal{L}} = \sum_{j=1}^{N} l_j(u)\phi_j$, where $\{\phi_j : j = 1, \ldots, N\}$ is an arbitrary set in X. The smallness of $\epsilon_{\mathcal{L}}(X, Y)$ is the main condition (see the results presented below) that guarantee the property of a set of functionals to be asymptotically determining. The so-called modes, nodes and local volume averages (the description of these functionals can be found in [3], for instance) are the main examples of sets of functionals with a small completeness defect. For further discussions and for other properties of the completeness defect we refer to [3, 4]. Here we point out the following estimate

$$\|u\|_Y \leq C_{\mathcal{L}} \cdot \max_{j=1,\ldots,N} |l_j(u)| + \epsilon_{\mathcal{L}}(X,Y) \cdot \|u\|_X, \quad u \in X, \tag{1}$$

where $C_{\mathcal{L}} > 0$ is a constant depending on $\mathcal{L}$. Below we also need the following assertion (see [3]).

Proposition 1. *Let Z be a reflexive Banach spaces such that $X \subset Z \subset Y$ and all the embeddings are continuous and dense. Assume that the inequality*

$$\|u\|_Z \leq a_\theta \|u\|_Y^\theta \|u\|_X^{1-\theta}, \quad u \in X,$$

is valid with some constants $a_\theta > 0$ and $0 < \theta < 1$. Then for any set $\mathcal{L}$ of the linear functionals on X the estimate $\epsilon_{\mathcal{L}}(X,Z) \le a_\theta \left[\epsilon_{\mathcal{L}}(X,Y)\right]^\theta$ holds.

Abstract model

We consider the following second-order abstract equation:

$$\begin{cases} Mu_{tt}(t) + \mathcal{A}u(t) + Du_t(t) = F(u(t)), \\ u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \end{cases} \tag{2}$$

under the following set of assumptions:

Assumption 1 (A1) *$\mathcal{A}$ is a closed, linear positive selfadjoint operator acting on a Hilbert space $\mathcal{H}$ with $\mathcal{D}(\mathcal{A}) \subset \mathcal{H}$. We shall denote by $|\cdot|$ and $\|\cdot\|$ the norm of $\mathcal{H}$ and $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, respectively; $(\cdot,\cdot)$ will denote a scalar product in $\mathcal{H}$. We shall use the same symbol to denote the duality pairing between $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$ and $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})'$.*

(A2) *Let V be another Hilbert space such that $\mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \subset V \subset \mathcal{H} \subset V' \subset \mathcal{D}(\mathcal{A}^{\frac{1}{2}})'$, all injections being continuous and dense, $M \in L(V,V')$, the bilinear form (Mu,v) is symmetric and $(Mu,u) \ge \alpha_0|u|_V^2$, where $\alpha_0 > 0$ and $(\cdot,\cdot)$ is understood as a duality pairing between V and V'. Hence, $M^{-1} \in L(V',V)$. Setting $\bar{M} = M|_{\mathcal{H}}$ with $\mathcal{D}(\bar{M}) = \{u \in V; Mu \in \mathcal{H}\}$ we have $D(\bar{M}^{\frac{1}{2}}) = V$.*

(A3) *The operator $D : \mathcal{D}(\mathcal{A}^{1/2}) \to [\mathcal{D}(\mathcal{A}^{1/2})]'$ is monotone and hemicontinuous with $D(0) = 0$ and $(Du - Dv, u - v) \ge 0$ for $u, v \in \mathcal{D}(\mathcal{A}^{1/2})$.*

(A4) *The nonlinear operator $F : \mathcal{D}(\mathcal{A}^{\frac{1}{2}}) \to V'$ is locally Lipschitz, i.e.:*

$$|F(u_1) - F(u_2)|_{V'} \;\le\; L(K) \,\| u_1 - u_2 \|, \quad \forall \, \| u_i \| \le K \,.$$

We also assume that F has the form $F(u) = -\Pi'(u)$, i.e. $F(u)$ is the Frechet derivative of C^1-functional $\Psi(u) = -\Pi(u)$ on $\mathcal{D}(\mathcal{A}^{\frac{1}{2}})$, where $\Pi(u)$ is bounded on bounded sets from $\mathcal{D}(\mathcal{A}^{1/2})$ and the function $\alpha(\mathcal{A}u,u) + \Pi(u)$ is bounded from below on $\mathcal{D}(\mathcal{A}^{1/2})$ for some $0 \le \alpha < 1/2$.

It can be shown (see, e.g., [13] and [6, Chap.2]) that under Assumption 1 there exists a global solution $u(t)$ to (2) from $C(\mathbf{R}_+, \mathcal{D}(\mathcal{A}^{\frac{1}{2}})) \cap C^1(\mathbf{R}_+, V)$ and the following energy relation

$$E(u(t), \dot{u}(t)) + \int_0^t (D\dot{u}(\tau), \dot{u}(\tau))d\tau = E(u_0, u_1) \tag{3}$$

holds, where $E(u_0, u_1) = \frac{1}{2}\left((Mu_1, u_1) + (\mathcal{A}u_0, u_0)\right) + \Pi(u_0)$.

Below we also need the following hypotheses, which are responsible for long time behaviour of solutions to (2).

Assumption 2 (A5) *There exists a continuous, increasing, concave function* $H : \mathbf{R}_+ \mapsto \mathbf{R}_+$, $H(0) = 0$, *such that*

$$(Mv, v) \leq H((Dv, v)), \tag{4}$$

We also assume that

$$(Dv, u) \leq C(|\mathcal{A}^{1/2}u|) \cdot (Dv, v) + c_2|\mathcal{A}^{1/2-\eta}u|^2, \tag{5}$$

for any $u, v \in \mathcal{D}(\mathcal{A}^{1/2})$, *where* $C(r)$ *is non-decreasing function of* $r > 0$, c_2 *is a positive constant and* $\eta \in (0, 1/2]$.

(A6) *There exist constants* $b_0 > 0$ *and* $\eta \in (0, 1/2]$ *such that*

$$(F(w + u) - F(w + z \cdot u), u) \leq (1 - z)b_0(\mathcal{A}^{1-2\eta}u, u)\ , \tag{6}$$

for any $z \in [0, 1]$, $u \in \mathcal{D}(\mathcal{A}^{1/2})$ *and* w *from the set* $\mathcal{N} = \{u \in \mathcal{D}(\mathcal{A}^{1/2}) : \mathcal{A}u = F(u)\}$ *of stationary solutions.*

Let $H_T(s) \equiv 3H(s/T)$, where $T > 0$. Since H_T is increasing, $cI + H_T$ is invertible for every $c > 0$. Therefore the function $p(s) \equiv (cI + H_T)^{-1}(s)$ is positive, continuous and strictly increasing with $p(0) = 0$. Finally we set $q(s) \equiv s - (I + p)^{-1}(s)$ for $s \geq 0$. It is clear that q is strictly increasing, positive and zero at the origin. With function q we associate the nonlinear differential equation:

$$\frac{d}{dt}S(t) + q(S(t)) = f(t),\ t > 0; \quad S(0) = S_0 \in \mathbf{R}. \tag{7}$$

Since q is monotone increasing, for any $f \in L_1(\mathbf{R}_+)$ there exists unique global solution $S \in C(\mathbf{R}_+)$. Moreover, if $f = 0$ then $S(t) \to 0$ when $t \to \infty$. We refer to [10, 13] for details.

Determining functionals

Our main result is the following assertion.

Theorem 3 *Let* $u(t)$ *be a solution to problem* (2) *and* $w \in \mathcal{D}(\mathcal{A}^{1/2})$ *be a solution to the stationary problem* $\mathcal{A}u = F(u)$. *Assume that* $\mathcal{L} = \{l_j : j = 1, ..., N\}$ *is a set of functionals on* $\mathcal{D}(\mathcal{A}^{1/2})$ *and* $\epsilon_{\mathcal{L}} \equiv \epsilon_{\mathcal{L}}(\mathcal{D}(\mathcal{A}^{1/2}), \mathcal{H})$ *is the corresponding completeness defect. Assume that Assumption* 1 *and Assumption* 2 *hold.*

Part I. We assume in addition that (A.5) *in Assumption* 2 *is satisfied with* $H(s) = c_1 s$, $c_1 > 0$. *Then the condition*

$$\lim_{t\to+\infty} l_j(u(t)) = l_j(w) \quad \textit{for all} \quad j = 1, \ldots, N,$$

implies that

$$\lim_{t\to\infty} \left(|M^{1/2}u_t(t)|^2 + \|u(t) - w\|^2\right) = 0 \tag{8}$$

provided $\epsilon_{\mathcal{L}}^{4\eta}(b_0 + 2c_2) < 1$, *where* η, b_0, *and* c_2 *are the constants from* (5) *and* (6).

Part II. Assume that there exists a positive function $h(t)$ *on* $\mathbf{R}_+$ *with the properties:* (a) $h(t)$ *is decreasing and* $\lim_{t\to\infty} h(t)=0$; (b) *for any* $T>0$ *there exists* $c_T>0$ *such that* $h(t)\le c_T h(t+T)$ *and* (c) $\max_j |l_j(u(t)-w)|^2 \le h(t)$ *for* $t > 0$. *Then there exists* $T \ge 2$ *such that*

$$|M^{1/2}u_t(t)|^2 + \|u(t) - w\|^2 \le C \cdot \left(S\left(\frac{t}{T} - 1\right) + h(t)\right)$$

for all $t > T$, *where* $S(t)$ *satisfies the nonlinear ODE* (7) (*with parameter* c *defining* q *depending on the values of constants assumed in Assumption* 1 *and Assumption* 2) *and with* $f(t) \equiv p(Ch(Tt))$ *and* $S(0)$ *depending on* u_0, u_1 *and* w. *Here* C *is a constant.*

Remark 4 The first part of Theorem 3 refers to the situation of strong dissipation when nonlinear damping Du_t leads to exponential decay rates for the unforced problems. In this case, the mere convergence to zero of $l_j(u(t) - w)$ when $t \to \infty$ guarantees the asymptotic convergence of $u(t)$ to an equilibrium.

In the case of weaker dissipation $D(u_t)$ (e.g., $H(s)$ is sublinear at the origin) and additional information available on the decay rates of functionals $l_j(u(t))$ to $l_j(w)$, the second part of Theorem 3 provides decay rates for the convergence of solution $u(t)$ to the equilibrium. These rates are described by solutions of a nonlinear ODE (7).

Proof. We rely here on some ideas developed in [5] for wave equation with nonlinear dissipation. Let $v(t) = u(t) - w$. Then for $v(t)$ we have the following equation

$$Mv_{tt} + D(v_t) + \mathcal{A}v = F(w + v(t)) - F(w), \ t > 0. \tag{9}$$

Multiplying equation (9) in $\mathcal{H}$ by v_t we obtain:

$$\frac{1}{2} \cdot \frac{d}{dt}\left(|M^{1/2}v_t(t)|^2 + |\mathcal{A}^{1/2}v(t)|^2\right) + (D(v_t), v_t) = (F(u) - F(w), v_t). \tag{10}$$

It is not difficult to see that

$$(F(u) - F(w), v_t) = (F(u), u_t) - (F(w), v_t) = -\frac{d}{dt}\Phi(v(t)),$$

where

$$\Phi(v) = \Pi(u) - \Pi(w) + (F(w), v) \equiv -\int_0^1 (F(w + zv) - F(w), v)dz.$$

Consequently from (10) we obtain the equality:

$$\frac{d}{dt}\widetilde{E}(t) + (D(v_t), v_t) = 0, \tag{11}$$

where

$$\widetilde{E}(t) = \frac{1}{2}\left(|M^{1/2}v_t(t)|^2 + |\mathcal{A}^{1/2}v(t)|^2\right) + \Phi(v(t)).$$

It follows from assumption (A6) that

$$\Phi(v) \geq -\frac{b_0}{2}|\mathcal{A}^{1/2-\eta}v|^2. \tag{12}$$

Since $|\mathcal{A}^{1/2-\eta}v| \leq |\mathcal{A}^{1/2}v|^{1-2\eta} \cdot |v|^{2\eta}$, $0 \leq \eta \leq 1/2$, using (1) with $X = \mathcal{D}(\mathcal{A}^{1/2})$ and $Y = \mathcal{H}$ and Proposition 1 we obtain that

$$|\mathcal{A}^{1/2-\eta}v|^2 \leq (1+\delta)\epsilon_{\mathcal{L}}^{4\eta}|\mathcal{A}^{1/2}v|^2 + C_{\mathcal{L},\delta} \max_{j=1,\dots,N} |l_j(u)|^2, \tag{13}$$

for each $\delta > 0$. From (12) and (13) we get the following estimate.

Lemma 5

$$\widetilde{E}(t) \geq \frac{1}{2}|M^{1/2}v_t|^2 + \left(\frac{1}{2} - \frac{b_0}{2}\epsilon_{\mathcal{L}}^{4\eta}(1+\delta)\right)|\mathcal{A}^{1/2}v|^2 - C_{\mathcal{L},\delta}\max_j |l_j(v)|^2. \tag{14}$$

Moreover, $\tilde{E}(t) \geq 0$ for all $t \geq 0$ provided $\epsilon_{\mathcal{L}}^{4\eta} b_0 < 1$.

Proof. The inequality in (14) is a consequence of inequalities (12) and (13). In order to prove positivity of $\tilde{E}$ we note that it follows from (11) and assumption (A3) that the function $\widetilde{E}(t)$ is monotony decreasing. Inequality (14) implies that if $\epsilon_{\mathcal{L}}^{4\eta} < b_0^{-1}$ and $\lim_{t\to+\infty} l_j(v(t)) = 0$ then $\lim_{t\to+\infty} \widetilde{E}(t) \geq 0$. Thus $\widetilde{E}(t) \geq 0$, $t > 0$, as desired. ∎

The following estimate is critical for the asymptotic behaviour.

Lemma 6 *Let $T > T_0$, where $T_0 > 0$ is sufficiently large. Then*

$$2p(\widetilde{E}(mT)/2) + \widetilde{E}(mT) \leq \widetilde{E}((m-1)T) + p(\mathcal{N}_{\mathcal{L}}(m,T))$$

for $m = 1, 2\ldots$, *where*

$$\begin{aligned}\mathcal{N}_{\mathcal{L}}(m,T) &= C_{\mathcal{L}}\left[\max_j |l_j(v(mT))|^2 + \max_j |l_j(v((m-1)T))|^2\right.\\ &\quad \left. + \int_{(m-1)T}^{mT} \max_j |l_j(v(s))|^2 ds\right] .\end{aligned}$$

In particular, when $H(s) = c_1 s$, *then we have*

$$\widetilde{E}(mT) \leq \gamma\widetilde{E}((m-1)T) + \mathcal{N}_{\mathcal{L}}(m,T), \quad m = 1, 2\ldots, \quad 0 < \gamma < 1.$$

Proof. Multiplying equation (9) by v we find

$$\frac{d}{dt}(Mv_t, v) = |M^{1/2}v_t|^2 - |A^{1/2}v|^2 + (F(v+w) - F(w), v) - (D(v_t), v).$$

Since $-\frac{1}{2}|A^{1/2}v|^2 = -\widetilde{E}(t) + \frac{1}{2}|M^{1/2}v_t|^2 + \Phi(v)$, we have

$$\begin{aligned}\frac{d}{dt}(Mv_t, v) &= \frac{3}{2}|M^{1/2}v_t|^2 - \frac{1}{2}|A^{1/2}v|^2\\ &\quad + \{\Phi(v) + (F(w+v) - F(w), v)\} - (D(v_t), v) - \widetilde{E}(t).\end{aligned}$$

By (A6) we have

$$\begin{aligned}\Phi(v) &+ (F(w+v) - F(w), v)\\ &= \int_0^1 (F(w+v) - F(w+zv)), v)dz \leq \frac{b_0}{2}|A^{1/2-\eta}v|^2.\end{aligned}$$

Therefore using (13) we obtain the following inequality

$$\begin{aligned}\frac{d}{dt}(Mv_t, v) &\leq -\widetilde{E}(t) - \frac{1}{2}\left(1 - \epsilon_{\mathcal{L}}^{4\eta}(1+\delta)b_0\right)|A^{1/2}v|^2\\ &\quad + C_{\mathcal{L},\delta}\max_j |l_j(v)|^2 + \frac{3}{2}|M^{1/2}v_t|^2 + (D(v_t), v).\end{aligned}$$

Integrating the last inequality from 0 to T with respect to t we obtain:

$$\begin{aligned}\int_0^T \widetilde{E}(s)ds &\leq -(Mv_t(T), v(T)) + (Mv_t(0), v(0)) + \frac{3}{2}\int_0^T |M^{\frac{1}{2}}v_s(s)|^2 ds\\ &\quad -\frac{1}{2}\left(1 - \epsilon_{\mathcal{L}}^{4\eta}(1+\delta)b_0\right)\int_0^T |A^{1/2}v(s)|^2 ds\\ &\quad + \int_0^T (D(v_t(s)), v(s))ds + C_{\mathcal{L},\delta}\int_0^T \max_j |l_j(v(s))|^2 ds. \quad (15)\end{aligned}$$

From Lemma 5 we obtain

$$|A^{1/2}v(t)|^2 + |M^{1/2}v(t)|^2 \leq C\widetilde{E}(t) + C_{\mathcal{L}}\max_j |l_j(v(t))|^2 \quad (16)$$

under the condition $\epsilon_{\mathcal{L}}^{4\eta} b_0 < 1$. Thus, by direct computations

$$\begin{aligned} |(Mv_t(T), v(T))| &+ |(Mv_t(0), v(0))| \le C[\widetilde{E}(T) + \widetilde{E}(0)] \\ &+ C_{\mathcal{L}} \left(\max_j |l_j(v(T))|^2 + \max_j |l_j(v(0))|^2 \right). \end{aligned}$$

By (5)

$$\int_0^T (D(v_t), v) ds \le \int_0^T C(|\mathcal{A}^{1/2} v|) \cdot (D(v_t), v_t) ds + c_2 \int_0^T |\mathcal{A}^{1/2-\eta} v|^2 ds.$$

As above, it is easy to see that $C(|\mathcal{A}^{1/2} v(s)|) \le C_0 \equiv C_0(E(u_0, u_1), w)$. Therefore, using (13) we obtain

$$\begin{aligned} \int_0^T (D(v_t), v) ds \le C_0 \int_0^T (D(v_t), v_t) ds + c_2 \epsilon_{\mathcal{L}}^{4\eta} (1+\delta) \int_0^T |\mathcal{A}^{1/2} v(s)|^2 ds \\ + C_{\mathcal{L},\delta} \int_0^T \max_j |l_j(v(s))|^2 ds. \end{aligned} \tag{17}$$

From (4) and Jensen's inequality we also have

$$\int_0^T (Mv_t, v_t) ds \le \int_0^T H((D(v_t), v_t)) ds \le \frac{T}{3} H_T \left(\int_0^T (D(v_t), v_t) ds \right). \tag{18}$$

Therefore applying inequalities (17) and (18) to main inequality in (15) we obtain

$$\int_0^T \widetilde{E}(s) ds \le C[\widetilde{E}(0) + \widetilde{E}(T)] + (C_0 I + \frac{T}{2} H_T) \left(\int_0^T (D(v_t), v_t) ds \right) + \mathcal{N}_{\mathcal{L}}(1, T)$$

provided $\epsilon_{\mathcal{L}}^{4\eta}(b_0 + 2c_2) < 1$. Since $t\widetilde{E}(t) \le \int_0^t \widetilde{E}(s) ds$ (by (11) $\widetilde{E}(t)$ is a decreasing function), we obtain the following inequality

$$T\widetilde{E}(T) \le C_1 \widetilde{E}(T) + (C_0 I + \frac{T}{2} H_T) \left(\int_0^T (D(v_t), v_t) ds \right) + \mathcal{N}_{\mathcal{L}}(1, T).$$

Using the relation

$$\int_0^T (D(v_t(s)), v_t(s)) ds = \widetilde{E}(0) - \widetilde{E}(T),$$

which follows from (11), and taking $T > \max[2C_1, 2]$ we obtain

$$\widetilde{E}(T) \le (C_0 I + H_T) \left(\widetilde{E}(0) - \widetilde{E}(T) \right) + \mathcal{N}_{\mathcal{L}}(1, T).$$

Using the concavity of H_T and recalling the definition of $p(s)$ applied with $c = C_0$ we obtain

$$2p(\widetilde{E}(T)/2) + \widetilde{E}(T) \leq \widetilde{E}(0) + p\left(\mathcal{N}_{\mathcal{L}}(1,T)\right) \ ,$$

which gives the inequality in Lemma 6 for the case $m = 1$. Reiterating this argument on each subinterval $((m-1)T, mT)$ gives the desired conclusion in Lemma 6.

In the special case when $H(s) = c_1 s$, $H_T(s) = \widetilde{c}s/T$, defining $\gamma < 1$ in an appropriate way we find that $\widetilde{E}(T) \leq \gamma\widetilde{E}(0) + \mathcal{N}_{\mathcal{L}}(1,T)$, which implies the statement in the second part of Lemma 6 with $m = 1$. Repeating the same argument on the interval $[(m-1)T, mT]$ yields the desired conclusion in the Lemma. ∎

To complete the proof of Theorem 3 it suffices to apply Lemma 6 on successive time intervals. This gives, for the case of linear $H(s)$

$$\widetilde{E}(nT) \leq \gamma^n \widetilde{E}(0) + C_{\mathcal{L}} \sum_{i=1}^{n} \gamma^i L_{n-i}, \tag{19}$$

where $L_i = \max_{t\in[(m-1)T,mT]} \max_j |l_j v(t)|^2$ and $0 < \gamma < 1$. The conclusion of Theorem 3 for linear $H(s)$ easily follows from (19). In general case we use a comparison theorem similar to Lemma 3.3 [10]. ∎

Remark 7 Assume that there exists a positive function $h(t)$ on $\mathbf{R}_+$ with the properties described in Part II of Theorem 3. In the case of linear $H(s)$ we can derive from (19) the relation

$$\widetilde{E}(t) \leq C\widetilde{E}(0)e^{-\widetilde{\gamma}t} + C_{\mathcal{L}} \int_0^t e^{-\widetilde{\gamma}(t-\tau)} h(\tau) d\tau.$$

for some $\widetilde{\gamma} > 0$. This formula describes the decay rates of the solution $u(t)$ to the equilibrium w. The corresponding decay rates are exponential when $h(t) = ce^{-\beta t}$ or polynomial if $h(t) = c(1+t)^{-\beta}$, $\beta > 0$. We note that in general $h(t)$ may depend crucially on the solutions $u(t)$ and w.

Theorem 3 implies immediately the following assertion.

Corollary 8 *Let $w_1, w_2 \in \mathcal{D}(\mathcal{A}^{1/2})$ be two stationary solutions to* (2). *Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a set of functionals on $\mathcal{D}(\mathcal{A}^{1/2})$ and $\epsilon_{\mathcal{L}}^{4\eta}(b_0 + 2c_2) < 1$, where η, b_0, and c_2 are the same as in Theorem* 3. *Then the condition $l_j(w_1) = l_j(w_2)$ for $j = 1, \ldots, N$ implies that $w_1 = w_2$.*

The two corollaries formulated below deal with the case when the problem possesses precompact trajectories (this property holds in the application considered below for the case $\alpha > 0$).

Let $u(t)$ be a solution to the problem (2). We recall that the set

$$\gamma_+(u_0, u_1) = \cup\{(u(t); u_t(t)) \ : \ t \geq 0\}$$

in the space $H = \mathcal{D}(\mathcal{A}^{1/2}) \times V$ is said to be the *semi-trajectory* emanating from $(u_0; u_1)$ of the dynamical system generated by (2) in H. We also define the ω-limit set of the semi-trajectory $\gamma_+(u_0, u_1)$ by the formula

$$\omega(\gamma_+) \equiv \omega(u_0, u_1) = \cap_{\tau>0}\left[\cup\{(u(t); u_t(t)) \ : \ t \geq \tau\}\right]_H,$$

where $[A]_H$ is the closure of the set A in H.

Corollary 9 *Let $\mathcal{L} = \{l_j : j = 1, ..., N\}$ be a set of functionals on $\mathcal{D}(\mathcal{A}^{1/2})$ such that $\epsilon_{\mathcal{L}}^{4\eta}(b_0+2c_2) < 1$, where η, b_0, and c_2 are the constants from* (5) *and* (6). *Assume that $u(t)$ is a solution to problem* (2) *with precompact semi-trajectory $\gamma_+ = \gamma_+(u_0, u_1)$ and there exists the finite limits $\lim_{t\to+\infty} l_j(u(t)) \equiv l_j$ for every $j = 1, \dots, N$. Then there exists a stationary solution $w \in \mathcal{D}(\mathcal{A}^{1/2})$ such that* (8) *holds.*

Proof. Precompactness of γ_+ implies that ω-limit set $\omega(\gamma_+)$ is non-empty compact set in H. The relation (3) implies that the functional $V(y) = E(u_0, u_1)$, $y = (u_0; u_1)$, is a strict Lyapunov function on H for system (2) (see books [1], [4] or [8] for the definition) and therefore the ω-limit set $\omega(\gamma_+)$ lies in the set $\mathcal{N} \equiv \{(w; 0) \in H \ : Aw = F(w)\ \}$ of equilibrium points to problem (2). From the convergence $l_j(u(t)) \to l_j$ we have that $l_j(w) = l_j$ for all $(w; 0) \in \omega(\gamma_+) \subset \mathcal{N}$. Consequently Corollary 8 implies that $\omega(\gamma_+)$ consists of a single point $(w; 0)$ and therefore (8) holds. ∎

Corollary 10 *Let the assumptions of Corollary* 9 *be valid. Assume that $u^{(1)}(t)$ and $u^{(2)}(t)$ are solutions to equation* (2) *with precompact semi-trajectories $\gamma_+^{(1)}$ and $\gamma_+^{(2)}$ and*

$$\lim_{t\to+\infty}\left(l_j(u^{(1)}(t)) - l_j(u^{(2)}(t))\right) = 0, \quad j = 1, \dots, N. \tag{20}$$

Then $\omega(\gamma_+^{(1)}) \equiv \omega(\gamma_+^{(2)})$. If the set $\mathcal{N}$ of equilibrium points is finite, then there exists a stationary solution $w \in \mathcal{D}(\mathcal{A}^{1/2})$ such that (8) *holds for both solutions $u_1(t)$ and $u_2(t)$.*

Proof. Let $(z_1, 0) \in \omega(\gamma_+^{(1)}) \subset \mathcal{N}$. Then there exists a sequence $\{t_m\}$ such that $t_m \to +\infty$ and $u^{(1)}(t_m) \to z_1$ in the space $\mathcal{D}(\mathcal{A}^{1/2})$ when $m \to \infty$. Since $\gamma_+^{(2)}$ is precompact set, we can choose a subsequence $\{t_{m_k}\} \subset \{t_m\}$ such that $u^{(2)}(t_{m_k}) \to z_2$, where $(z_2, 0) \in \omega(\gamma_+^{(2)}) \subset \mathcal{N}$. Property (20) gives that $l_j(z_1) = l_j(z_2)$ for all $j = 1, 2, \ldots, N$. Consequently Corollary 8 implies that $z_1 = z_2$ and therefore we have $(z_1, 0) \in \omega(\gamma_+^{(2)})$. This implies that $\omega(\gamma_+^{(1)}) = \omega(\gamma_+^{(2)})$. If $\mathcal{N}$ is finite, then it is easy to see that $\omega(\gamma_+^{(1)}) = \omega(\gamma_+^{(2)})$ consists of a single equilibrium point. This implies the assertion of the corollary. ∎

Applications

In this section we deal with the following problem

$$\begin{cases} (1 - \alpha \cdot \Delta)\partial_t^2 u + d_0(x) \cdot g_0(u_t) - \alpha \mathrm{div}(d(x) g(\nabla u_t)) \\ +\Delta^2 u - [u, v + F_0] = p(x), \qquad x \in \Omega,\ t > 0, \\ \quad u|_{\partial\Omega} = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \quad u|_{t=0} = u_0(x), \quad \partial_t u|_{t=0} = u_1(x), \end{cases} \tag{21}$$

where $[u, v] = \partial_{x_1}^2 u \cdot \partial_{x_2}^2 v + \partial_{x_2}^2 u \cdot \partial_{x_1}^2 v - 2 \cdot \partial_{x_1 x_2}^2 u \cdot \partial_{x_1 x_2}^2 v$ and $v = v(u)$ is a solution to the problem

$$\Delta^2 v + [u, u] = 0, \quad v|_{\partial\Omega} = \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0. \tag{22}$$

Here Ω is a smooth bounded domain in $\mathbf{R}^2$, $F_0(x) \in H^4(\Omega)$ and $p(x) \in L_2(\Omega)$ are given functions determined by the mechanical loads. The parameter $\alpha \geq 0$ takes into account the rotational inertial momenta of the elements of the plate. The terms $g_0(u_t)$ and $g(u_t)$ represent mechanical (potentially nonlinear) damping in the system with damping parameters d_0, d nonnegative and bounded in Ω.

We introduce the following spaces and operators:

- $\mathcal{H} \equiv L_2(\Omega)$, $V \equiv H_0^1(\Omega)$ in the case $\alpha > 0$ and $V = \mathcal{H}$ for $\alpha = 0$.
- $\mathcal{A}u \equiv \Delta^2 u, u \in \mathcal{D}(\mathcal{A})$ with $\mathcal{D}(\mathcal{A}) \equiv H_0^2(\Omega) \cap H^4(\Omega)$.
- $Mu \equiv Iu - \alpha\Delta u, u \in \mathcal{D}(M) \equiv H_0^1(\Omega) \cap H^2(\Omega)$.
- Hence $V' = H^{-1}(\Omega)$ $(\alpha > 0)$ and $V' = L_2(\Omega)$ $(\alpha = 0)$, $\mathcal{D}(\mathcal{A}^{1/2}) = H_0^2(\Omega)$, $[\mathcal{D}(\mathcal{A}^{1/2})]' = H^{-2}(\Omega)$.
- $F(u) \equiv [u, v(u) + F_0] + p$, where $v(u)$ satisfies (22).
- $D(u) \equiv d_0(x) g_0(u) - \alpha div\ (d(x) g(\nabla u))$.

The nonlinear term $F(u)$ has the form $F(u) = -\Pi'(u)$ with

$$\Pi(u) = \frac{1}{4}\|\Delta v(u)\|^2 - \frac{1}{2}([u, u], F_0) - ([p, u),$$

where $v(u) \in H_0^2(\Omega)$ is defined by (22).

We assume that the mappings $g_0 : \mathbf{R} \mapsto \mathbf{R}$ and $g : \mathbf{R}^2 \mapsto \mathbf{R}^2$ are locally Lipschitz and possess the properties $g(s_1, s_2) = (g_1(s_1); g_2(s_2))$ for $(s_1; s_2) \in \mathbf{R}^2$, and $g_i(0) = 0$, $i = 0, 1, 2$, and $g_i(s)$ is non-decreasing for each $i = 0, 1, 2$.

Under these conditions one can show (see, e.g., [6], [12] or [13]) that Assumption 1 holds for both $\alpha > 0$ and $\alpha = 0$ cases. Thus problem (21) and (22) has a unique solution $u(t)$ in $C(\mathbf{R}_+; H_0^2(\Omega)) \cap C(\mathbf{R}_+; V_\alpha(\Omega))$, where $V_\alpha(\Omega) = H_0^1(\Omega)$ for $\alpha > 0$ and $V_\alpha(\Omega) = L_2(\Omega)$ for $\alpha = 0$.

To check assumption (A5) we need some additional hypotheses concerning $d_0(x)g_0(v)$ and $d(x)g(v)$. We assume that (a) there exist positive constants d_0 and d_1 such that $d_0 \leq d_0(x) \leq d_1$ and $d_0 \leq d(x) \leq d_1$ and (b) there exist positive constants a, b and $q \geq 1$ such that

$$sg_i(s) \geq as^2, \; i = 0, 1, 2, |s| \geq 1; \quad |g_i(s)| \leq b\left(1 + |s|^q\right), \; i = 1, 2. \tag{23}$$

As for assumption (A6), we can show that

$$\begin{aligned}(F(w+u) \quad &- \quad F(w + z \cdot u), u) \leq -\frac{1-z}{2}\|\Delta v(u)\|^2 \\ &+ \quad c_0(1-z)\|u\|^2_{H^1(\Omega)}\left(\|\Delta w\|^2 + \|F_0\|^2_{H^4(\Omega)}\right)\end{aligned}$$

for any $z \in [0, 1]$ and $u, w \in H_0^2(\Omega)$, where c_0 is a constant depending on embedding theorems. Since the set of stationary solutions to (21) is bounded (see, e.g., [6]), the assumption (A6) holds in this case.

Remark 11 In the case considered above, when we assume that the dissipation g_i, $i = 0, 1, 2$, is differentiable near the origin, we can make the corresponding constant c_2 arbitrary small. We also have $\eta = 3/8$ and $b_0 = c_0\left(\|\Delta w\|^2 + \|F_0\|^2_{H^4(\Omega)}\right)$ for the system under consideration.

We also note that if (23) holds for all $s \in \mathbf{R}$, then $H(s) = c_1 s$ with a suitable c_1. In general, construction of appropriate function $H(s)$ relies on monotonicity of $g(s)$ and asymptotic growth condition in (23) and can be done in the same way as in [10] for wave equation.

Under all these assumptions Theorem 3 is applicable here and we have the following assertion.

Theorem 12 *Let $u(t)$ be a solution to problem* (21) *and* (22) *and $w \in H_0^2(\Omega)$ be a solution to the problem*

$$\Delta^2 u - [u, v + F_0] = p(x), \quad x \in \Omega, \quad u|_{\partial\Omega} = \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0,$$

where v solves (22). *Assume that $\mathcal{L} = \{l_j : j = 1, ..., N\}$ is a set of functionals on $H_0^2(\Omega)$ with the completeness defect $\epsilon_{\mathcal{L}} \equiv \epsilon_{\mathcal{L}}(H_0^2(\Omega), L_2(\Omega))$.*

Then there exists $\epsilon_0 > 0$ *such that the conditions* $\epsilon_{\mathcal{L}} < \epsilon_0$ *and*

$$\lim_{t\to+\infty} l_j(u(t)) = l_j(w) \quad \textit{for all} \quad j = 1, \ldots, N$$

imply that

$$\lim_{t\to+\infty} \left(\|u_t(t)\|^2_{L^2(\Omega)} + \alpha\|\nabla u_t(t)\|^2_{L^2(\Omega)} + \|\Delta(u(t) - w)\|^2_{L^2(\Omega)} \right) = 0.$$

Remark 13 Theorem 12 implies the property of stationary solutions to von Karman equations which is similar to Corollary 8. However the precompactness of trajectories of the dynamical system generated by (21) and (22) we can guarantee in the case $\alpha > 0$ only. Therefore this is only the case when Corollaries 9 and 10 can be applied. For details we refer to [6]. We also note that for the case $\alpha = 0$, $g_0(s) = g_0 \cdot s$ determining functionals for problem (21) and (22) were studied in [2].

The situation when the completeness defect $\epsilon_{\mathcal{L}}$ can be easily estimated from above is the following. Assume that $\{\phi_j : j = 1, ..., N\}$ be a certain set of linearly independent functions from $H^2_0(\Omega)$. Let us define an interpolation operator $R_{\mathcal{L}}$ by the formula $R_{\mathcal{L}}w = \sum_{j=1}^N l_j(w)\phi_j$. If $R_{\mathcal{L}}w$ is a "good" approximation for w, we have

$$\|w - R_{\mathcal{L}}w\|_{L_2(\Omega)} \le Ch^{\alpha}\|w\|_{H^2(\Omega)},$$

where C and α are positive constants and $h > 0$ is small enough. In this case we obviously have $\epsilon_{\mathcal{L}} \le Ch^{\alpha}$. This observation allows us to give the following examples (for details and further discussion we refer to [3, 4]).

Local volume averages. Assume that $\lambda(x) \in L^{\infty}(\mathbf{R}^2)$ has compact support and $\int_{\mathbf{R}^2} \lambda(x)\, dx = 1$. For $h > 0$ let us define functionals

$$\mathcal{L} = \left\{ l_j \ : \ l_j(w) = \frac{1}{h^2} \cdot \int_{\Omega} w(x)\lambda\left(\frac{x}{h} - j\right) dx, \quad j \equiv (j_1, j_2) \in \mathcal{J} \right\},$$

where $\mathcal{J} \equiv \{(j_1, j_2) \in \mathbf{Z}^2 : (j_1 h, j_2 h) \in \Omega\}$. In this case we have the estimate $\epsilon_{\mathcal{L}}(H^2_0(\Omega), L_2(\Omega)) \le ch^2$.

Nodes. Let $\mathcal{T}^h$ be a triangulation of the domain Ω, made of triangles with sides less than h and let $\{x_j : j = 1, ..., N_h\}$ be the set of all vertices of the triangles from $\mathcal{T}^h$. Then the completeness defect for the set

$$\mathcal{L} = \{l_j : l_j(w) = w(x_j),\ j = 1, ..., N_h\}$$

admits the estimate $\epsilon_{\mathcal{L}}(H^2_0(\Omega), L_2(\Omega)) \le ch^2$.

Modes. Let $\{e_k\}$ be the basis in $L^2(\Omega)$ consisting of the eigenvectors of the biharmonic operator Δ^2 with the Dirichlet boundary conditions. We suppose

$$\mathcal{L} = \left\{ l_j : l_j(w) = \int_\Omega w(x) \cdot e_j(x)\, dx, \quad j = 1, ..., N \right\}.$$

Then the estimate $\epsilon_{\mathcal{L}}(H_0^2(\Omega), L_2(\Omega)) \le cN^{-1}$ holds.

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