ASYMPTOTICALLY OPTIMAL CONTROLS OF HYBRID LQG PROBLEMS: SUMMARY OF RESULTS

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Abstract: This paper is concerned with hybrid control of a class of linear quadratic Gaussian systems modulated by a finite-state Markov chain. It develops approximation schemes for systems involving singularly perturbed Markov chains with weak and strong interactions. Computation results indicate that our approximation schemes perform quite well.

1 INTRODUCTION

This work develops asymptotically optimal controls of a class of hybrid LQG (linear quadratic Gaussian) systems, in which the systems are modulated by a Markov chain (or Markov jump process). The main feature of the hybrid systems is imbedded in the Markovian jump process. Many such systems arise in various applications in speech recognition, telecommunications, and manufacturing, for treating various system noise and uncertainties. Roughly, a hybrid system shows both "continuous" and "discrete" characteristics. In our formula-

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tion, random environment continuously (in time) disturbs the system (through the system coefficient matrices), whereas the resulting effect takes place at a set of discrete values (discrete events). It is thus natural to introduce a Markov chain model. Recent research effort on LQG problems has been devoted to the study of hybrid linear quadratic Gaussian (LQG) systems for over a decade; see [1], [2] and the references cited therein for a literature review.

To account for uncertainty, unlike the traditional LQG problem in which the system matrices A and B are fixed, we allow these matrices to depend on a Markov jump process with finite-state space in this work. Thus corresponding to different states of the Markov chain, the system itself displays different configuration. The situation is more involved as compared to the traditional setting. It is typical in many applications that the state space of the Markov chain is very large. In such a case, it is difficult to obtain solutions to the associated Riccati equations. To overcome the difficulty, we use singular perturbation techniques in the modeling, control design, and optimization. The resulting systems naturally display certain two-time-scale behavior, a fast time scale and a slowly varying one. To put this in a manageable framework, we introduce a small parameter $\varepsilon > 0$, and model the underlying system as one involving singularly perturbed Markov chains.

This paper concentrates on asymptotic and near optimality of hybrid systems involving singularly perturbed Markov chains. To treat the systems, we use an averaging approach to analyze the system in which the underlying Markov chain involves weak and strong interactions. The idea is to aggregate the states according to their jump rates and replace the actual system with its average. Using the optimal control of limit (average) problem as a guide, we then construct controls for the actual systems leading to feasible approximation schemes. We show that these approximation schemes give us nearly optimal controls. By focusing on approximate optimality, we succeed to reduce the complexity of the underlying systems drastically. The reduction of dimensionality is the major advantage of the averaging approach. To demonstrate how the average schemes work, we provide a numerical example of a one-dimensional system. This paper only deals with Markov chains with recurrent states. Similar results can be obtained for models with transient states and absorbing states. Proofs of results and technical complements are omitted and referred to [1].

2 PROBLEM FORMULATION

Let $\varepsilon > 0$ be a small parameter. Consider the Markov $\alpha^\varepsilon(t)$ generated by Q^ε, which consists of two parts, a rapidly changing part and a slowly varying one, i.e.,

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q}.$$

Consider the linear system

$$\begin{aligned} &dx(t) = [A(\alpha^\varepsilon(t))x(t) + B(\alpha^\varepsilon(t))u(t)]dt + \sigma dw(t), \\ &x(s) = x, \text{ for } s \le t \le T, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_1}$ is the state, $u(t) \in \mathbb{R}^{n_2}$ is the control, $A(i) \in \mathbb{R}^{n_1 \times n_1}$ and $B(i) \in \mathbb{R}^{n_1 \times n_2}$ are well defined and have finite values for $i \in \mathcal{M}$, and $w(\cdot)$ is a standard Brownian motion. The objective is to find the optimal control $u(\cdot)$ so that the expected quadratic cost function

$$J^\varepsilon(s,x,\alpha,u(\cdot)) = E\left\{\int_s^T [x'(t)M(\alpha^\varepsilon(t))x(t)+u'(t)N(\alpha^\varepsilon(t))u(t)]dt+x'(T)Dx(T)\right\}$$

is minimized, where $M(i)$, $N(i)$ are well defined and have finite values for $i \in \mathcal{M}$, $M(i)$ are symmetric nonnegative definite matrices for $i \in \mathcal{M}$, D and $N(i)$ are symmetric and positive definite for $i \in \mathcal{M}$, E is the expectation given $\alpha(s) = \alpha$ and $x(s) = x$, and the processes $\alpha^\varepsilon(\cdot)$ and $w(\cdot)$ are independent.

3 HJB AND RICCATI EQUATIONS

We consider the optimal LQG control problem by using the dynamic programming approach, and derive the associated Hamilton-Jacobi-Bellman (HJB) and Riccati equations. Let $v^\varepsilon(s,i,x)$ be the value function, i.e., $v^\varepsilon(s,i,x) = \inf_{u(\cdot)} J^\varepsilon(s,i,x,u(\cdot))$. Then v^ε satisfies the following HJB equation:

$$\begin{cases} 0 = \dfrac{\partial v^\varepsilon(s,i,x)}{\partial s} + \min\limits_u \Big\{ (A(i)x + B(i)u)' \dfrac{\partial v^\varepsilon(s,i,x)}{\partial x} + x'M(i)x + u'N(i)u \\ \qquad\qquad + \dfrac{1}{2}\mathrm{tr}\left(\sigma\sigma' \dfrac{\partial^2 v^\varepsilon(s,i,x)}{\partial x^2}\right) + Q^\varepsilon v^\varepsilon(s,\cdot,x)(i) \Big\}, \quad 0 \le s \le T \\ v^\varepsilon(T,i,x) = x'Dx, \ i \in \mathcal{M}, \end{cases}$$

where

$$Q^\varepsilon v^\varepsilon(s,\cdot,x)(i) = \sum_{j \neq i} q_{ij}^\varepsilon (v^\varepsilon(s,j,x) - v^\varepsilon(s,i,x)).$$

Let

$$v^\varepsilon(s,i,x) = x'K^\varepsilon(s,i)x + q^\varepsilon(s,i), \tag{2}$$

for some $m \times m$ matrix K^ε and a scalar function q^ε. Substituting (2) into the Riccati equation and comparing the coefficients of x leads to the following Riccati equation for $K^\varepsilon(s,i)$,

$$\begin{cases} \dot{K}^\varepsilon(s,i) = -K^\varepsilon(s,i)A(i) - A'(i)K^\varepsilon(s,i) - M(i) \\ \qquad\qquad + K^\varepsilon(s,i)B(i)N^{-1}(i)B'(i)K^\varepsilon(s,i) - Q^\varepsilon K^\varepsilon(s,\cdot)(i) \\ K^\varepsilon(T,i) = D, \end{cases} \tag{3}$$

where $Q^\varepsilon K^\varepsilon(s,\cdot)(i) = \sum_{j \neq i} q_{ij}^\varepsilon (K^\varepsilon(s,j) - K^\varepsilon(s,i))$, and the equation for q^ε,

$$\begin{cases} \dot{q}^\varepsilon(s,i) = -\mathrm{tr}(\sigma\sigma' K^\varepsilon(s,i)) - Q^\varepsilon q^\varepsilon(s,\cdot)(i), \\ q^\varepsilon(T,i) = 0. \end{cases} \tag{4}$$

The optimal control $u^{\varepsilon,*}(s,i,x)$ has the form:

$$u^{\varepsilon,*}(s,i,x) = -N^{-1}(i)B'(i)K^\varepsilon(s,i)x. \tag{5}$$

To find the optimal control, one has to solve the Riccati equations. However, in many problems in telecommunication and manufacturing, such solutions are very difficult to obtain due to the large dimensionality. In this case, one has to resort to approximation schemes.

4 APPROXIMATION SCHEMES

Consider the generator of the Markov chain given by

$$Q^\varepsilon = \frac{1}{\varepsilon}\widetilde{Q} + \widehat{Q} = \frac{1}{\varepsilon}\begin{pmatrix} \widetilde{Q}^1 & & & \\ & \widetilde{Q}^2 & & \\ & & \ddots & \\ & & & \widetilde{Q}^l \end{pmatrix} + \widehat{Q}, \tag{6}$$

where $\widetilde{Q}^k \in \mathbb{R}^{m_k \times m_k}$ are weakly irreducible, for $k = 1, 2, \ldots, l$, and $\sum_{k=1}^{l} m_k = m$. Let $\mathcal{M}_k = \{s_{k1}, \ldots, s_{km_k}\}$, for $k = 1, \ldots, l$, denote the states corresponding to $\widetilde{Q}^k$ and let $\mathcal{M}$ denote the state space of the underlying chains. Then

$$\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_l = \left\{ s_{11}, \ldots, s_{1m_1}, \ldots, s_{l1}, \ldots, s_{lm_l} \right\}.$$

Note that $\widetilde{Q}$ governs the rapidly changing part and $\widehat{Q}$ describes the slowly varying components. The slow and fast components are coupled through weak and strong interactions in the sense that the underlying Markov chain fluctuates rapidly within a single group $\mathcal{M}_j$ and jumps less frequently among groups $\mathcal{M}_k$ and $\mathcal{M}_j$ for $k \neq j$. The states in $\mathcal{M}_k$, $k = 1, \ldots, l$, are not isolated or independent of each other. More precisely, if we consider the states in $\mathcal{M}_k$ as a single "state," then these "states" are coupled through the matrix $\widehat{Q}$, and transitions from $\mathcal{M}_k$ to $\mathcal{M}_j$, $k \neq j$ are possible. By aggregating the states s_{kj} in $\mathcal{M}_k$ as one state k, we obtain an aggregated process $\{\overline{\alpha}^\varepsilon(\cdot)\}$ defined by $\overline{\alpha}^\varepsilon(t) = k$ when $\alpha^\varepsilon(t) \in \mathcal{M}_k$. The process $\overline{\alpha}^\varepsilon(\cdot)$ is not necessarily Markovian. However, using certain probabilistic argument, we have shown in [2] that $\overline{\alpha}^\varepsilon(\cdot)$ converges weakly to a Markov chain $\overline{\alpha}(\cdot)$ generated by

$$\overline{Q} = \operatorname{diag}(\nu^1, \ldots, \nu^l)\widehat{Q}\operatorname{diag}(\mathbb{1}_{m_1}, \ldots, \mathbb{1}_{m_l}),$$

where $\mathbb{1}_n = (1, \ldots, 1)' \in \mathbb{R}^n$. Based on such a result, we obtain the following theorem.

Theorem 4.1. *For $k = 1, 2, \ldots, l$ and $j = 1, 2, \ldots, m_k$,*

$$K^\varepsilon(s, s_{kj}) \to \overline{K}(s, k) \text{ and } q^\varepsilon(s, s_{kj}) \to \overline{q}(s, k),$$

uniformly on $[0,T]$ *as* $\varepsilon \to 0$, *where* $\overline{K}(s,k)$ *and* $\overline{q}(s,k)$ *are the unique solutions to the following differential equations: For* $k = 1,2,\ldots,l$,

$$\begin{cases} \dot{\overline{K}}(s,k) = -\overline{K}(s,k)\left(\sum_{j=1}^{m_k} \nu_j^k A(s_{kj})\right) - \left(\sum_{j=1}^{m_k} \nu_j^k A'(s_{kj})\right)\overline{K}(s,k) \\ \quad - \sum_{j=1}^{m_k} \nu_j^k M^{-1}(s_{kj}) + \overline{K}(s,k)\left(\sum_{j=1}^{m_k} \nu_j^k B(s_{kj})N^{-1}(s_{kj})B'(s_{kj})\right)\overline{K}(s,k) \\ \qquad\qquad -\overline{Q}\,\overline{K}(s,\cdot)(k), \\ \overline{K}(T,k) = D, \end{cases}$$

and

$$\begin{cases} \dot{\overline{q}}(s,k) = -\mathrm{tr}\left(\sigma\sigma'\overline{K}(s,k)\right) - \overline{Q}\overline{q}(s,\cdot)(k), \\ \overline{q}(T,k) = 0, \end{cases}$$

respectively.

The convergence of $K^\varepsilon(s,i)$ and $q^\varepsilon(s,i)$ leads to that of

$$v^\varepsilon(s,i,x) = x'K^\varepsilon(s,i)x + q^\varepsilon(s,i),$$

where $K^\varepsilon(s,i)$ and $q^\varepsilon(s,i)$ denote the solutions to the differential equations (3) and (4), respectively. In fact,

$$v^\varepsilon(s,s_{kj},x) \to v(s,k,x), \text{ for } j = 1,\ldots,m_k, \text{ as } \varepsilon \to 0,$$

where

$$v(s,k,x) = x'\overline{K}(s,k)x + \overline{q}(s,k),$$

corresponds to the value function of a limit problem. Let $\mathcal{U}$ denote the control set for the limit problem:

$$\mathcal{U} = \left\{U = (U^1,\ldots,U^l) :\ U^k = (u^{k1},\ldots,u^{km_k}), u^{kj} \in \mathbb{R}^{n_2}\right\}.$$

Define

$$f(s,k,x,U) = \sum_{j=1}^{m_k} \nu_j^k \left(A(s_{kj})x + B(s_{kj})u^{kj}\right)$$

$$\overline{M}(k) = \sum_{j=1}^{m_k} \nu_j^k M(s_{kj}), \text{ and } \widetilde{N}(k,U) = \sum_{j=1}^{m_k} \nu_j^k \left(u^{kj,\prime}N(s_{kj})u^{kj}\right).$$

Then it can be shown that $v(s,k,x)$ satisfies the following HJB equations:

$$\begin{cases} 0 = \dfrac{\partial v(s,k,x)}{\partial s} + \min\limits_{U\in\mathcal{U}}\Big\{ f(s,k,x,U)\dfrac{\partial v(s,k,x)}{\partial x} + x'\overline{M}(k)x + \widetilde{N}(k,U) \\ \qquad\qquad + \dfrac{1}{2}\mathrm{tr}\left(\sigma\sigma'\dfrac{\partial^2 v(s,k,x)}{\partial x^2}\right) + \overline{Q}v(s,\cdot,x)(k)\Big\}, \\ v(T,k,x) = x'Dx. \end{cases} \tag{7}$$

The corresponding control problem is

$$\begin{cases} \text{Min } J(s,k,x,U(\cdot)) = E\Big\{\int_s^T \Big(x'(t)\overline{M}(\overline{\alpha}(t))x(t) + \widetilde{N}(\overline{\alpha}(t),U(t))\Big)dt \\ \qquad\qquad +x'(T)Dx(T)\Big\} \\ \text{s.t. } dx(t) = f(t,\overline{\alpha}(t),x(t),U(t))dt + \sigma dw(t),\ x(s) = x,\ s \le t \le T, \end{cases}$$

where $\overline{\alpha}(\cdot) \in \{1,2,\ldots,l\}$ is a Markov chain generated by $\overline{Q}$ with $\overline{\alpha}(s) = k$.

The optimal control for this limit problem is:

$$U^*(s,k,x) = (U^{1*}(s,x),\ldots,U^{l*}(s,x)), \quad \text{with}$$

$$U^{k*}(s,x) = (u^{k1*}(s,x),\ldots,u^{km_k*}(s,x)), \quad \text{and}$$
$$u^{kj*}(s,x) = -N^{-1}(s_{kj})B'(s_{kj})\overline{K}(s,k)x.$$

Using such controls, we construct

$$\overline{u}^*(s,\alpha,x) = \sum_{k=1}^{l}\sum_{j=1}^{m_k} I_{\{\alpha=s_{kj}\}}u^{kj*}(s,x) \tag{8}$$

for the original problem. Note that this control can also be written as

$$\overline{u}^*(s,\alpha,x) = -N^{-1}(\alpha)B'(\alpha)\overline{K}(s,k)x, \text{ if } \alpha \in \mathcal{M}_k.$$

Apparently, this control is identical to the optimal control in (5) except K^ϵ is replaced by $\overline{K}$.

Theorem 4.2. *The control $u^\epsilon(t) = \overline{u}^*(t,\alpha^\epsilon(t),x(t))$ is nearly optimal, i.e.,*

$$\lim_{\epsilon\to 0}|J^\epsilon(s,\alpha,x,u^\epsilon(\cdot)) - v^\epsilon(s,\alpha,x)| = 0.$$

Remark 4.3. The most attractive feature of such an approximation scheme is that it requires much less computation effort. For instance, if the dimension of the system is $n_1 = 3$ and the Markov chain has $m = 40$ states divided into 5 groups with each group consisting 8 states, then the optimal scheme requires compute the Riccati equations of dimension $(n_1(n_1+1)/2)\times m = 6\times 40 = 240$. The dimension of the limit Riccati equation is only $(n_1(n_1+1)/2)\times 5 = 30$.

5 A NUMERICAL EXAMPLE

Let $\mathcal{M} = \{1,2\}$ and $Q^\epsilon = \varepsilon^{-1}\begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$. Consider the one-dimensional system with the following specifications: $A(1) = 0.5, A(2) = -0.1, B(1) = 1, B(2) = 2, \sigma = 1, M(1) = M(2) = N(1) = N(2) = D = 1$. To solve the problem numerically, we discretize the equations with step size h. The time horizons in the continuous-time model is $T = 5$ and in the corresponding discrete-time setting is $T_h = 10/h$ with $h = 0.0001$. The results below are based on computations using 100 sample paths.

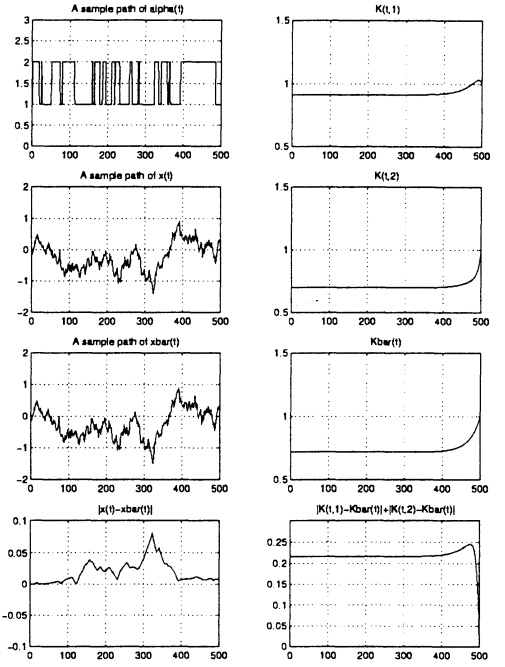

Fig. 1. Various sample paths with $\varepsilon = 0.1$

Take $s = 0$, $\alpha(0) = 1$, and $x(0) = x = 0$. Let $v^\varepsilon = v^\varepsilon(0,1,0)$, $J^\varepsilon = J^\varepsilon(0,1,0,u^\varepsilon(\cdot))$, and $v = v(0,0)$. For various ε we have the error bounds given in the following table.

ε	$\lvert K^\varepsilon - \overline{K}\rvert$	$\lvert x^\varepsilon - \overline{x}^\varepsilon\rvert$	$\lvert v^\varepsilon - v\rvert$	$\lvert J^\varepsilon - v^\varepsilon\rvert$
0.1	2.17ε	0.10ε	4.23ε	$2.21\sqrt{\varepsilon}$
0.01	2.47ε	0.19ε	5.01ε	$0.36\sqrt{\varepsilon}$
0.001	2.50ε	0.13ε	5.09ε	$0.46\sqrt{\varepsilon}$
0.0001	2.50ε	0.12ε	5.10ε	$4.17\sqrt{\varepsilon}$

Table 1. Error Bounds

Sample paths of various trajectories of $\alpha^\varepsilon(\cdot)$, $x^\varepsilon(\cdot)$, $\overline{x}^\varepsilon(\cdot)$, $K^\varepsilon(\cdot,1)$, $K^\varepsilon(\cdot,2)$, $\overline{K}(\cdot)$ are given in Fig. 1 for $\varepsilon = 0.1$ and in Fig. 2 for $\varepsilon = 0.01$. It can be seen from these graphs the smaller the ε the more rapidly $\alpha^\varepsilon(\cdot)$ jumps, which results better approximations. To summarize, the numerical simulations indicate that our algorithm gives a very good approximation to exact optimal solutions with only one half of the computation effort.

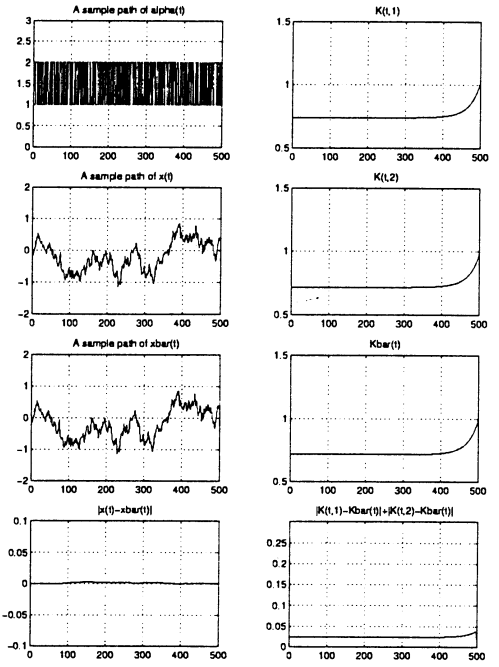

Fig. 2. Various sample paths with $\varepsilon = 0.01$

References

[1] G. Yin and Q. Zhang, *Continuous-Time Markov Chains and Applications: A Singular Perturbation Approach*, Springer-Verlag, 1998.

[2] Q. Zhang and G. Yin, On nearly optimal controls of hybrid LQG problems, preprint.