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Constructing Bezier Conic Segments With Monotone Curvature

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Abstract

Techniques for assisting the construction of Bezier conic segments are given a mathematical foundation. Two situations are resolved. When the endpoints of disjoint curves must be connected, a symmetric and complete characterization of allowable weights and locations of the second control point is given. The second case, where only one tangent is fixed, yields a corresponding nonsymmetric treatment. In both cases, if the angle opposite the base of the Bezier triangle is obtuse, then there exist values of the weight that guarantee monotone curvature of the conic segment. In both cases the location of the control point determines whether parabolic, elliptic of hyperbolic segments can be constructed.

Keywords

Bezier curve, conic segment, monotone curvature

Introduction

Dividing curves into segments with monotonically changing curvature offers visual cues for interrogating curves in a Computer Aided Design environment [1, 4]. Sapidis and Frey [5] dealt exclusively with parabolic segments in providing guidelines for the middle control point in a Bezier triangle. Frey and Field [3] added a weight to this control point and provided regions for locating this point to create all possible conic sections with monontonically changing curvature. This paper focuses on the asymmetric construction of monotone conic segments, namely, the extension of a curve beyond its endpoint with a C^1 curve having monotone curvature. This paper also considers the case where the curvature at this endpoint has been specified.

The next section presents rational Bezier curves and derives their monotonicity conditions. Specializations to Bezier segments of conics comprises the next section on the symmetric and asymmmetric cases that correspond respectively to reconstructing and extending curves.

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The final section contains an application of the results on conic segments with monotone curvature. These results complete the mathematical treatment found in [3,5]. Thus this paper encompasses both the reconstruction of interior segments and the extensions of curves with segments of monotonically changing curvature.

Monotonic Curvature Discriminant

The curvature $\kappa(t)$ of a regular space curve $\mathbf{C}(t)$, a curve where $\mathbf{C}'(t) \neq 0$, satisfies

$$\kappa(t) = \frac{||\mathbf{C}'(t) \times \mathbf{C}''(t)||}{||\mathbf{C}'(t)||^3} \quad . \tag{1}$$

When $\kappa(t)$ varies, a local minimum in the curvature of $\mathbf{C}(t)$ occurs if $\kappa'(t^*) = 0$ for some t^* in its parametric domain. In this case either $\mathbf{C}''(t^*) = 0$ or $\mathbf{C}'(t^*)$ and $\mathbf{C}''(t^*)$ are parallel so that $\mathbf{C}(t)$ has a inflection point when $t = t^*$. Setting the derivative of $\kappa(t)$ in (1) to zero yields the extremum condition

$$s(t) = \mathbf{B}(t) \cdot (\mathbf{C}'(t) \times \mathbf{C}'''(t))(\mathbf{C}'(t) \times \mathbf{C}'(t)) - 3||\mathbf{C}'(t) \times \mathbf{C}''(t)||\mathbf{C}'(t) \cdot \mathbf{C}''(t) = 0 \quad , \tag{2}$$

where

$$\mathbf{B}(t) = \frac{\mathbf{C}'(t) \times \mathbf{C}''(t)}{||\mathbf{C}'(t) \times \mathbf{C}''(t)||}$$

denotes the unit binormal vector perpendicular to the osculating plane of $\mathbf{C}(t)$.

The most general rational Bezier curves typically assume the form

$$\frac{\sum_{i=0}^{n} \binom{n}{i} w_i \mathbf{b}_i (1-t)^{n-i} t^i}{\sum_{i=0}^{n} \binom{n}{i} w_i (1-t)^{n-i} t^i}$$

where $\mathbf{b}_i$, a vertex on the control polygon, denotes a *control point* and w_i denotes the *weight* at $\mathbf{b}_i$. Let

$$\mathbf{q}(t) = \frac{\mathbf{x}(t)}{\beta(t)} = \frac{(1-t)^2\mathbf{b}_0 + 2wt(1-t)\mathbf{b}_1 + t^2\mathbf{b}_2}{(1-t)^2 + 2wt(1-t) + t^2}, \quad -\infty < t < \infty \quad , \tag{3}$$

express a conic in standard Bezier form with Bezier control points $\mathbf{b}_0 = (x_0, y_0)$, $\mathbf{b}_1 = (x_1, y_1)$ and $\mathbf{b}_2 = (x_2, y_2)$. $\mathbf{q}(t)$ traces the *Bezier segment* of this conic when $t \in [0,1]$. The triangle formed by the three control points in (3) contains the entire Bezier segment and its sides $\mathbf{b}_1 - \mathbf{b}_0$, and $\mathbf{b}_2 - \mathbf{b}_1$ are tangent to the Bezier segment at $\mathbf{b}_0$ and $\mathbf{b}_2$ respectively; see Figure 1 and [2] for figures that display higher degree Bezier curves.

For a planar curve $\mathbf{C}(t)$ the vectors $\mathbf{C}'(t)$, $\mathbf{C}''(t)$ and $\mathbf{C}'''(t)$ all lie in the plane of the curve. Therefore both $\mathbf{C}'(t) \times \mathbf{C}''(t)$ and $\mathbf{C}'(t) \times \mathbf{C}'''(t)$ are perpendicular to this plane and, furthermore, since the conic $\mathbf{q}(t)$ must be planar,

$$\mathbf{q}'(t) \times \mathbf{q}'''(t) = -3\frac{\beta'(t)}{\beta(t)}(\mathbf{q}(t)' \times \mathbf{q}''(t)) \quad . \tag{4}$$

The extremum condition for $\mathbf{q}(t)$ in (2) then simplifies to

$$s(t) \equiv \mathbf{q}'(t) \cdot \frac{d}{dt}(\beta(t)\mathbf{q}'(t)) = 0 \quad . \tag{5}$$

Differentiation of the terms in (5) yields, see [3] for details,

$$\begin{aligned} s'(t) = p(t) = & -2w^2(1-2t)\left[w - 2(w-1)t(1-t)\right](\mathbf{b}_1 - \mathbf{b}_0) \cdot (\mathbf{b}_1 - \mathbf{b}_0) \\ & + w\left[1 - 4t - 8(w-1)t^3 - 4(w-1)^2t^4\right](\mathbf{b}_1 - \mathbf{b}_0) \cdot (\mathbf{b}_2 - \mathbf{b}_0) \\ & + t\left[1 + (w-1)t\right]\left[1 + 2(w-1)t(1+wt)\right](\mathbf{b}_2 - \mathbf{b}_0) \cdot (\mathbf{b}_2 - \mathbf{b}_0) = 0. \end{aligned} \tag{6}$$

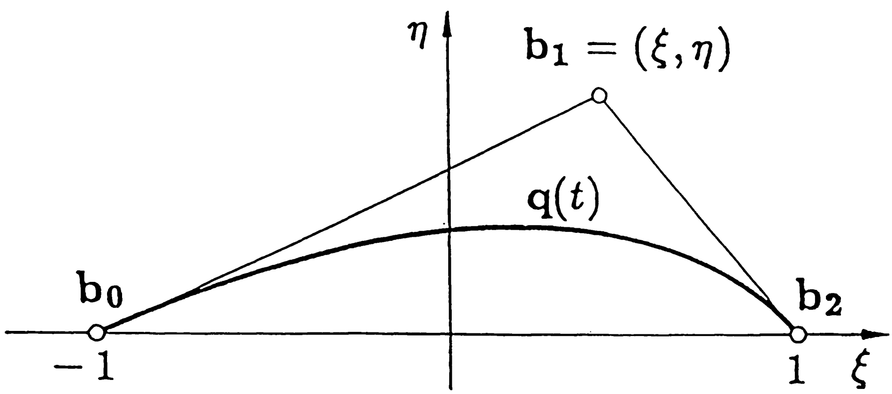

Figure 1 A conic Bezier segment, its control polygon and a local (ξ, η)-coordinate system for the symmetric case.

Monotone Curvature

Under scaling, rotations and translations of their control polygons, Bezier curves continue to preserve monotonicity of curvature. Therefore,we select the coordinates of $\mathbf{b}_0$, $\mathbf{b}_1$ and $\mathbf{b}_2$ to simplify $p(t)$ but not reduce its generality. The symmetric selection fixes $\mathbf{b}_0$ and $\mathbf{b}_2$ and lets $\mathbf{b}_1$ vary arbitrarily and the asymmetric selection fixes $\mathbf{b}_0$ and $\mathbf{b}_1$ and lets $\mathbf{b}_2$ vary arbitrarily. These selections correspond respectively to reconstructing and extending curves.

In [3] the symmetric case,

$$\mathbf{b}_0 = (-1, 0) \quad , \quad \mathbf{b}_1 = (\xi, \eta) \quad \text{and} \quad \mathbf{b}_2 = (0, -1) \quad , \tag{7}$$

produced

$$K(w, t, \xi, \eta) = \xi^2 + \eta^2 - 2\lambda(w, t)\xi + c(w) = 0 \quad , \tag{8}$$

where

$$\gamma(w, t) = w + 1 - \beta(t) \quad , \quad \tau = 2t - 1 \quad , \quad c(w) = \frac{w^2 - 1}{w^2} \tag{9}$$

and

$$\lambda(w, t) = \frac{1}{2w}\left(\frac{\gamma(w, t)}{\tau} + (w^2 - 1)\frac{\tau}{\gamma(w, t)}\right) \quad . \tag{10}$$

$K(w, t, \xi, \eta)$ represented a coaxal family of circles with respect to t in the (ξ, η)-plane. All the circles had centers on the η-axis and the critical circles, those for which $t = 0$ and $t = 1$, passed through $(-1, 0)$ and $(0, 1)$ respectively.

In this paper we investigate the asymmetric case with the control points

$$\mathbf{b}_0 = (-1, 0) \quad , \quad \mathbf{b}_1 = (0, 0) \quad \text{and} \quad \mathbf{b}_2 = (\xi, \eta) \quad . \tag{11}$$

These coordinates reduce $p(t)$ in (6) to critical lines for $t = 0$ and to critical circles for $t = 1$;

$$\xi = 2w^2 - 1 \quad \text{and} \quad \left(\xi - \frac{1}{2(2w^2 - 1)}\right)^2 + \eta^2 = \frac{1}{4(2w^2 - 1)^2} \quad . \tag{12}$$

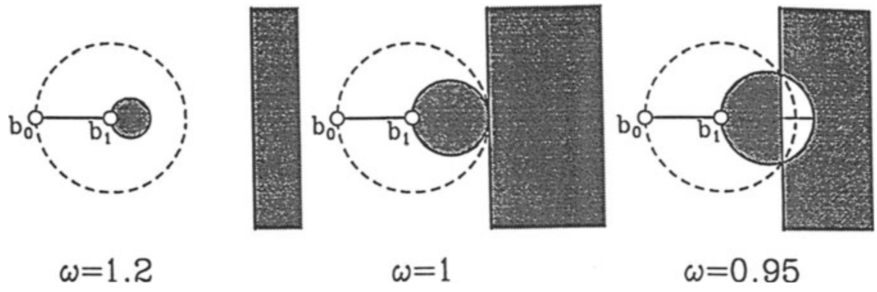

Figure 2 $w > 1$, $w = 1$ and to $w < 1$ correspond respectively to hyperbolas, parabolas and ellipses. All critical circles pass through the middle control point $\mathbf{b}_1$. When the third control point $\mathbf{b}_2$ lies in the shaded region, the Bezier conic segment has monotone curvature.

All the critical circles pass through the origin and, when the critical line and the critical circle intersect for a fixed value of $w \leq 1$, their intersection lies on the unit circle. Figure 2 displays the cases $w > 1$, $w = 1$ and $w < 1$,which correspond respectively to hyperbolas, parabolas and ellipses. In Figure 2, for fixed w, $w \geq 1/\sqrt{2}$, the conic segment monotonically increases in curvature with t when $\mathbf{b}_2$ lies simultaneously on or to the left of the critical line and inside or on the critical circle. The conic section monotonically decreases in curvature with t when $\mathbf{b}_2$ lies simultaneously on or to the right of the critical line and outside or on the critical circle. For all cases the inverse with respect to the unit circle of a point in the shaded region inside the unit circle lies to the right of the critical line and outside the unit circle and vice versa.

For control points in general position the normalizing factor in the asymmetric case equals $|\mathbf{b}_0 - \mathbf{b}_1|$. η corresponds to the normalized distance from $\mathbf{b}_2$ to the line defined by $\mathbf{b}_0$ and $\mathbf{b}_1$. ξ corresponds to the normalized distance from the line passing through $\mathbf{b}_1$ and perpendicular to the line defined by $\mathbf{b}_0$ and $\mathbf{b}_1$; see Figure 3.

For any fixed values of w and t, choosing a coordinate (ξ^*, η^*) which satisfies the equation (6) guarantees an extremum of curvature at the given value of t. For fixed w the goal thus becomes the determination of locations for the coordinates (ξ^*, η^*) that cannot lie on any circle or line corresponding to a value t, $0 < t < 1$.

Theorem 1 (asymmetric case): Let $w \geq 1/\sqrt{2}$. If (ξ, η), $\eta \neq 0$, lies in the closed right half plane and outside or on the circle defined in (12) or if (ξ, η), $\eta \neq 0$, lies on or inside the circle and in the closed left half plane defined in (12), then the $p(t)$ in (6) cannot be zero for any value of t, $0 < t < 1$.

We give a short proof of this theorem that uses a corresponding theorem for the symmetric case. In [3] we proved the following theorem.

Theorem 2 (symmetric case): Let $\hat{w} \geq \frac{1}{\sqrt{2}}$. If $(\hat{\xi}, \hat{\eta})$, $\hat{\eta} \neq 0$, lies inside or on one of the critical circles defined by $t = 0$ and $t = 1$ in (8) and outside or on the other, then no root of $K(\hat{w}, t, \hat{\xi}, \hat{\eta})$ satisfies $0 < t < 1$.

Proof of Theorem 1: Let t, $0 < t < 1$, and w, $w \geq 1/\sqrt{2}$. Define the function of a complex variable $s(z) = (1 - z)/(1 + z)$, $z = \xi + i\eta$. $s(z)$ maps the critical circles in (8) into the straight line and the circle in (12). In particular, $s(z)$ maps the interior of the critical circle for $t = 0$

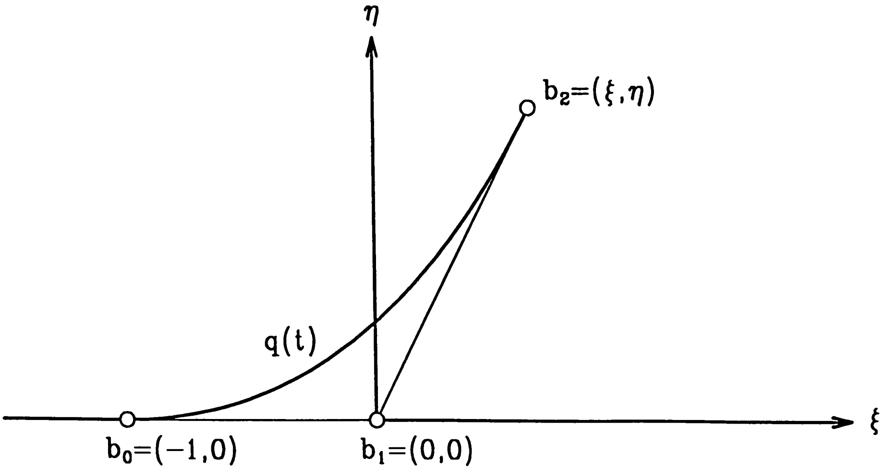

Figure 3 The local (ξ,η)-coordinate system for the asymmetric case.

onto the right half plane and maps the interior of the critical circle for $t = 1$ onto the interior of the circle in (12). Since linear fractional transformations are conformal mappings, $s(z)$ preserves the intersections and the angles of intersections of curves. Furthermore, $s(s(z)) = z$ makes $s(z)$ idempotent.

Let $(\tilde{\xi},\tilde{\eta})$ satisfy Theorem 2 for the symmetric case. Note that $(\tilde{\xi},\tilde{\eta})$ cannot satisfy (8) for t, $0 < t < 1$. Moreover, for any curve passing through $(\tilde{\xi},\tilde{\eta})$ with all of its points satisfying Theorem 2, say curve C, no circle satisfying (8) with $0 < t < 1$ can intersect C. Apply $s(z)$ to C and to the circles in (8) such that $0 < t < 1$. Since $s(z)$ is a linear fractional transformation, the image of C must be contained in the regions defined in Theorem 1 and cannot intersect the image of any circle in (8) with $0 < t < 1$. But under $s(z)$ the image of $K(w,t,\xi,\eta)$ in (8) is $p(t)$ in (6) with b_0, b_1, and b_2 defined in (12). The idempotency of $s(z)$ makes $(\tilde{\xi},\eta)$ easily determined for any (ξ,η) satisfying the hypothesis of Theorem 1 and this proves the theorem. Note that $s(z)$ maps the η-axis onto the unit circle. Therefore, the intersections of the critical circles from Theorem 2 on the η-axis, see [3], ensures that the critical line and critical circles in the asymmetric case intersect on the unit circle as shown in Figure 2. ∎

Imposing a prescribed value of curvature, say k, at b_0 $(t = 0)$ in the asymmetric case, restricts the placement of b_2 . Since the curvature of $q(t)$ at $t = 0$ equals $\eta/2w^2$, b_2 must lie on one of the horizontal lines $L_k^+ : \eta = 2kw^2$ or $L_k^- : \eta = -2kw^2$. For each admissable value of w the locations of b_2 must also satisfy Theorem 1. L_k intersects the critical line at $(2w^2-1, 2kw^2)$ and the critical circle when

$$\xi = \frac{1}{2(2w^2-1)}\left(1 \pm \sqrt{1 - 16k^2w^2(2w^2-1)^2}\right) \tag{13}$$

Figure 2 clearly demonstrates that, for any value of curvature k, there will always exist locations of b_2 to the right of the vertical critical line. The existence of a location to the left of the critical line, where curvature will increase monotonically from $\mathbf{b}_0$ to $\mathbf{b}_2$, depends on the discriminant in (13). The discriminant restricts k and w so that

$$k \leq \frac{1}{4w(2w^2 - 1)} \quad . \tag{14}$$

For any finite value of k, $1/\sqrt{2} < w$ guarantees that there will always exist locations of $\mathbf{b}_2$ which define a quadratic rational Bezier curve with curvature increasing monotonically from the curvature at $\mathbf{b}_0$. For values of curvature $k < 1/4$ only ellipses can be used and for values of curvature $k \leq 1/4$ ellipses, parabolas and hyperbolas may be used. Figure 4 displays a typical set of regions in which $\mathbf{b}_2$ may be placed to produce a quadratic rational Bezier curve with a given curvature at $\mathbf{b}_0$.

As shown in Figure 5, the symmetric case in Theorem 2 yields comparable regions. In this case to produce a quadratic rational Bezier curve with a given curvature k at $\mathbf{b}_1$, $\mathbf{b}_1$ must lie on either of two lines $\mathcal{L}_k^+ : \eta = k/w$ or $\mathcal{L}_k^- : \eta = -k/w$. Intersections of $\mathcal{L}_k^+$ and $\mathcal{L}_k^-$ with the critical circles of Theorem 2 produces the following intervals for the x-coordinate of the segments on $\mathcal{L}_k^+$ and $\mathcal{L}_k^-$ that satisfy Theorem 2;

$$\frac{1}{2w^2}\left(1 - 2w^2 - \sqrt{1 - 4k^2w^2}\right) \leq \xi \leq \frac{1}{2w^2}\left(2w^2 - 1 - \sqrt{1 - 4k^2w^2}\right) \tag{15a}$$

and

$$\frac{1}{2w^2}\left(1 - 2w^2 + \sqrt{1 - 4k^2w^2}\right) \leq \xi \leq \frac{1}{2w^2}\left(2w^2 - 1 + \sqrt{1 - 4k^2w^2}\right) \tag{15b}$$

The discriminant in (15) imposes the contraint that $w \leq 1/2k$. Since $1/\sqrt{2} < w$, k must then satisfy $k < 1/\sqrt{2}$. Figure 5 displays a typical set of regions in which $\mathbf{b}_1$ may be placed to produce a quadratic rational Bezier curve with a given curvature at $\mathbf{b}_1$.

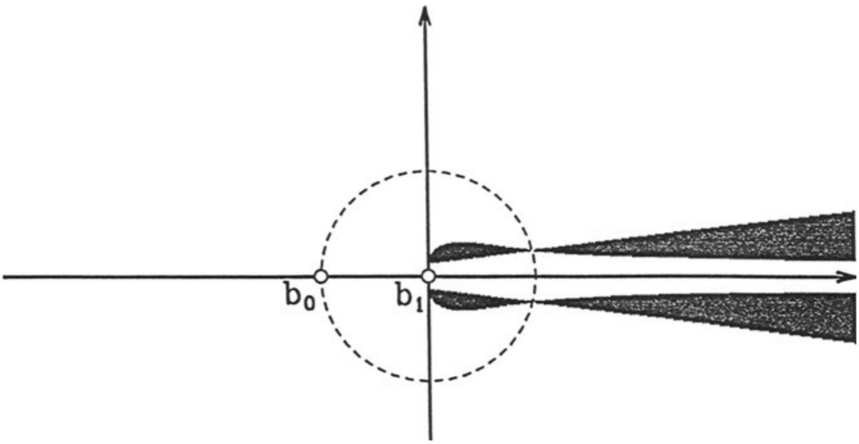

Figure 4 Shaded areas denote where to place $\mathbf{b}_2$ for a given curvature at $\mathbf{b}_0$.

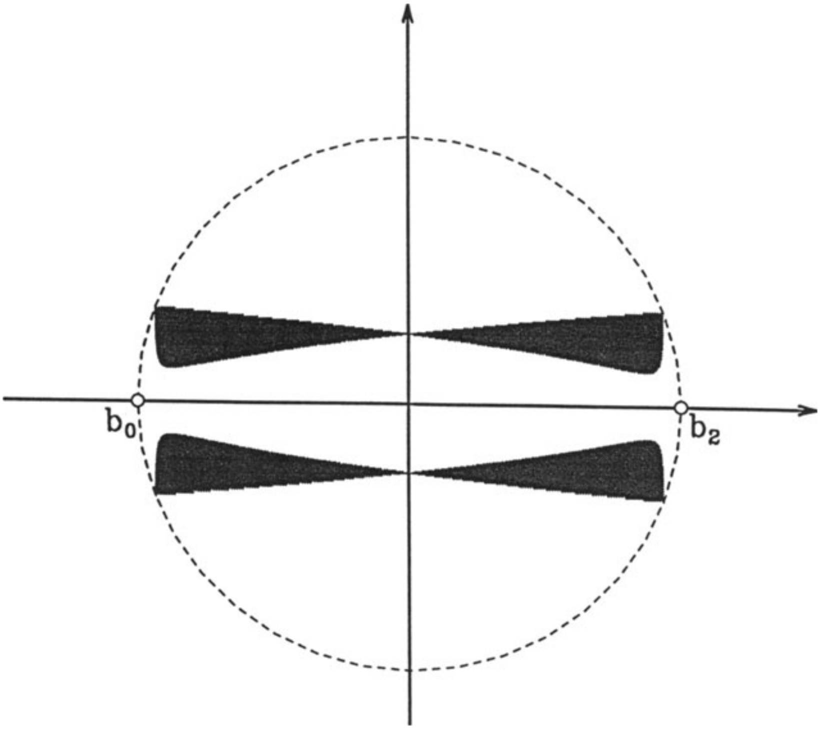

Figure 5 Shaded areas denote where to place $\mathbf{b}_1$ for a given curvature at $\mathbf{b}_1$.

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