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On a class of composite plates of maximal compliance

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Abstract

Within the class of thin, transversely loaded plates of composite microstructure undergoing bending as well as transverse shearing an effective elastic potential of the weakest plate is found and its nonlinear characterization is explicitly given. The optimal design of the plate of maximal compliance turns out to coincide with the optimal design of the weakest plate of microstructure upon which Kirchhoff' s constraints are imposed.

Keywords

Optimum plate design, composite plates, homogenization

1 INTRODUCTION

A correct designing of thin two-phase elastic plates of extremal compliance requires a relaxation, cf Kohn and Vogelius(1986). In the relaxed problem one admits mixing the phases at the microstructural level. The plate is then characterized by effective stiffnesses given by Duvaut-Metellus formulae for Kirchhoff plate homogenization. Such optimal designs can be found in Lur'e and Cherkaev(1986) .This approach goes along similar lines as that developed for plane elasticity (cf Allaire and Kohn(1993)). However, its application to plate problems is controvertible, since the three-dimensional analysis of accuracy of Duvaut-Metellus formulae shows that they assess the stiffnesses incorrectly, in particular those responsible for torsion. An essential improvement in evaluating stiffnesses can be achieved if one admits transverse shear deformation of the microstructure, see Lewiński (1992). Therefore it is proposed in this paper to endow the optimized plate with a Hencky-Reissner (not Kirchhoff) microstructure by using homogenization formulae derived by the conformal (or refined) scaling, cf Lewiński (1992), Telega (1992). The local subproblem is solved by the translation method, cf Lur'e and Cherkaev (1986).

2 FINDING THE WEAKEST TWO-PHASE PLATE

Consider a thin plate whose middle plane occupies a plane domain Ω parametrized by a Cartesian coordinate system x_α, e_α being its versors, α and other small Greek letters run over 1,2. Tensors (a_i) defined by

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$$a_1 = a_{11} + a_{22}\ , \quad a_2 = a_{11} - a_{22}\ , \quad a_3 = a_{12} + a_{21}\ , \quad a_4 = a_{12} - a_{21} \tag{1}$$

where $a_{\alpha\beta} = 2^{-\frac{1}{2}} e_\alpha \otimes e_\beta$ constitute an orthonormal basis in the space of second order tensors. The stiffness tensor of a two-phase thin plate is given by

$$D(x) = D_1\chi_1(x) + D_2\chi_2(x) \tag{2}$$

χ_α being indicator functions of the plate phases characterized by the isotropic stiffnesses

$$D_\alpha = 2k_\alpha a_1 \otimes a_1 + 2\mu_\alpha(a_2 \otimes a_2 + a_3 \otimes a_3)\ ; \tag{3}$$

k_α and μ_α represent elastic moduli. The ordered case is dealt with: $k_2 > k_1 > 0$ and $\mu_2 > \mu_1 > 0$. The constitutive relationships are of Kirchhoff' s type

$$M^{\alpha\beta} = D^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}(w)\ , \qquad \kappa_{\lambda\mu}(w) = -w_{,\lambda\mu}\ , \tag{4}$$

$D^{\alpha\beta\lambda\mu}$ being components of D in the basis $e_\alpha \otimes e_\beta \otimes e_\lambda \otimes e_\mu$ and $()_{,\lambda} = \partial/\partial x_\lambda$. A scalar field v defined on Ω is said to be kinematically admissible (kin.adm.) if $v \in H_0^2(\Omega)$, i.e. the plate is clamped along its boundary. The plate is subject to transverse loading $p = p(x)$, $x = (x_\alpha)$. The compliance of the plate

$$J(\chi_1) = \int_\Omega pw dx \tag{5}$$

is viewed as a functional over all possible $\chi_1(\chi_2 = 1 - \chi_1)$, i.e. over all layouts of the phases; the deflection function w represents the solution to the Kirchhoff plate problem

$$-\frac{1}{2}J(\chi_1) = \inf\left\{\frac{1}{2}\int_\Omega [\kappa_{\lambda\mu}(v)D^{\lambda\mu\alpha\beta}\kappa_{\alpha\beta}(v) - 2pv]dx \,\middle|\, v \text{ kin.adm.}\right\} \tag{6}$$

with D depending upon χ_1 according to Eq.(2).

Assume that the amounts of both plate materials are given, i.e.

$$\int_\Omega \chi_1 dx = A_1 \tag{7}$$

A_1 being a constant. Define

$$J_\lambda(\chi_1) = \frac{1}{2}J(\chi_1) - \lambda\int_\Omega \chi_1 dx \tag{8}$$

λ being a Lagrangian multiplier. Since the weakest plate is characterized by a maximal compliance $J(\chi_1)$ it is sufficient to consider the auxiliary problem $\sup\{J_\lambda(\chi_1)|\chi_1\}$ that by (6) can be cast in the form

$$\inf_{\chi_1}\ \inf_{v\ kin.adm.} \left\{\frac{1}{2}\int_\Omega [\kappa_{\alpha\beta}(v)D^{\alpha\beta\lambda\mu}(x)\kappa_{\lambda\mu}(v) - 2pv + 2\lambda\chi_1]dx\right\} \tag{9}$$

This problem is ill-posed, cf Kohn and Strang (1986), Lipton (1994). To relax and thus make the problem well-posed one should endow the plate with a two-phase microstructure of thin plate (Kirchhoff) type, cf Lipton(1994). This formulation will not be recalled. In the present paper we shall consider a seemingly slight modification of the relaxed formulation (9). In place of the Kirchhoff's microstructure we assume Hencky-Reissner's microstructure undergoing additionally a transverse shear deformation. Let us introduce indispensable notions. The mid-plane of the periodicity cell is denoted by $Y = (0, l_1) \times (0, l_2)$. The layout of the bending stiffnesses within Y is determined by indicator functions $\chi^Y_\alpha(x, \cdot)$

$$D(x,y) = D_1\chi^Y_1(x,y) + D_2\chi^Y_2(x,y)\ , \quad y \in Y\ , \tag{10}$$

where D_α are given by (3). The transverse shear stiffness tensor H is assumed to be common for both phases: $H = 2^{\frac{1}{2}} g a_1$, where g is the value of the transverse shear stiffness and tensor a_1 is given by (1) . Such a mismatch between the layouts of D and H is possible if the microstructure is transversely isotropic . The averages over Y

$$\langle \chi^Y_\alpha(x,y)\rangle = \theta_\alpha(x)\ , \quad \langle\cdot\rangle = \frac{1}{l_1 l_2}\int_Y (\cdot)dy \tag{11}$$

are functions : $\theta_\alpha : \Omega \to [0,1]$ interrelated with χ_α by the relations

$$\int_\Omega \theta_\alpha dx = \int_\Omega \chi_\alpha dx\ . \tag{12}$$

Let us define bending and shearing deformations as functions of the fields $\phi = (\phi_\alpha)$, w defined on Y:

$$e_{\alpha\beta}(\phi) = \frac{1}{2}(\phi_{\alpha|\beta} + \phi_{\beta|\alpha})\ , \ \gamma_\alpha(\phi, w) = \phi_\alpha + w_{|\alpha}\ , \tag{13}$$

where $()_{|\alpha} = \partial/\partial y_\alpha$. Problem (9) is replaced with

$$\inf_{\theta_1:\Omega\to[0,1]}\ \inf_{v\ kin.adm.}\left\{\frac{1}{2}\int_\Omega [W_H(\kappa(v);\theta_1) - 2pv + 2\lambda\theta_1]dx\right\}\ , \tag{14}$$

where

$$W_H(\kappa;\theta_1) = \inf\left\{W(\chi^Y_1;\kappa,\theta_1)\,\middle|\,\langle\chi^Y_1\rangle = \theta_1\right\} \tag{15}$$

and

$$\begin{aligned} W(\chi^Y_1;\kappa,\theta_1) = \inf\Big\{&\underline{\langle e_{\alpha\beta}(\phi)D^{\alpha\beta\lambda\mu}(x,y)e_{\lambda\mu}(\phi) + \gamma_\alpha(\phi,w)H^{\alpha\beta}\gamma_\beta(\phi,w)\rangle}\ \Big| \\ &e = (e_{\alpha\beta})\ ,\ \gamma = (\gamma_\alpha) \text{ such that they are of the form (13), are } Y-\text{periodic and} \\ &\langle e_{\alpha\beta}(\phi)\rangle = \kappa_{\alpha\beta}\Big\}\ , \end{aligned} \tag{16}$$

where $H = H^{\alpha\beta} e_\alpha \otimes e_\beta$. On can show that $\langle \gamma_\alpha \rangle = 0$ due to assumption of H being constant. The expression underscored represents a sum of bending and transverse shearing energies. Definition of the effective potential W follows from the formula for effective stiffnesses of moderately thick plates, cf Lewiński (1992), Telega (1992). Formulation (14) is correct similarly as the relaxed problem within Kirchhoff's theory (then $\phi_\alpha = -w_{|\alpha}$ in (16)). Obviously, formulation (14) is neither full nor partial relaxation of problem (9) (notions introduced in Kohn and Vogelius (1986), see also Bonnetier and Conca (1994)), since it exceeds the framework of the Kirchhoff' s theory. This problem is an approximation of Hencky-Reissner type of the optimization problem of the two-phase transversely homogeneous plate within the three-dimensional description.

3 BOUNDING THE EFFECTIVE POTENTIAL $W_H(\chi_1^Y; \kappa, \theta_1)$

To take into account differential constraints (13) involved in (16) it is helpful to apply the translation method, see Lur'e and Cherkaev (1986), Allaire and Kohn (1993), Cherkaev and Gibianski (1993). For technical aims we define

$$\begin{aligned} &E = [\phi_{1|1} \ , \ \phi_{1|2} \ , \ \phi_{2|1} \ , \ \phi_{2|2} \ , \ w_{|1} + \phi_1 \ , \ w_{|2} + \phi_2]^T \\ &\mathcal{D} = diag[D, \tilde{H}] \ , \quad \tilde{H} = diag[H, H] \end{aligned} \tag{17}$$

Then one can write

$$e_{\alpha\beta} D^{\alpha\beta\lambda\mu} e_{\lambda\mu} + \gamma_\alpha H^{\alpha\beta} \gamma_\beta = E^T \mathcal{D} E \tag{18}$$

We need the estimate

$$\langle E^T T E \rangle \geq \langle E^T \rangle T \langle E \rangle \quad \forall E \text{ of the form (17)} \ , \tag{19}$$

where T is a constant 6×6 matrix. A good candidate for T is

$$T = \begin{bmatrix} c & & & d & b & b \\ & c & -d & & -b & b \\ & -d & c & & b & -b \\ d & & & c & b & b \\ b & -b & b & b & a & \\ b & b & -b & b & & a \end{bmatrix} \tag{20}$$

where if

(a) $c = 0$, then $b = 0$, $a \geq 0$

(b) $c = 0$, then $a \geq 0$, $c > 0$, $2b^2 \leq ac$

If $c = a = b = 0$, then (19) becomes an equality for all d and E . The proof of (19) can be performed by Fourier analysis. It is omitted to save space.

Now let us change the basis

$$\phi_{\beta|\alpha} e_\alpha \otimes e_\beta = \delta_i a_i \ , \ \kappa_{\alpha\beta} e_\alpha \otimes e_\beta = \kappa^i a_i \ , \ i = 1, \ldots, 4 \tag{21}$$

hence

$$E^T T E = \tilde{E}^T \tilde{T} \tilde{E} \ , \ \tilde{E} = [\delta, \gamma] \ , \ \delta = [\delta_1, \delta_2, \delta_3, \delta_4] \ , \gamma = [\gamma_1, \gamma_2] \tag{22}$$

where

$$\tilde{T} = \begin{bmatrix} 2r & & & & t & t \\ & 2s & & & & \\ & & 2s & & & \\ & & & 2r & -t & t \\ t & & & -t & a & \\ t & & & t & & a \end{bmatrix}$$

and

$$t = \sqrt{2} b \ , \quad 2r = c + d \ , \quad 2s = c - d \tag{23}$$

Parameter t couples bending and transverse shearing deformations. The constitutive relationships assume a diagonal form

$$\sigma = \tilde{D} \tilde{E} \ , \quad \sigma = [M^1, M^2, M^3, M^4, Q^1, Q^2] \tag{24}$$

$$\tilde{D} = diag[2k, 2\mu, 2\mu, 0, g, g] \tag{25}$$

and

$$M = M^{\alpha\beta} e_\alpha \otimes e_\beta = M^i a_i \ , \quad Q = Q^\alpha e_\alpha \tag{26}$$

with k, μ defined similarly as D in Eq. (10). Now let us express W in a new form

$$\begin{aligned} & W(\chi_1^Y; \kappa, \theta_1) = \inf \left\{ \langle \delta^T \tilde{D} \delta + \gamma^T \tilde{H} \gamma \rangle \,\middle|\, \langle \delta_i \rangle = \kappa^i \text{ for } i = 1,2,3 \ ; \right. \\ & \left. \delta, \gamma \text{ are } Y-\text{periodic and defined by}(22); \ \langle \delta_4 \rangle = 0 \ , \ \langle \gamma_\alpha \rangle = 0 \right\} \end{aligned} \tag{27}$$

or

$$\begin{aligned} & W(\chi_1^Y; \kappa, \theta_1) = \inf \left\{ \langle \tilde{E}^T \tilde{\mathcal{D}} \tilde{E} \rangle \,\middle|\, \tilde{E} \text{ given by (22)with conditions given above} \right\} \\ & \tilde{\mathcal{D}} = diag[\tilde{D}, \tilde{H}] \end{aligned} \tag{28}$$

The idea of the translation method is to decompose $\langle \tilde{E}^T \tilde{\mathcal{D}} \tilde{E} \rangle$

$$\langle \tilde{E}^T (\tilde{\mathcal{D}} - \tilde{T}) \tilde{E} \rangle + \langle \tilde{E}^T \tilde{T} \tilde{E} \rangle \ , \tag{29}$$

omit differential constraints while estimating the first term and apply the estimate (19) in which these differential constraints are fully used. Hence

$$W(\chi_1^Y; \kappa, \theta_1) \geq E_0^T [\langle (\tilde{\mathcal{D}} - \tilde{T})^{-1} \rangle^{-1} + \tilde{T}] E_0 \quad E_0 = \langle \tilde{E} \rangle = [\kappa^1, \kappa^2, \kappa^3, 0, 0, 0] \tag{30}$$

and parameters involved in $\tilde{T}$ should satisfy condition $(\tilde{\mathcal{D}} - \tilde{T}) \geq 0$ (in particular: $-2r \geq 0$, etc.). Note that the right hand-side is independent of χ_1^Y .

By virtue of a specific form of matrices $\tilde{\mathcal{D}}$ and $\tilde{T}$ one can evaluate the right hand side of (30) explicitly

$$\frac{1}{2}W(\chi_1^Y;\kappa,\theta_1) \geq \eta^2 \max\{(K^0+\xi^2 G^0)\Big|\text{admissible } r,s,t,a\} \tag{31}$$

where $\eta = \kappa^1$, $\xi = [(\kappa^2)^2+(\kappa^3)^2]^{\frac{1}{2}}//\kappa^1$

$$K^0 = \frac{(g-a)(k_1k_2 - r\bar{k}) - \bar{k}t^2}{(g-a)(\tilde{k}-r)-t^2} \;, \quad G^0 = \frac{\mu_1\mu_2 - s\bar{\mu}}{\tilde{\mu}-s} \tag{32}$$

and

$$\tilde{f} = f_1\theta_2 + f_2\theta_1 \;, \quad \bar{f} = f_1\theta_1 + f_2\theta_2 \;, \quad \Delta f = f_2 - f_1 \tag{33}$$

for $f \in \{k,\mu\}$.

Note that η and ξ are invariants of κ ; η represents the spherical part of κ , ξ being the ratio of the deviatoric part of κ to η . One can show that maximum in (31) is attained for $t=0$ and then $c=0, r=-s$; condition $-2r \geq 0$ becomes irrelevant. Function $f(r) = K^0(r) + \xi^2 G^0(r)$ attains maximum at

$$r_0 = (\tilde{k}\Delta\mu\cdot\xi - \Delta k\tilde{\mu})//(\Delta\mu\cdot\xi + \Delta k) \tag{34}$$

Since $K^0 > 0$ and $G^0 > 0$, we have

$$-r \leq \mu_1 \Rightarrow \xi \geq \xi_2 = \frac{\tilde{\mu}+k_1}{\theta_1\Delta\mu} \tag{35}$$

$$-r \geq -k_1 \Rightarrow \xi \leq \xi_1 = \frac{\theta_1\Delta k}{\tilde{k}+\mu_1} \tag{36}$$

Thus $W \geq \underline{W}(\kappa,\theta_1)$ and

$$\underline{W}(\kappa,\theta_1) = \begin{cases} W_2(\theta_1\;,\eta,\xi) & \text{for} \quad \xi \leq \xi_2 \\ W_{int}(\theta_1\;,\eta,\xi) & \text{for} \quad \xi_2 \leq \xi \leq \xi_1 \\ W_1(\theta_1\;,\eta,\xi) & \text{for} \quad \xi \geq \xi_1 \end{cases} \tag{37}$$

where $W_J(\theta_1\;,\eta,\xi) = 2\eta^2(a_J + c_J\xi^2)$, $J=1,2$;

$$W_{int}(\theta_1\;,\eta,\xi) = 2\eta^2[a_J + c_J\xi^2 - A_J(\xi-\xi_J)^2] \;, \tag{38}$$

the last equation being true for both $J=1,2$; coefficients a_J, c_J, A_J depend on the data and will not be reported here. We see that $\underline{W}$ is composed of three parabolas ; the intermediate one is stitched smoothly with W_2 at $\xi=\xi_2$ and with W_1 at $\xi=\xi_1$. Hence the effective constitutive relationships are continuous.

4 CONCLUSIONS

The final result (37) coincides with that concerning entirely Kirchhoff relaxation (substitution $\phi_\alpha = -w_{|\alpha}$, in (16)), cf Lur'e and Cherkaev (1986). Thus the lower bound $\underline{W}$ can be attained by second rank laminates (regimes: $\xi \leq \xi_2$ and $\xi \geq \xi_1$) and by first rank laminates $(\xi_2 \leq \xi \leq \xi_1)$. In all cases eigendirections of κ coincide with directions of layering. Consequently torsion is absent along these directions, hence formulae for $D^{\alpha\alpha\beta\beta}_{hom}$ suffice for proving attainability of the bound. Since these effective stiffnesses are common for both conformal (refined) scaling- based approach and plane scaling-based Duvaut-Metellus' approach (see Lewiński (1991)) one concludes that the design of the weakest Kirchhoff plate with Kirchhoff's microstructure coincides with the weakest Kirchhoff's plate with Hencky-Reissner's microstructure.

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