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On design of H_∞ optimal controls for uncertain nonlinear systems

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Abstract

In this paper, H_∞ optimal control problems for some uncertain nonlinear systems are discussed. By the applications of LSDA-approach and game theory, this kind of optimal problems is evoluted to some convex programmings. Therefore global optimization algorithms for the problems can be obtained.

Keywords

Disturbance attenuation problem (LSDA), game theory, uncertain nonlinear system, robust controller.

1 INTRODUCTION

The true nature of H_∞ optimal control theory is to design a controller for the considered system. This controller should stabilize the closed loop system and meanwhile optimize the H_∞ norm of the gain from disturbance acting the system to a given output (which is often called a penalty) (see Francis, 1987). For details, consider a system:

$$\dot{x} = f(x,u,w), \tag{1}$$
$$z = h(x,u,w), \qquad t \geq t_0, \tag{2}$$

where x, u, w is state vector, control input and disturbance, respectively, $w \in L_2[t_0, \infty)$, f and h are continuous functions with $f(0) = 0, h(0) = 0$, z is the penalty output.

For this system, if there exists a feedback controller, say $u = u^*$, such that,

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(1) the closed loop system

$$\dot{x} \;=\; f(x,u^*,0), \qquad t \geq t_0 \tag{3}$$

is asymptotically stable at the equilibrium point $x_0 = 0$;
(2) whenever $w \neq 0$, L_2-norm of the gain from w to z is not bigger than a given positive constant γ , i.e.,

$$\int_{t_0}^{T} \|z\|^2 dt \leq \gamma^2 \int_{t_0}^{T} \|w\|^2 dt, \qquad T \geq t_0. \tag{4}$$

We will call controller $u = u^*$ a solution of the locally stable disturbance attenuation problem with an attenuation parameter (γ-LSDA, or LSDA- problem, for short) for the system Σ.

It is clear that the H_∞ optimal control theory is just to optimize all the number with which the γ-LSDA problems are solvable. So, both the LSDA problem and the optimization of all attenuation parameters are of importance in the theory of H_∞ optimal control.

In recent years, much attention has been paid to γ-LSDA problems (see, *e.g.* , Hill, 1992; Isidori, 1992; Tong, 1994; Van der Schaft, 1992). For linear system, the existence of a state feedback solution to the LSDA problem is equivalent to the solvability of a matrix Riccatti equation (Scherer, 1989). The linearization approach is still the first step in the study of LSDA problems for nonlinear systems (Van der Schaft, 1991). To deal with affine nonlinear systems, differential game theory has been used by several authors, specially, in a recent paper (Tong, 1994), the authors have investigated this problem directly for some much generalized nonlinear systems by means of convex game theory. Some interesting results have been reached.

On the other hand, LSDA problems will become very difficult to deal with when there exist some uncertainties (such as structured uncertainties) involving in the considered systems. Generally speaking, there is no feasible schedule of design to such problems, especially as concerns the global optimization of attenuation parameters, even for uncertain linear systems. Fortunately, papers (Geromel, 1992) and (Peres, 1993) offered a kind of design methods to the H_∞ optimal problems for a class of uncertain linear systems. They provided an algorithm which ensured a global optimization.

In this paper, we deal with both the LSDA problems and the attenuation parameters optimizations for the nonlinear systems with a convex structured uncertainty. We use the results obtained by (Tong, 1994) to solve the LSDA problem for each nominal nonlinear system. Then by some interesting operations, we transfer the optimization of the attenuation parameters for the whole class of the uncertain nonlinear systems into a mathematical programming on some convex sets. Therefore, it is much easy to provide global optimization algorithms for the problems. By the way, such optimizations also have strong robustness. Certainly, when the systems degenerate to linear cases, our results coincident comparatively with that of papers (Peres, 1993 and Geromel, 1992) and moreover, it can be found that, in this case, the reliability of such inference provided here has been strengthened.

The paper is organized in the following manner. In section 2 the features of LSDA problem and uncertain nonlinear system are recalled at first. We also make some conventions in the section. The formulation of the problem we discuss and the main results are presented in section 3. We just give the outlines of the proofs for the results here. In the last section, section 4, we use the obtained results to discuss uncertain linear systems, this can be considered as a comparison of our newly obtained conceptions with the known ones on the optimization problems.

2 FORMULATION OF PROBLEMS

We will explain some necessary conceptions at the beginning of the section. Then describe the considered problems in detail.

2.1 LSDA Problem

Consider nonlinear system

$$\dot{x} = f(x,u,w) \tag{5}$$
$$z = h(x,u,w) \qquad t \geq t_0 \tag{6}$$

where $x \in R^n$ is the state vector, u, w denote the control input and the disturbance input. We suppose that u and w take values in convex bounded closed sets U and W respectively, also, $w(t) \in L_2[t_0, \infty)$. Furthermore, f and h are continuous functions with $f(0) = 0, h(0) = 0$. These assumptions, will, if not said otherwise, be preserved for all the systems considered.

The output z is always called cost or penalty output for the reason of its practical explanation of a certain kind of cost the system pays because of the occurrence of disturbance w.

LSDA problem has been defined in last section ((1)-(4)). It is clear that the objective of the H_∞ control theory is to search a solution for the LSDA problem which with the minimum attenuation parameter. Therefore, the understanding and investigation on LSDA problem is of much importance and value.

2.2 Description of uncertain nonlinear systems

Consider the following group of nonlinear systems Σ:

$$\dot{x} = f(x) + g(x)u + q(x)w \tag{7}$$
$$z = (h(x), u)^\tau \qquad t \geq t_0 \tag{8}$$

where the function vector $(f, q)^\tau$ belongs to the set D:

$$D = \{\sum_{i=1}^N \xi_i (f_i, q_i)^\tau \mid \xi_i \geq 0, \sum_{i=1}^N \xi_i = 1\},$$

and the symbol τ means transportion of matrix or vector.

Suppose that function vectors (f_i, q_i) form the following systems Σ_i:

$$\dot{x} = f_i(x) + g(x)u + q_i(x)w \tag{9}$$
$$z = (h(x), u)^\tau \qquad t \geq t_0 \tag{10}$$

we will call systems $\Sigma_i, i = 1, \cdots, N$, the nominal systems of uncertain system Σ. This means that the uncertainty of Σ is formed by a convex combination of Σ_i. Clearly Σ is just a common determined nonlinear system when $N = 1$.

The fixed matrix function $g(x)$ implies that we hope that we can control the system in a certain fixed way.

For simplicity, we make the following conventions:

Given $(f, q)^\tau$, the corresponding form of Σ is denoted as F, and we write $F \in \Sigma$. So, for nominal system Σ_i, we say $F_i \in \Sigma$.

3 MAIN RESULTS

3.1 Robust Controllers and Attenuation Parameters

1. The robust solutions of LSDA problems for Σ

For $F \in \Sigma$, let $\mathcal{S}_F$ represents the set

$$\mathcal{S}_F = \{\mathcal{T}\colon R^n \to U \mid \text{such that closed- loop system } \dot{x} = f(x) + g(x)\mathcal{T}(x) \text{ is asymptotically stable at } 0\}.$$

Define a functional h_F: $\mathcal{S}_F \to R^n$ as

$$h_F(\mathcal{T}) = \sup_{w \neq 0} \frac{\|z\|}{\|w\|}, \mathcal{T} \in \mathcal{S}_F,$$

where $\|.\|$ is L_2-norm. Therefore, the functional $h_F(\mathcal{T})$ is actually the value of the L_2-gain of closed-loop system from w to z.

Another set is the epigragh of h_F:

$$\text{epi}h_F = \{(\mathcal{T}, \gamma) \mid h_F(\mathcal{T}) \leq \gamma, \gamma > 0, \mathcal{T} \in \mathcal{S}_F\}.$$

So, if $(\mathcal{T}, \gamma) \in \text{epi}h_F$, the feedback $u = \mathcal{T}(x)$ can stabilize the system F when $w = 0$, and otherwise

$$\|z\| \leq \gamma \|w\|$$

for any feasible disturbance w. This means that $u = \mathcal{T}(x)$ is a solution to the γ-LSDA problem of F.

Define set:

$$\mathcal{P}_1 = \{\gamma > 0 \mid \exists \mathcal{T}\colon (\mathcal{T}, \gamma) \in \text{epi}h_F, \forall F \in \Sigma\}.$$

If $\mathcal{P}_1$ is not empty, then, for $\gamma \in \mathcal{P}_1$, there exists a $\mathcal{T}$ such that $u = \mathcal{T}(x)$ is a solution to LSDA problem for every system $F \in \Sigma$. This means we get a robust LSDA controller for the whole group of systems Σ.

2. H_∞ optimization problem for the uncertain systems

We remember that H_∞ control problem is principally to optimize attenuation parameters with which the LSDA problems are solvable.

Now we discuss the optimization problem for Σ:

$$(P_1) \qquad\qquad \min\{\gamma > 0 \mid \gamma \in \mathcal{P}_1\}.$$

It is worthy to make the following remark:

Remark 1. If there exists a solution, say γ_0 , to problem (P_1), then for the case of $N = 1$, there must exist a feedback controller which reaches the objective of the H_∞ optimal control for system F_1. When $N > 1$, according to the meaning of γ_0, we can find a operator T_ε, for every $\varepsilon > 0$, which solves the common $(\gamma_0 + \varepsilon)$-LSDA problems of system group Σ.

It should be pointed out, meanwhile, that problem (P_1) is usually difficult to deal with, especially there is no feasible method to get a global solution for the problem.

3.2 LSDA-approach Based Method to the Optimization

It is helpful to recall some results about the LSDA problem before we start our inference. In papers (Hill 1992, Van der Schaft 1991 and Veillette 1989), one can find more detailed descriptions of LSDA-problems.

For system $F \in \Sigma$, its Hamilton functional is defined as:

$$H_F(x,p,u,w)(\mu) = \\ p^\tau[f(x) + g(x)u + q(x)w] + z^\tau z - \mu^{-1}w^\tau w,$$

where $x, p \in R^n, u \in U, w \in W, \mu > 0$.

Based on the inference of Tong(1994), $\forall x, p \in R^n$, $H_F(x,p,u,w)$ must has a saddle point respecting to every pair of (u, w), moreover, the saddle point can be expressed as $u^* = -\frac{1}{2}g(x)^\tau p$, and $w^* = \frac{\mu}{2}q(x)^\tau p$. Therefore,

$$\begin{aligned} H_F^*(x,p)(\mu) &= H_F(x,p,u^*,w^*)(\mu) \\ &= p^\tau f(x) + \tfrac{1}{4}[p^\tau(\mu q(x)q(x)^\tau - g(x)g(x)^\tau)p] + h(x)^\tau h(x). \end{aligned}$$

Theorem 1. *For a system $F \in \Sigma$, if there exists a differential positive functional V such that*

$$H_F^*(x, V_x)(\mu) \leq 0 \tag{11}$$

for every $x \in R^n$ where V_x stands for $\frac{dV(x)}{dx}$.

Then, feedback controller $u = u^*(x, V_x) = -\frac{1}{2}g(x)^\tau V_x$ is a solution to the LSDA problem of system F, the attenuation parameter of which is $\gamma = \sqrt{\mu^{-1}}$.

Let us make following assumption:

Assumption (A) All the LSDA-problems will be considered for such parameters:
A=$\{\gamma \mid \gamma = \sqrt{\mu^{-1}}, \mu q(x)q(x)^\tau - g(x)g(x)^\tau \geq 0, \forall x\}$

Define set $\mathcal{E}_F$ as

$\mathcal{E}_F = \{(V, \mu) \mid \mu > 0,\ V \text{ satisfying theorem 1 }\}.$

Theorem 2. *For any $F \in \Sigma$, the following statements are valid*
(i). $\mathcal{E}_F$ is convex;
(ii).If $(V,\mu) \in \mathcal{E}_F$, then $u = u^*(x, V_x)$ is a solution to the LSDA problem of system F with an attenuation parameter $\gamma = \sqrt{\mu^{-1}}$.

This conclusion can be verified from the definition of the set $\mathcal{E}_F$ easily.

It is a natural ideal to find a way of describing the H_∞ optimal problems of systems Σ by the use of nominal systems F_i. To do so, the following set is important.

$$\mathcal{E} = \cap_{i=1}^N \mathcal{E}_{F_i} = \{(V,\mu) \mid (V,\mu) \in \mathcal{E}_{F_i}, i = 1,\cdots,N\}.$$

Surely, $\mathcal{E}$ is a convex set.

Theorem 3.

$$\{(u^*(x,V_x), \sqrt{\mu^{-1}}) \mid (V,\mu) \in \mathcal{E}\} \subseteq \cap_{F\in\Sigma}\mathrm{epi}h_F.$$

Proof. for r arbitrary $F = \sum_{i=1}^N \xi_i F_i \in \Sigma$, when $(V,\mu) \in \mathcal{E}$, we have, for $x \in R^n$,

$$H_{F_i}^*(x,V_x)(\mu) \leq 0$$

because $(V,\mu) \in \mathcal{E}_{F_i}$. So,

$$\begin{aligned} H_F^*(x,V_x)(\mu) &= V_x^\tau f(x) + \frac{1}{4}[V_x^\tau(\mu q(x)q(x)^\tau - g(x)g(x)^\tau)V_x]h(x)^\tau h(x) \\ &= V_x^\tau \sum_{i=1}^N \xi_i f_i(x) + \frac{1}{4}[V_x^\tau(\mu(\sum_{i=1}^N \xi_i q_i(x))(\sum_{i=1}^N \xi_i q_i(x))^\tau - g(x)g(x)^\tau)V_x] \\ &\quad + h(x)^\tau h(x). \end{aligned}$$

The above formula is convex with respect to q, so that

$$H_F^*(x,V_x)(\mu) \leq \sum_{i=1}^N \xi_i H_{F_i}^*(x,V_x)(\mu) \leq 0,$$

So, $(u^*(x,V_x), \sqrt{\mu^{-1}}) \in \mathrm{epi}h_F$ is valid.

Remark. Theorem 3 implies that the set $\mathcal{P}_1$ in nonempty when $\mathcal{E}$ in nonempty.

Let $\mathcal{P}_2$ stands for the set

$$\mathcal{P}_2 = \{\mu > 0 \mid \exists V : (V,\mu) \in \mathcal{E}\}.$$

Theorem 4. *Set $\mathcal{P}_2$ is a convex set and*

$$\{\sqrt{\mu^{-1}} \mid \mu \in \mathcal{P}_2\} \subseteq \mathcal{P}_1.$$

This statement is right clearly.

The convexity of the set $\mathcal{P}_2$ is very important for our inference. Theorem 4 implies that we have constructed a convex subset for set $\mathcal{P}_1$. This makes it possible to discuss optimal problem (P_1) by means of convex programming.

3.3 Global Optimization Algorithms

Consider a convex programming problem:

$$(P_2) \qquad \max\{\mu > 0 \mid \mu \in \mathcal{P}_2\}.$$

Definition. If μ^* is a solution of problem (P_2), then we call number $\gamma^* = \frac{1}{\sqrt{\mu^*}}$ a quasi-optimal attenuation coefficient of systems Σ.

Remark 1.. Programming (P_2) is principally a convex optimal problem, so, generally speaking, it can be easily solved. Unfortunately, in general case, $\mathcal{P}_2$ is just a subset of $\mathcal{P}_1$, hence (P_2) and (P_1) are not equivalent. This is why we present the name of quasi-optimal coefficient here. However, it is worthy to point out that quasi-optimization is rightly the optimization when Σ degenerates to linear systems (See the next section).

The problem (P_2) is a simple convex linear programming. For such a problem, there are many well-developed algorithms, for instance, one can find some detailed discussion in book (Rockafellar,1970). For this reason and for simplicity, we don't list any of the algorithms here.

4 AN EXAMPLE: LINEAR CASE

Consider a family of linear systems Σ_l

$$\dot{x} = Ax + B_1 w + B_2 u \qquad (12)$$

$$z = Cx + Du \qquad t \geq t_0, \qquad (13)$$

where the matrix pair of (A, B_1) belongs to set

$$\{\textstyle\sum_{i=1}^N \xi_i (A_i, B_{1i}) \mid \xi_i \geq 0, \sum_{i=1}^N \xi_i = 1\},$$

and the systems related to (A_i, B_{1i}) are denoted by $F_i \in \Sigma_l$.

If we require the controller for linear system having the linear form of feedback of state, say, $u = -Kx$, then with the using of the same arguments as before we get the following descriptions.

Lemma. *For any system $F \in \Sigma_l$, and $\gamma > 0$, the following statements are equivalent:*
(1) there exists matrix K such that $u = -Kx$ is the solution of γ-LSDA problem of system F;
(2) there exists a positive symmetric matrix $\mathcal{W}$ such that

$$H_F^*(x, \mathcal{W})(\mu) \leq 0, \quad x \in R^n$$

where $H_F^*(x, \mathcal{W}) = H_F^*(x, V_x)$, $V = \frac{1}{2} x^\tau \mathcal{W} x$, and $\mu = \frac{1}{\gamma^2}$.

This conclusion was presented in paper (Scherer, 1989) where it was proved for all linear systems. Now we can get a interpretation of theorem 4 to the linear case:

Theorem 5. *The following statements are valid:*
(1) $\mathcal{P}_2$ is a convex set;
(2) the set $\mathcal{P}_2$ is isomorphic to the set $\mathcal{P}_1$, i.e.,

$\{\frac{1}{\sqrt{\mu}} \mid \mu \in \mathcal{P}_2\} = \mathcal{P}_1$.

It is clear that the implication of theorem 5 can be stated as following.

Theorem 6. *For the uncertain linear systems Σ_l, its H_∞ optimization problems is equivalent to a convex linear problem (P_2).*

Remark 2. This conclusion means that, on the set A, optimization problem (P_2) is equivalent to convex programming (P_1). Infact, whenever $A \cap P_1$ is nonempty, assumption (A) becomes trivial. To the other cases, the relative problems have been investigated in my Ph.D thesis.

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